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Instanton Action for Two-Dimensional Black Hole in the Rainbow Gravity

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ABSTRACT

In this work, we find the exact solution of the Duffin-Kemmer-Petiau (DKP) equation for a two-dimensional black hole model in the framework of rainbow gravity. Subsequently, we discuss the transmission probability by making use of the instanton action method and study briefly the effect of rainbow functions on a two-dimensional black hole instanton. Also, we introduce graphical analysis of our result.

***Keywords*:** DKP equation, Rainbow Gravity, Black Hole, Instanton Action, Cauchy's residue theorem

1. INTRODUCTION

In recent years, the theory of Loop Quantum Gravity (LQG) has been drawn significant attention since it merges Quantum Mechanics (QM) and General Relativity (GR) [1-3]. This idea has mainly been used to explain the existence of a minimum measurable length eliminating the big bang singularity problem. Nowadays, the paradox emerged from between the Lorentz symmetry and the existence of minimal length is believed to be solved via the help of doubly special relativity (DSR) formalism, which is a deformed structure of the special relativity (SR) theory [4-7]. However, such a modification brings a number of problems such as threshold anomalies of TeV photons and ultra-high energy cosmic rays. Furthermore, the modified

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dispersion relation (MDR) has been directly applied for the curved spacetime by using a deformed equivalence principle (DEP) of the GR [8]. The heart of this idea is basically based on the belief that spacetime geometry behaves as the energy of the test particle. From Ref [9-10], the generalized form of the MDR is given by [11]

$$f^2(\varepsilon)E^2 - g^2(\varepsilon)P^2 = m^2 \tag{1}$$

with $\varepsilon = \frac{E}{E_{pl}}$. Here, E_{pl} indicates the Plank energy while E represents particle's energy. Also, $f(\varepsilon)$ and $g(\varepsilon)$ are known as rainbow functions, which satisfy the following requirements

$$\lim_{\varepsilon\to 0} f(\varepsilon) = \lim_{\varepsilon\to 0} g(\varepsilon) = 1. \tag{2}$$

Magueijo and Smolin [11] has proposed the rainbow functions with the help of the varying light speed idea as given below

$$f(\varepsilon) = \frac{1}{1-\varepsilon\gamma}, \qquad g(\varepsilon) = 1\ , \tag{3}$$

where γ is known as the rainbow parameter. Other existing scenarios related to the rainbow functions presented in literature are listed as follows [6, 12]

$$f(\varepsilon) = \sqrt{1-\varepsilon^2}, \qquad g(\varepsilon) = 1, \tag{4.a}$$

$$f(\varepsilon) = 1, \qquad g(\varepsilon) = 1 + \frac{\varepsilon}{2}. \tag{4.b}$$

The modified equivalence principle states that tetrads obey the following relation [9]

$$g_{\mu\nu} = e^i_\mu(\varepsilon) \otimes e^j_\nu(\varepsilon)\eta_{ij} \tag{5}$$

with the expressions

$$e^0_0(\varepsilon) = \frac{1}{f(\varepsilon)}\tilde{e}^0_0 \tag{6a}$$

and

$$e^i_i(\varepsilon) = \frac{1}{g(\varepsilon)}\tilde{e}^i_i. \tag{6b}$$

where: the tilde indicates the energy-independent tetrads and η_{ij} is the Minkowski spacetime.

Recently, the GR theory has gained considerable increasing popularity since it opens a new window to understand the gravity from an interesting perspective [13-15]. In this model, the gravity was interpreted as a result of geometry by Einstein [16-17]. He formulated his idea by introducing the following equation

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu} \tag{7}$$

where $T_{\mu\nu}$ denotes the stress-energy tensor of matter fields, G indicates the Newtonian constant of gravitation and $G_{\mu\nu}$ is the Einstein field tensor written as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \quad . \tag{8}$$

In the above definition, $R_{\mu\nu}$ and R represent the Ricci curvature tensor and the scaler curvature, respectively.

Black hole is an astronomical object with a strong gravitational. It pulls cosmological objects strongly so that even the light cannot escape from it, thus such astrophysical objects cannot be observed directly. However, astronomers detect its existence by making use of a variety of clever techniques. Also, it is not always easy to analyze the physical properties of a black hole in a four-dimensional curved spacetime because of some difficulties in quantizing gravity. Generally, such difficulties are overcome theoretically by reducing four-dimensional spacetime into two-dimensional spacetime [18-19]. In this case, the corresponding Einstein field equation is found as

$$R - \Lambda = 8\pi G\, T \tag{9}$$

where T shows the trace of two-dimensional energy momentum tensor and Λ is the well-known cosmological constant. Mann [20] proposed the following line-element describing a two-dimensional black hole:

$$ds^2 = -\alpha(x)\, dt^2 + \frac{1}{\alpha(x)} dx^2 \tag{10}$$

where

$$\alpha(x) = -\frac{\Lambda}{2} x^2 + 2M|x| - C\,, \tag{11}$$

C denotes an arbitrary constant and M represents positive energy source. If we rewrite two-dimensional black hole metric by making use of the rainbow gravity formalism with the help of Eq. 6, we reach at the following result

$$ds^2 = -\frac{\alpha(x)}{f^2}\, dt^2 + \frac{1}{\alpha(x) g^2} dx^2. \tag{12}$$

The Wentzel-Kramers-Brillouin (WKB) approach is a method used for determining the tunneling behavior of quantum particles in Quantum Field Theory (QFT) [21]. As known that, the quantum mechanics always say that a particle has a non-zero tunneling probability penetrating in region where be classically forbidden and this transmission probability can be easily calculated with the Feynman path integral [21]

$$\langle n \left| e^{-iHt/\hbar} \right| n \rangle = N e^{\frac{i}{\hbar} S[x(t)]} \tag{13}$$

where $S[x(t)] = \int_{-a}^{a} dx\,\sqrt{2(V(x) - E)}$ is the classical action, N indicates a normalization constant, $\hbar$ is the Planck's constant and a is known as an intersection point for system potential.

If the form of the system's potential acts like a well and a particle sits at the bottom of this well, its corresponding energy will be close to zero. For this reason, the classical action is converted into instanton action as the following form [21]

$$S[x(t)] = \int_{-a}^{a} dx\sqrt{2V(x)} \tag{14}$$

where $V(x)$ represents the system potential.

The DKP equation was firstly suggested by Duffin, Kemmer and Petiau and is morphologically similar to the general form of Dirac equation except from γ matrices since they are replaced with β matrices [22-24]. However, we are generally faced by some various challenges when getting the solution of the DKP equation in (3 + 1) dimensions since it has a 16-component wave equation. From this mathematical point of view, it can be clearly said that assuming a lower number of dimensions is an alternative idea and a self-consistent method. Thus, it is used to obtain a clue about the behavior of the DKP equation in four-dimensional spacetime. There are also many studies based on such a reduction in the literature [25-28].

We organize this paper as follows. Firstly, we find the exact solution of the DKP equation for the two-dimensional black hole metric written as equation (12). In the third section, we calculate transmission probability based on the signs of spatial variable by focusing on the instanton action method, subsequently, the obtained result is graphically analyzed for different cases of the rainbow functions. The last section is devoted to final remarks.

2. EXACT SOLUTION OF THE DKP EQUATION

The DKP equation for a curved spacetime is described by the following equation [29-31]

$$\left[i\beta^{\mu}\left(\partial_{\mu} - \Sigma_{\mu}\right) - m\right]\Psi(t,\vec{x}) = 0 \tag{15}$$

where m is the particle mass, $\beta^{\mu} = \gamma^{\mu} \otimes I + I \otimes \gamma^{\mu}$ represent the Kemmer matrices, $\Psi(t,\vec{x})$ indicates the 16-component Kemmer wave function and Σ_{μ} is the spinorial connections, which are given by

$$\Sigma_{\mu} = \Gamma_{\mu} \otimes I + I \otimes \Gamma_{\mu}. \tag{16}$$

where, I is the $4 \otimes 4$ identity matrices and Γ_{μ} are the spin connections for spin-1/2 particles.

Spin connections are computed via the subsequent relation,

$$\Gamma_{\lambda} = -\frac{1}{8} g_{\mu\alpha}\Gamma^{\alpha}_{\nu\lambda}[\gamma^{\mu},\gamma^{\nu}] \tag{17}$$

where $\Gamma^{\alpha}_{\nu\lambda}$ are the Christoffel symbols, which are written directly by means of the metric tensor as follows

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}). \tag{18}$$

Unal [32-34] illustrated that the Kemmer wave function is an invariant under the local Lorentz transformations, when the DKP particles are mathematically equivalent to the product of a two-identical particle systems of spin-1/2. Thus, the Kemmer wave function and Kemmer matrices are transformed into

$$\Psi(t,\vec{x}) = \Psi_D(t,\vec{x}) \otimes \Psi_D(t,\vec{x}) = \begin{bmatrix} h_1 \\ h_0 \\ h_0 \\ h_2 \end{bmatrix} e^{-iwt} \tag{19}$$

and

$$\beta^\mu = \sigma^\mu \otimes I + I \otimes \sigma^\mu, \tag{20}$$

where: since the usual Dirac matrices are replaced with the Pauli spin matrices and the Kemmer wave function oscillates with time. If we prefer to take the Pauli matrices as $\sigma^\mu = (\sigma^3, -i\sigma^2)$ and insert Eqs.19 and 20 into Eq. 15, we obtain the following coupled differential equations

$$\left(2ifw\alpha^{-\frac{1}{2}} - m\right) h_1 + 2g\left(\alpha^{\frac{1}{2}}\partial_x - \frac{\alpha'}{2\alpha^{\frac{1}{2}}}\right) h_0 = 0, \tag{21}$$

$$\left(2ifw\alpha^{-\frac{1}{2}} + m\right) h_2 + 2g\left(\alpha^{\frac{1}{2}}\partial_x - \frac{\alpha'}{2\alpha^{\frac{1}{2}}}\right) h_0 = 0, \tag{22}$$

$$h_0 = \frac{g\alpha^{\frac{1}{2}}}{m} \partial_x(h_2 - h_1). \tag{23}$$

After performing some mathematical algebra, we reach at the following expression and differential equation;

$$h_1 + h_2 = \frac{2iwf\alpha^{-\frac{1}{2}}}{m}(h_1 - h_2) \qquad , \tag{24}$$

$$\left[\frac{\partial^2}{\partial x^2} + \left(\frac{wf}{g}\right)^2 \frac{1}{\alpha^2} + \left(\frac{m}{2g}\right)^2 \frac{1}{\alpha}\right] \Phi(x) = 0, \tag{25}$$

where $\Phi(x) = h_2 - h_1$. If we substitute Eq. 11 into Eq. 25, we get

$$\left[\frac{\partial^2}{\partial x^2} + \left(\frac{2wf}{g}\right)^2 \frac{1}{(\Lambda x^2 - 4M|x| + 2C)^2} - \left(\frac{m}{g\sqrt{2}}\right)^2 \frac{1}{(\Lambda x^2 - 4M|x| + 2C)}\right] \Phi(x) = 0 \tag{26}$$

One can easily conclude that the exact solution of Eq. 26 is characterized by the sings of x variable. Following ref. [abb], the whole solution of the above equation is obtained by means of the Associated Legendre polynomials [35]

$$\Phi_{\mp}(x) = \sqrt{\Lambda x^2 - 4Mx + 2C}\left\{C_1 P_l^m\left(\frac{x \mp \frac{M}{\Lambda}}{\sqrt{\left(\frac{2M}{\Lambda}\right)^2 - \frac{2C}{\Lambda}}}\right) + C_2 Q_l^m\left(\frac{x \mp \frac{M}{\Lambda}}{\sqrt{\left(\frac{2M}{\Lambda}\right)^2 - \frac{2C}{\Lambda}}}\right)\right\}, \tag{27}$$

where $\mp$ indicates the positive and negative spatial solution, *m* and *n* are parameters of the Associated Legendre polynomials, and their values are found as [35]

$$l = \frac{1}{2}\left(-1 + \sqrt{1 + \frac{2m^2}{\Lambda g^2}}\right), \tag{28}$$

$$m = \sqrt{-1 + \frac{\left(\frac{wf}{\Lambda g}\right)^2}{\left(\frac{M}{\Lambda}\right)^2 - \frac{C}{2\Lambda}}}\ . \tag{29}$$

where, the existence of a non-zero solution of Eq.29 causes *m* to be an integer.

Therefore, the energy spectrum is quantized as:

$$E_n = \frac{g\hbar}{f}\sqrt{2M^2 - C\Lambda}\, n, \tag{30}$$

where *n* is a positive inter number.

3. THE CALCULATION OF TRANSMISSION PROBABILITY USING INSTANTON ACTION

From Eq.26, it can easily be said that the remaining terms excluding the second derivative term and the potential of the system is written as the following form

$$V(x) = \left(\frac{2wf}{g}\right)^2 \frac{1}{(\Lambda x^2 - 4M|x| + 2C)^2} - \left(\frac{m}{g\sqrt{2}}\right)^2 \frac{1}{(\Lambda x^2 - 4M|x| + 2C)}. \tag{31}$$

After writing the instanton action for this potential by making use of Eq. 14, we get the subsequent conclusion

$$S[x(t)] = \int_{-\infty}^{\infty} \frac{\sqrt{\left(\frac{2wf}{\Lambda g}\right)^2 - \frac{2m^2}{\Lambda g^2}\left(x^2 - 4\frac{M}{\Lambda}|x| + \frac{2C}{\Lambda}\right)}}{x^2 - 4\frac{M}{\Lambda}|x| + \frac{2C}{\Lambda}}\, dx. \tag{32}$$

where, according to the signs of spatial variable, Eq. 32 becomes as follows

$$S[x(t)] = \begin{cases} \int_{-\infty}^{\infty} \frac{\sqrt{\left(\frac{2wf}{\Lambda g}\right)^2 - \frac{2m^2}{\Lambda g^2}(x^2 - 4\frac{M}{\Lambda}x + \frac{2C}{\Lambda})}}{\left(x - \frac{2M}{\Lambda} - A\right)\left(x - \frac{2M}{\Lambda} + A\right)} dx \quad for \quad x > 0 \\ \\ \int_{-\infty}^{\infty} \frac{\sqrt{\left(\frac{2wf}{\Lambda g}\right)^2 - \frac{2m^2}{\Lambda g^2}(x^2 + 4\frac{M}{\Lambda}x + \frac{2C}{\Lambda})}}{\left(x + \frac{2M}{\Lambda} - A\right)\left(x + \frac{2M}{\Lambda} + A\right)} dx \quad for \quad x < 0 \end{cases} \tag{33}$$

where

$$A = \sqrt{\left(\frac{2M}{\Lambda}\right)^2 - \frac{2C}{\Lambda}}. \tag{34}$$

From Eq.33, each integral has a mathematical discontinuity at the following points

$$\begin{aligned} x &= \frac{2M}{\Lambda} \mp A \qquad && x > 0 \\ x &= -\frac{2M}{\Lambda} \mp A \qquad && x < 0 \end{aligned} \tag{35}$$

and they lead the above integrals to be diverges. If we use Cauchy's residue theorem to take the integral placed at the Eq. 33 [36], we get the following results for transmission probability,

$$|T| = \begin{cases} e^{\frac{\pi wf}{g}\sqrt{4M^2 - 2C\Lambda}} & x > 0 \\ \\ e^{-\frac{\pi wf}{g}\sqrt{4M^2 - 2C\Lambda}} & x < 0 \end{cases} \tag{36}$$

where Planck's constant is adopted to 1 and positively oriented contours has been taken in account for the residual calculation. Furthermore, if the rainbow functions in Eq. 4a and 4.b are inserted into Eq. 35, we find semi-classical transmission probability as follows

- First Scenario:

$$|T| = \begin{cases} e^{\pi w\sqrt{(4M^2 - 2C\Lambda)(1-\varepsilon^2)}} & x > 0 \\ \\ e^{-\pi w\sqrt{(4M^2 - 2C\Lambda)(1-\varepsilon^2)}} & x < 0 \end{cases} \tag{37}$$

- Second Scenario:

$$|T| = \begin{cases} e^{\frac{2\pi w\sqrt{4M^2 - 2C\Lambda}}{2+\varepsilon}} & x > 0 \\ \\ e^{-\frac{2\pi w\sqrt{4M^2 - 2C\Lambda}}{2+\varepsilon}} & x < 0 \end{cases} \tag{38}$$

where $2M^2$ is bigger than $C\Lambda$ to have a physical meaning of semi-classical transmission probability for both scenarios. If we create a plot of both scenarios, we get

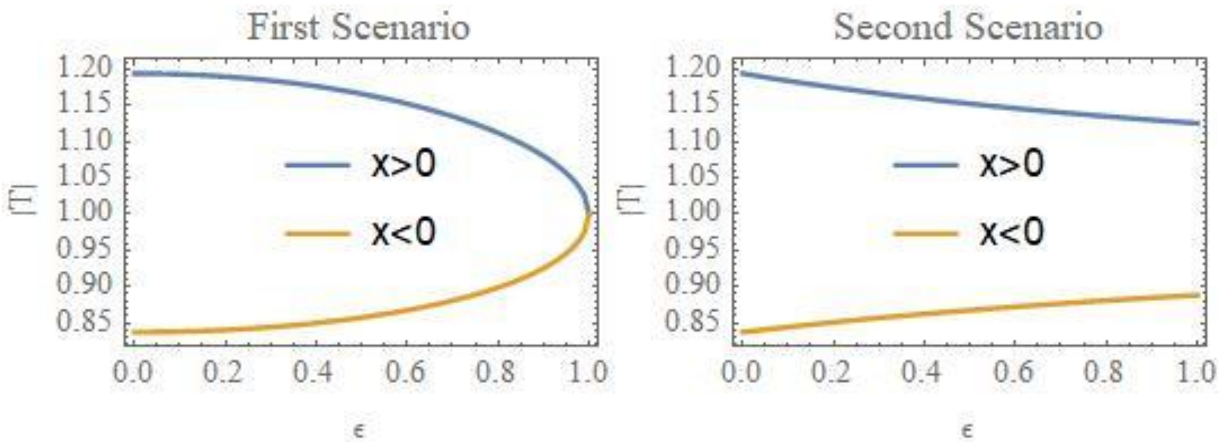

Figure 1. Semi-classical transmission probability is plotted by ε for both scenarios.

As shown in the figure, semi-classical transmission probability for positive spatial values is decreasing with ε while it for negative spatial values is increasing with ε in first scenario. Similarly, such a behaviour is observed in the second scenario. However, transmission probability depends on a non-linear structure of function in first scenario while it has a linear behaviour of function in second scenario.

4. CONCLUSIONS

In this study, we solved the DKP equation for a two-dimensional black hole model in gravity's rainbow framework. In this case, the general solution was written by means of the associated Legendre Polynomials related to the signs of x variables. Subsequently, we calculated instanton action for the two-dimensional black hole metric to find semi-classical transmission probability and made a plot related to it. According to the obtained results, ε that is directly related to the Planck energy acts as a reducing and increasing factor for the positive and negative region of both scenarios, respectively.

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