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A Study of Some Characteristics of the Accelerated Expansion of the Universe in the Framework of Brans-Dicke Theory

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ABSTRACT

In the framework of the generalized Brans-Dicke theory of gravity, time dependence of various cosmological parameters has been determined in the present study for a spatially flat, homogeneous and isotropic universe filled with pressure-less matter. Mathematical formulations have been carried out with the help of two models, based upon two different expressions for the scale factor. In the first model, the entire matter content (dark matter + baryonic matter) of the universe has been assumed to be conserved. A smooth transition from a state of decelerated expansion to a state of accelerated expansion of the universe has been obtained from an exact solution of the field equations, without incorporating any parameter in the theoretical formulation that represents the dark energy. Time dependence of the scalar field has been determined from this solution with the help of astrophysical characteristics of the expanding universe. The nature of dependence of the Brans-Dicke parameter upon time and also upon the scalar field has been found. The Brans-Dicke parameter has been found to have a small negative value and it becomes more negative as the scalar field decreases with time. It has been found in the present study that the gravitational constant, which is reciprocal of the scalar field parameter, increases with time. In the second of the two models discussed here, an ansatz has been assumed regarding the mode of change of the dark energy content of the universe with time. Using this model, the time dependence of the densities of matter and dark energy and also the density parameters corresponding to these two constituents of the universe has been determined.

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1. INTRODUCTION

The accelerated expansion of the universe has become one of the most widely explored phenomena in the field of cosmology following a number of astrophysical observations and their interpretations in recent times [1-6]. For this acceleration, the *dark energy* (DE), an exotic entity having a negative pressure, has been held responsible. Throughout the world, an extensive research is being carried out to find the properties of DE which constitutes approximately seventy percent of the universe. Several models, involving DE, have been constructed to explain the acceleration which followed a phase of deceleration in cosmic expansion. This transition from deceleration to acceleration, of the process of expansion, is indicated by the change of sign of the deceleration parameter from positive to negative, as per interpretations of data obtained from supernova [7]. The present phase of accelerated expansion is known to be a recent phenomenon in the history of the universe and it follows a phase of deceleration, as revealed by astrophysical observations [8]. Nucleosynthesis of the universe and its structure formation are dependent upon this sequence of events. It has clearly been demonstrated by many recent works that during the span of time when the redshift (z) was greater than a certain value (slightly smaller than one), universe was expanding with deceleration [3, 9].

The calculations for the present article are based on the Brans-Dicke (BD) theory of gravitation. On the basis of an earlier research by P. Jordan, this theory was proposed by R.H. Dicke and C.H. Brans in 1961 [10]. This BD theory had provided a new theoretical framework to explain gravitational phenomena, like Albert Einstein's theory of General Relativity. The gravitational interactions, according to BD theory, is governed by a scalar field and also a tensor field evident in general relativity. The scalar field parameter (ϕ), according to BD theory, is the reciprocal of what is known as gravitational constant (G), which can no longer be treated as a constant. The properties of space-time geometry are controlled here by a coupling parameter, denoted by ω. The behaviours of cosmological parameters, obtained from general relativity, can be reproduced by BD theory using a constant value of ϕ and an infinite value of ω [11]. Among all theories of gravitation, BD theory has a position of prominence for its ability to account for the properties of cosmic expansion since the early phase of inflation [12]. The parameter ω has to be treated as dependent upon the scalar field parameter ϕ, according to a modified (or generalized) version of the BD theory [13-15]. Several models, based on the BD theory of gravity, have been formulated to account for the characteristics of the expanding universe in terms of the properties of cosmological parameters [16-19].

For the accelerated expansion of the universe, the coupling parameter (ω) needs to have a very low negative value, as obtained from models based on the generalized BD theory, contrary to what was found during the applications of the initial version of the theory [12, 20-22]. In the framework of BD theory, the findings from the FRW models with a dynamic Λ-term and a variable deceleration parameter have been analyzed in detail in a recent study in this field [23]. Through a recent research, based on an interaction between matter and scalar field, it has been shown that the generalized form of BD theory can successfully explain the accelerated expansion of the universe, preceded by deceleration, for a high value of the Brans-Dicke parameter (ω) [24].

The purpose of the present study is to obtain the time evolution of various cosmological parameters in the light of generalized BD theory. Model-1 is based on the conservation of the matter content (dark + baryonic) of the universe, having zero pressure. A change of sign of the deceleration parameter, from positive to negative, is obtained without considering contributions from anything that represents dark energy. Time evolution of the gravitational constant (G) and other quantities and also the dependence of the BD parameter (ω) upon the scalar field (ϕ) have been determined through this model. G is found to rise with the change of time. In Model-2, we have used a scale factor which is not based upon a theory where matter has been treated as conserved. Time variation of the deceleration parameter, based on this scale factor, shows the transition from a phase of decelerated to accelerated expansion. In accordance with facts obtained from astrophysical observations, an ansatz has been assumed to describe the time dependence of the dark energy content of the universe. Time evolutions of the densities of dark energy and matter and the corresponding density parameters have been determined.

2. SOLUTION OF FIELD EQUATIONS

For a homogeneous and isotropic universe with zero spatial curvature (flat space), the field equations of the generalized BD theory are given by [17],

$$3\frac{\dot{a}^2}{a^2} = \frac{\rho_m}{\phi} + \frac{\omega(\phi)}{2}\frac{\dot{\phi}^2}{\phi^2} - 3\frac{\dot{a}}{a}\frac{\dot{\phi}}{\phi} \tag{1}$$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = -\frac{\omega(\phi)}{2}\frac{\dot{\phi}^2}{\phi^2} - 2\frac{\dot{a}}{a}\frac{\dot{\phi}}{\phi} - \frac{\ddot{\phi}}{\phi} \tag{2}$$

here: a, ϕ, ω and ρ_m are respectively the scale factor, scalar field parameter, Brans-Dicke coupling parameter and the density of matter (*dark* + *baryonic*). A single dot (.) over any quantity denotes its time derivative of the first order and double dot (..) over any quantity denotes its time derivative of the second order (i.e., $\dot{x} = \frac{dx}{dt}$ and $\ddot{x} = \frac{d^2x}{dt^2}$). In obtaining these equations, the pressure of the cosmic fluid has been taken to be zero, which is consistent with the model of the present matter-dominated universe which is constituted by pressure-less dust. Combining equations (1) and (2) we get,

$$4\frac{\dot{a}^2}{a^2} + 2\frac{\ddot{a}}{a} = \frac{\rho_m}{\phi} - 5\frac{\dot{a}}{a}\frac{\dot{\phi}}{\phi} - \frac{\ddot{\phi}}{\phi} \tag{3}$$

<u>MODEL - 1</u>

On the basis of some previous studies, we use the following ansatz for the scalar field for solving equation (3) [17, 23].

$$\phi = \phi_0 \left(\frac{a}{a_0}\right)^n \tag{4}$$

where, ϕ_0 and a_0 are the values of ϕ and a at the present time (i.e., at $t = t_0$) and n is a constant parameter.

Considering the matter content to be conserved, we write,

$$\rho_m = \rho_{m_0} \left(\frac{a_0}{a}\right)^3 \tag{5}$$

where, ρ_{m_0} is the value of ρ_m at the present time.

Using equations (4) and (5) in equation (3), one gets the following equation.

$$\frac{\ddot{a}}{a} + (n+2)\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{(n+2)} \frac{\rho_{m_0}}{\phi_0} \left(\frac{a}{a_0}\right)^{-(n+3)} \tag{6}$$

Equation (6) can be expressed as,

$$-qH^2(n+2) + (n+2)^2 H^2 = \frac{\rho_{m_0}}{\phi_0} \left(\frac{a}{a_0}\right)^{-(n+3)} \tag{7}$$

For obtaining equation (7) from (6), we have used the expressions for the Hubble parameter $\left(H = \frac{\dot{a}}{a}\right)$ and the deceleration parameter $\left(q = -\frac{\ddot{a}a}{\dot{a}^2}\right)$.

Equation (7) is a relation between a, H and q, which are functions of time. Their values at any instant of time must satisfy the equation. Taking that instant to be $t = t_0$, we put $a = a_0$, $H = H_0$ and $q = q_0$ in equation (7), leading to the following form.

$$-q_0 H_0^2 (n+2) + (n+2)^2 H_0^2 = \frac{\rho_{m_0}}{\phi_0} \tag{8}$$

Equation (8) is quadratic in n. Using the known values of the cosmological parameters (q, H) at present time (i.e., at $t = t_0$), we get, $n = -1.94, -2.08$.

The solution of equation (6) is,

$$a = a_0 \left[\frac{(3+n)\{(t+C_2)^2 \rho_{m_0}^2 - C_1 \phi_0^2 (2+n)^2\}}{2(2+n)\phi_0 \rho_{m_0}}\right]^{\frac{1}{(3+n)}} \tag{9}$$

where C_1 and C_2 are the constants of integration.

Putting $a = a_0$ at $t = t_0$ in equation (9) we get,

$$C_1 = \frac{\left[(t_0+C_2)^2 \rho_{m_0}^2 - \frac{2\rho_{m_0}\phi_0(2+n)}{(3+n)}\right]}{\phi_0^2 (2+n)^2} \tag{10}$$

Thus, C_1 is a function of the two parameters, n and C_2.

Substituting for C_1 in equation (9), from equation (10), one gets,

$$a = a_0 \left[\frac{(n+3)\rho_{m_0}\{(C_2+t)^2-(C_2+t_0)^2\}+2\phi_0(n+2)}{2\phi_0(n+2)}\right]^{\frac{1}{(n+3)}} \tag{11}$$

Using equation (11) we get the following expression for the Hubble parameter,

$$H = \frac{\dot{a}}{a} = \frac{2\rho_{m_0}(C_2+t)}{(n+3)\rho_{m_0}\{(C_2+t)^2-(C_2+t_0)^2\}+2\phi_0(n+2)} \tag{12}$$

Putting $H = H_0$ at $t = t_0$, in equation (12), one obtains,

$$H_0 = \frac{\rho_0}{\phi_0}\frac{(C_2+t_0)}{(n+2)} \tag{13}$$

Using equation (13) we get,

$$C_2 = \frac{H_0\phi_0}{\rho_0}(n+2) - t_0 \tag{14}$$

From equation (8) we obtained, $n = -1.94, -2.08$. For these two values we get $C_2 = 3.00 \times 10^{17}, \ -1.43 \times 10^{18}$ respectively from equation (14).

It is evident from equation (13) that, if $C_2 + t_0 > 0$, we must have $n > -2$ to keep $H_0 > 0$ (a necessary condition for the expanding universe).

Using equation (11), one obtains the following expression for the deceleration parameter.

$$q = \frac{n+1}{2} + \frac{(n+3)\rho_{m_0}(C_2+t_0)^2-2\phi_0(n+2)}{2\rho_{m_0}(C_2+t)^2} \tag{15}$$

Putting $q = q_0$ at $t = t_0$ in equation (15), we get,

$$q_0 = \frac{n+1}{2} + \frac{n+3}{2} - \frac{\phi_0}{\rho_{m_0}}\frac{(n+2)}{(C_2+t_0)^2} \tag{16}$$

Using equation (16) we get,

$$C_2 = \left(\frac{\phi_0}{\rho_{m_0}}\frac{n+2}{n+2-q_0}\right)^{1/2} - t_0 \tag{17}$$

From equation (8) we obtained, $n = -1.94, -2.08$. For the first value of n, we have $C_2 = 3.00 \times 10^{17}$. As per equation (17), C_2 can be imaginary for $n < -2$ (since $q_0 < 0$).

As per astrophysical observations, the universe has undergone a smooth transition from a phase of decelerated expansion to a phase of accelerated expansion [7, 8, 17]. Therefore, the deceleration parameter must have decreased from a positive value to its present value, which is negative.

Taking q to be constantly decreasing with time, i.e., $\dot{q} < 0$ for all values of t, we get the following inequality from eqn. (15).

$$\frac{-(n+3)\rho_{m_0}(C_2+t_0)^2 + 2\phi_0(n+2)}{\rho_{m_0}(C_2+t)^3} < 0$$

Putting $t = t_0$, the above relation becomes,

$$\frac{-(n+3)\rho_{m_0}(C_2+t_0)^2 + 2\phi_0(n+2)}{\rho_{m_0}(C_2+t_0)^3} < 0$$

If $C_2 + t_0 > 0$, the above relation implies that,

$$n < -2 + \left\{\frac{2\phi_0}{\rho_{m_0}}\frac{1}{(C_2+t_0)^2} - 1\right\}^{-1} \tag{18}$$

Equation (18) gives the upper limit for n, for $C_2 + t_0 > 0$.

For an expanding universe we always have $H > 0$, since $\frac{da}{dt} = aH$,where $a > 0$. Therefore, for $C_2 + t_0 > 0$, equation (12) leads to the following inequality.

$$2\phi_0\frac{n+2}{n+3} > \rho_{m_0}\{(C_2+t_0)^2 - (C_2+t)^2\} \tag{19}$$

The above relation can be valid for all values of t if $2\phi_0\frac{n+2}{n+3}$ is greater than the largest value of $\rho_{m_0}\{(C_2+t_0)^2 - (C_2+t)^2\}$. Its largest value is $\rho_{m_0}\{(C_2+t_0)^2 - {C_2}^2\}$, corresponding to $t = 0$. Therefore, the condition becomes,

$$n > -2 + \left[\frac{2\phi_0}{\rho_{m_0}}\frac{1}{(C_2+t_0)^2 - {C_2}^2} - 1\right]^{-1} \tag{20}$$

Equation (20) gives the lower limit for n, for $C_2 + t_0 > 0$.

Using equations (18) and (20) we write,

$$n_1 < n < n_2 \tag{21}$$

where, $n_1 = -2 + \left[\frac{2\phi_0}{\rho_{m_0}}\frac{1}{(C_2+t_0)^2 - {C_2}^2} - 1\right]^{-1}$ and $n_2 = -2 + \left\{\frac{2\phi_0}{\rho_{m_0}}\frac{1}{(C_2+t_0)^2} - 1\right\}^{-1}$ (21A)

Equation (21) clearly shows the range of variation for n (for $C_2 + t_0 > 0$), where its lower and upper limits are n_1 and n_2 respectively.

For $C_2 + t_0 < 0$, one obtains the following inequalities from the conditions of $\dot{q} < 0$ and $H > 0$ respectively.

$$n < -2 + \left[\frac{2\phi_0}{\rho_{m_0}} \frac{1}{(C_2+t_0)^2 - {C_2}^2} - 1\right]^{-1} \text{ and } n > -2 + \left\{\frac{2\phi_0}{\rho_{m_0}} \frac{1}{(C_2+t_0)^2} - 1\right\}^{-1}$$

These relations can be expressed as,

$$n < n_1 \text{ and } n > n_2 \tag{22}$$

where, n_1 and n_2 are given by equation (21A). Since $n_2 > n_1$, no value of n satisfies these two conditions (of eqn. 22) together. It means that, for the sake of consistency with astrophysical observations, the choice of the constant C_2 in the present model should be such that we have,

$$C_2 + t_0 > 0 \tag{23}$$

Equation (23) leads to the conclusion that one must use equation (21) for choosing the values for n for all calculations. It is now confirmed that $n > -2$.

Let T be the instant of time in the past at which the universe had $q = 0$. Putting $q = 0$ at $t = T$ in equation (15) we get,

$$T = -C_2 + \sqrt{\frac{2(2+n)\phi_0 - (3+n)\rho_{m_0}(t_0+C_2)^2}{(1+n)\rho_{m_0}}} \tag{24}$$

The universe is known to have a phase of decelerated expansion before the acceleration began [7, 8, 17]. Thus, we must have $T < t_0$. Applying this condition, upon equation (24), we get,

$$(t_0 + C_2)^2 > \frac{2(2+n)\phi_0 - (3+n)\rho_{m_0}(t_0+C_2)^2}{(1+n)\rho_{m_0}} \tag{25}$$

Based on our initial findings from equation (8), regarding the two values of$n$ (i.e., $n = -1.94, -2.08$), it is clear that $(1+n)\rho_{m_0}$ is a negative quantity. Multiplying both sides of the above inequality by $(1+n)\rho_{m_0}$, we arrive at the following expression.

$$(2+n)\rho_{m_0}(t_0 + C_2)^2 < (2+n)\phi_0 \tag{26}$$

Taking $(2+n) > 0$ (i.e., $n > -2$, as per eqn. 21) for equation (26), one can write,

$$\rho_{m_0}(t_0 + C_2)^2 = \alpha\phi_0 \text{ with } 0 < \alpha < 1 \tag{27}$$

Based on astrophysical observations, the time at which the signature flip of the deceleration parameter (q) took place, can be expressed as $T = \beta t_0$, where $0 < \beta < 1$. Therefore, using equation (24), we write,

$$\beta = \frac{1}{t_0}\left[-C_2 + \sqrt{\frac{2(2+n)\phi_0 - (3+n)\rho_{m_0}(t_0+C_2)^2}{(1+n)\rho_{m_0}}}\right] \tag{28}$$

Accelerated expansion of the universe leads to the observation that, $q_0 < 0$. Subjecting equation (16) to this condition, we get $(2+n)\rho_{m_0}(t_0+C_2)^2 < (2+n)\phi_0$, which is the same as equation (26).

To set the values of n, based on equation (21), we propose the following relation.

$$n = n_1 + \lambda(n_2 - n_1) \text{ with } 0 < \lambda < 1 \tag{29}$$

To set the values of C_2, from equation (27), we write,

$$C_2 = \left(\frac{\alpha\,\phi_0}{\rho_{m0}}\right)^{1/2} - t_0 \;\text{ with }\; 0 < \alpha < 1 \tag{30}$$

The parameters, λ and α, in the above relations are positive fractions. The value of C_2 depends upon α. The values of n_1 and n_2 (in eqn. 29) depend upon C_2 (as per eqn. 21A). Subtracting equation (2) from equation (1), we get,

$$\omega\frac{\dot{\phi}^2}{\phi^2} = 2\frac{\dot{a}^2}{a^2} + \frac{\dot{a}}{a}\frac{\dot{\phi}}{\phi} - \frac{\rho_m}{\phi} - 2\frac{\ddot{a}}{a} - \frac{\ddot{\phi}}{\phi} \tag{31}$$

Using equations (4), (5) and the definitions of the Hubble parameter $\left(H = \frac{\dot{a}}{a}\right)$ and the deceleration parameter $\left(q = -\frac{\ddot{a}a}{\dot{a}^2}\right)$ in equation (31), one gets,

$$\omega = \frac{2}{n^2} + \frac{2}{n} - \frac{\rho_{m_0}}{\phi_0 n^2 H^2}\left(\frac{a_0}{a}\right)^{(3+n)} - 1 + q\left(\frac{1}{n} + \frac{2}{n^2}\right) \tag{32}$$

Using $H = \frac{\dot{a}}{a}$ and equation (4), in equation (1), we get,

$$\rho_m = \phi H^2\left[3 - \frac{\omega}{2}n^2 + 3n\right] \tag{33}$$

BD parameter ω, from equation (32), has to be substituted into equation (33) to determine ρ_m.

Gravitational constant is the reciprocal of the BD scalar field (ϕ). Therefore, using equation (4) we get,

$$G = \frac{1}{\phi} = \frac{1}{\phi_0}\left(\frac{a}{a_0}\right)^{-n} \tag{34}$$

Using equation (34) one obtains,

$$\frac{\dot{G}}{G} = -\frac{\dot{\phi}}{\phi} = -nH \tag{35}$$

Time dependence of ω, ρ_m, G and $\frac{\dot{G}}{G}$ can be obtained from equations (32), (33), (34) and (35) respectively. For this purpose, one has to use the expressions for a, H and q, given by equations (11), (12) and (15) respectively.

As per equation (29), we have $n < 0$, causing an increase of G with time, as evident from equations (34). Similar results were obtained from models based on general relativity [25, 26].

Using the expressions for a, H, q (eqns. 11, 12, 15) along with equation (4), in equation (32), one obtains the following expression for ω as a function of ϕ.

$$\omega(\phi) = \frac{2}{n^2} + \frac{2}{n} - 1 - \frac{\phi_0(n+2)^2\rho_0(\phi/\phi_0)^{1+\frac{3}{n}}}{n^2\rho_0^2 K} + \frac{\rho_0 K(n+2)+(n+3)\rho_0(C_2+t_0)^2 - 2\phi_0(n+2)}{n^2\rho_0 K} + \frac{\rho_0 K(n+2)+\rho_0(n+3)(C_2+t_0)^2 - 2\phi_0(n+2)}{2n\rho_0 K}$$

where, $K = (C_2 + t_0)^2 + \frac{2\phi_0(n+2)\left(\frac{\phi}{\phi_0}\right)^{1+\frac{3}{n}} - 2\phi_0(n+2)}{\rho_0(n+3)}$

Using equation (4) in equation (3) and also using the definitions of H & q, we get the following expression for ρ_m which allows one to determine its value using any expression for the scale factor (a).

$$\rho_m = \phi_0 \left(\frac{a}{a_0}\right)^n H^2(4 - 2q + 4n + n^2 - nq) \tag{36}$$

MODEL - 2

The purpose of formulating this model is to find the time dependence of densities and density parameters regarding matter (dark + baryonic) and dark energy. For these calculations we have used the following scale factor obtained from some recent studies [23, 24].

$$a = a_0[\sinh(\gamma t)]^{\varepsilon} \tag{37}$$

where, γ, ε are positive constants and a_0 is the present value of scale factor.

The scale factor, used in Model-1, was obtained as the solution of the BD field equations, based on an assumption of the conservation of matter (dark + baryonic). For the formulation of the present model, the dark energy content of the universe has been assumed to increase with time at the cost of the matter content (dark + baryonic), as obtained from some interpretations of the astrophysical observations [21, 27, 28].

The scale factor (eqn. 37), used for this model, was shown to be a solution of a differential equation formed on the basis of an assumption that the deceleration parameter (q) is a function of time [23]. This assumption was based on a very strong evidence, obtained from recent observations, in favour of the fact that the deceleration parameter has changed its sign from positive to negative, indicating a change of phase of the phenomenon of cosmic expansion from deceleration to acceleration [1, 2-4, 7].

Expressions for the Hubble parameter (H) and the deceleration parameter (q), based on equation (37), are given by,

$$H = \gamma\varepsilon[\coth(\gamma t)] \tag{38}$$

$$q = \frac{cosech^2(\gamma t)}{\varepsilon\, coth^2(\gamma t)} - 1 \tag{39}$$

Combining equation (38) with (39) and considering their validity at all instants of time (including $t = t_0$) one obtains,

$$\gamma = \left[\frac{{H_0}^2}{\varepsilon^2}\{1 - \varepsilon(q_0 + 1)\}\right]^{\frac{1}{2}} \tag{40}$$

Equation (40) expresses the parameters γ as a function of the parameter ε.

According to some cosmological studies, the change of phase of the cosmic expansion from deceleration to acceleration took place due to the dark energy content of the universe, which has been shown to be increasing with time [21, 27, 28]. Based on this observation, we propose the following ansatz for describing the time dependence of the dark energy content of the universe.

$$D(t) = D_0\, Exp\left[\frac{\tau}{t_0} - \frac{\tau}{t}\right] \tag{41}$$

where, $D(t)$ & D_0 denote respectively the instantaneous dark energy content of the universe and its present value. τ is a constant parameter having the dimension of time. For $D(t)$ to be increasing with time we must have $\tau > 0$. Thus, for $t \to 0$ we have $D(t) \to 0$.

To set the value of τ for calculations, let us define a dimensionless constant κ such that,

$$\tau = \kappa t_0 \text{ where } \kappa > 0 \tag{42}$$

Taking ρ_T to be denoting the total energy density and ρ_m to be the matter density, we can write, $D(t) = (\rho_T - \rho_m)a^3$ and $D_0 = (\rho_{T_0} - \rho_{m_0})a_0^3$. Here, ρ_{T_0} and ρ_{m_0} are the values of ρ_T and ρ_m respectively at the present time (i.e., $t = t_0$). Substituting these expressions into equation (41) we have,

$$\rho_T a^3 - \rho_m a^3 = (\rho_{T_0} - \rho_{m_0})a_0^3 Exp\left[\frac{\tau}{t_0} - \frac{\tau}{t}\right] \tag{43}$$

Using equation (43) one gets the following expression for ρ_T.

$$\rho_T = \rho_m + (\rho_{T_0} - \rho_{m_0})\frac{a_0^3}{a^3} Exp\left[\frac{\tau}{t_0} - \frac{\tau}{t}\right] \tag{44}$$

Dark energy density can be expressed as $\rho_D = \rho_T - \rho_m$, because matter and dark energy are regarded as the main constituents of the present universe [9, 29].

The density parameters for dark energy and matter are given by,

$$\Omega_{\text{D}} = \frac{\rho_T - \rho_m}{\rho_T} = \frac{(\rho_{T_0} - \rho_{m_0})\frac{a_0^3}{a^3} Exp\left[\frac{\tau}{t_0} - \frac{\tau}{t}\right]}{\rho_m + (\rho_{T_0} - \rho_{m_0})\frac{a_0^3}{a^3} Exp\left[\frac{\tau}{t_0} - \frac{\tau}{t}\right]} \tag{45}$$

$$\Omega_m = \frac{\rho_m}{\rho_T} = \frac{\rho_m}{\rho_m + (\rho_{T_0} - \rho_{m_0})\frac{a_0^3}{a^3} Exp\left[\frac{\tau}{t_0} - \frac{\tau}{t}\right]} \tag{46}$$

In the above expressions for density parameters, ρ_m needs to be obtained from equation (36). The values of the cosmological parameters used for the present study are [9, 29],

$H_0 = 2.33 \times 10^{-18} sec^{-1}$, $t_0 = 4.415 \times 10^{17} sec$, $\phi_0 = \frac{1}{G_0} = 1.498 \times 10^{10}\ kg\ sec^2/m^3$,

$q_0 = -0.55$, $\rho_{m_0} = 2.91 \times 10^{-27}\ kg/m^3$, $\rho_{T_0} = 9.83 \times 10^{-27}\ Kg/m^3$.

3. RESULTS AND DISCUSSION

The characteristics of time variation of cosmological quantities are determined by the parameters $\alpha\ \&\ \lambda$ in Model-1 and, the parameters $\varepsilon\ \&\ \kappa$ in Model-2. For the graphical representation of theoretical findings, these parameters are so tuned that the values of H_0 and q_0 (based on these models) are close to their values obtained from astrophysical observations. Time dependence of cosmological quantities have been shown by plotting them against dimensionless cosmic time (t/t_0), where t_0 denotes the age of the universe.

Using equation (28), we obtained $\beta = 0.32$ (for $\alpha = 0.45$ and $\lambda = 0.01$), indicating the transition of the phase of cosmic expansion (from deceleration to acceleration) to have taken place in the past at around $t = 0.32t_0$ where t_0 denotes the present age of the universe. The same set of values for $\alpha\ \&\ \lambda$ has been used for the graphical depiction of the findings from Model-1. The phenomenon of signature flip of the deceleration parameter and an almost correct value of H_0 have been obtained from this model, using these values.

A large part of the calculation has been devoted to the finding of the permissible range of values of the parameter n, which play a very important role in determining the time evolution of the scalar field ($\phi \equiv 1/G$) through the ansatz of equation (4). It has been shown that, in order to have consistency with astrophysical observations, the parameter n must have a negative value, implying a decrease of ϕ (and a consequent rise in G) with time.

In Figure 1, the scale factor (a) has been plotted as a function of time, based on Model-1. It increases with time, as expected for an expanding universe [11].

Figure 2 shows the time variation of deceleration parameter (q), based on Model-1. The curve shows a signature flip indicating a transition from a phase of deceleration to acceleration, without considering any contribution from anything like dark energy. At the present time (i.e., at $t = t_0$), the value of the deceleration parameter obtained from this plot is -0.15 which (although indicating acceleration) is different from the value of -0.55 obtained from recent observations [9]. The assumption of the power-law function of ϕ (eqn. 4), though successful in many studies, may be one of the reasons for this discrepancy. The nature of time evolution of q is consistent with the findings of other recent studies [9, 25, 26].

Figure 3 shows the time variation of Hubble parameter (in sec^{-1}), based on Model-1. Hubble parameter is found to be decreasing with time. We have used log_{10} scale along the vertical axis for clarity. At the present time (i.e., at $t = t_0$) the value of the Hubble parameter, obtained from this plot, is around $2.32 \times 10^{-18} sec^{-1}$ which is close to its value obtained from astrophysical observations. Its value and the nature of time variation are consistent with a recent study on BD theory by G. K. Goswami, based on a model involving the cosmological constant (Λ) [9].

Figure 4 shows the time variation of Gravitational constant (G), based on Model-1. It is found to increase with time. The observation, that G increases with time, is consistent with the findings of other recent studies based on different models [17, 21, 25, 26].

Figure 5 shows the variation of the BD parameter (ω) as a function of time, based on Model-1. Here ω has a negative value and it becomes more negative with time, at a gradually decreasing rate. This behaviour is consistent with an observation found in a recent study (based on a different space-time) on the time variation of the BD parameter [20]. The value of ω_0 is close to -1.52.

Figure 6 shows the time variation of $\frac{\dot{G}}{G}$, based on Model-1. We have used log_{10} scale along the vertical axis for clarity. According to this plot, $\frac{\dot{G}}{G}$ is positive and it decreases with time. Here, $\left(\frac{\dot{G}}{G}\right)_{t=t_0} = 1.38 \times 10^{-10} yr^{-1}$ which is consistent with the findings obtained from various sources and reported by recent studies [30].

Figure 7 shows the variation of Brans-Dicke parameter (ω) as a function of the scalar field parameter (ϕ), based on Model-1. We have used log_{10} scale along the horizontal axis for clarity. According to the generalized version of Brans-Dicke theory, ω is no longer regarded as a constant. It depends upon the scalar field which changes with time [17]. According to our plot, ω becomes less negative as ϕ increases. Its values are close to -1.5, which is consistent with other studies [12, 21, 22]. Since n is negative, ϕ decreases with time, as per equation (4). This figure shows that, near the present time (i.e., at $t = t_0$ $when$ $\phi = \phi_0$) the variation of ω as a function of ϕ is almost linear.

Figure 8 shows the variation of the deceleration parameter (q) as a function of the scalar field parameter (ϕ), based on Model-1. Deceleration parameter decreases, from positive to negative values, as the scalar field decreases. Here we have used log_{10} scale along the horizontal axis for clarity. This plot is not based upon any expression of q written in terms of ϕ. It has been obtained from the datasets of q and ϕ generated separately from their functional dependence upon time.

Dependence of deceleration parameter (q) upon the BD parameter (ω) can be found from the Figures 7 and 8. As ω decreases, the value of q decreases, which is also evident from their time variation plots (Figures 2 & 5). According to equation (32), ω has a linear dependence upon q. The signature flip of q takes place when $\omega = -1.502$ and $= 1.18 \times 10^{11}\ kg\ sec^2/m^3$.

Figure 9 shows the variation of matter density (ρ_m) with time, based on Model-1. Here we have used log_{10} scale along the vertical axis for clarity. Matter density decreases with time. It is consistent with the finding of another model formulated by us [21].

Figure 10 shows the time variation of deceleration parameter (q), based on Model-2. Signature flip of q, from positive to negative, is clearly visible here, confirming the validity of the expression for scale factor (eqn. 37) used for all calculations of Model-2. The transition from decelerated expansion to accelerated expansion takes place at around $t = 0.34t_0$. The value of q obtained at $t = t_0$ (i.e., around -0.46) and its nature of time variation are consistent with findings from recent studies based on different cosmological models [9, 26].

Figure 11 shows the time variations of the density of matter (ρ_m) and the density of dark energy (ρ_D), based on Model-2. Here we have used log_{10} scale along the vertical axis for clarity. The density of matter (ρ_m) decreases with time. The dark energy density (ρ_D) rises to a peak value at around $t = 0.8t_0$ and keeps decreasing thereafter. Similar behaviours were obtained by us from a different cosmological model [21].

Figure 12 shows the time variations of the density parameters, for dark energy and matter (dark matter + baryonic matter), based on Model-2. The parameters, Ω_m and Ω_D, are respectively found to be decreasing and increasing with time. Their values at the present time (i.e., at $t = t_0$) and their natures of variations with time are consistent with those obtained from astrophysical observations and recent theoretical models [9, 24, 29]. According to this plot, the proportions of matter and dark energy was equal in the universe at around $t = 0.65t_0$. According to equations (41) and (42), the parameter κ determines how fast $D(t)$ changes with time. We have chosen $\kappa = 3$ for our calculations because it causes $\Omega_m = \Omega_d = 0.5$ when the redshift (*z*) is close to 0.5, which corresponds to a period of time around which the transition of phase from deceleration to acceleration took place, as per recent astrophysical observations [31, 32]. It is consistent with the fact that dark energy is responsible for this change of phase.

The nature of time variations of the deceleration parameter (q) and the Hubble parameter (*H*), as determined in the present article on the basis of Model-1 (shown in Figures 2 & 3 respectively, where both are found to be decreasing with time and the deceleration parameter shows a signature flip), are in qualitative agreement with a study by G. K. Goswami, in the BD framework incorporating the cosmological constant (Λ), where the author has shown the corresponding variations with respect to the redshift parameter (z) [9]. The redshift parameter ($z = \frac{a_0}{a} - 1$) decreases as the scale factor (*a*) increases with time. All these findings of Model-1 have come from a formulation where dark energy has not been taken into account.

4. CONCLUDING REMARKS

The present study is based on a spatially flat, homogeneous and isotropic space-time, in the framework of Brans-Dicke theory of gravity. Model-1 shows that, without considering the contribution of any parameter representing dark energy, a smooth transition from the decelerated phase to the accelerated phase of expansion of the universe can be obtained from the theory, assuming the universe to be filled with pressure-less matter. Although the nature of time evolution of different cosmological quantities, obtained from our study, are consistent with the findings of studies carried out with many other models, this formulation has generated a value of the deceleration parameter (at $t = t_0$) which is not sufficiently close to the range of values obtained from astrophysical observations [9]. One of the reasons for this result may be the ansatz of equation (4) which shows a power-law relation between the scalar field (ϕ) and the scale factor (*a*). This ansatz governs the time dependence of the scalar field (ϕ) which plays a very significant role in controlling the accelerated expansion of the universe. For a more accurate result one may use a new empirical relation between these parameters, which might make the calculations a little more difficult than the present one. The natures of variation of ω and q, as functions of ϕ, are found to be quite similar, indicating a probable role of the BD coupling parameter in causing the transition from deceleration to acceleration and, this role can be further explored in future, with greater accuracy, by choosing empirical expressions (different from eqn. 4) serving as relations between the scalar field and the scale factor. In Model-2, we have used a different scale factor which has been chosen in a way such that the deceleration parameter, obtained from it, changes its sign from positive to negative as time progresses, to have consistency with astrophysical observations. As a future extension of this work, one may think of employing more such scale factors to determine the time variations of cosmological parameters, such as the density parameters and the densities of matter and dark energy, and find the average nature of their time dependence from these results. Based on the observation that the dark energy content of the universe increases with time, we proposed an ansatz (represented by eqn. 41) for describing the time dependence of the dark energy content of the universe. To keep it physically viable, its functional form has been so chosen that $D(t)$ does not have a negative value and it would be increasing with time at a gradually slower rate, only to be consistent with the fact that a decreasing proportion of matter will have slower interactions with dark energy. Combining this ansatz with the expression for ρ_m (eqn. 36), obtained from BD field equations, we have determined the time evolution of dark energy density and the corresponding density parameter. More such theoretical investigations in this regard will be undertaken in future to connect the parameters ε and κ to an interaction between matter and dark energy which is capable of causing the changes of the proportions of these constituents of the universe with time.

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FIGURES

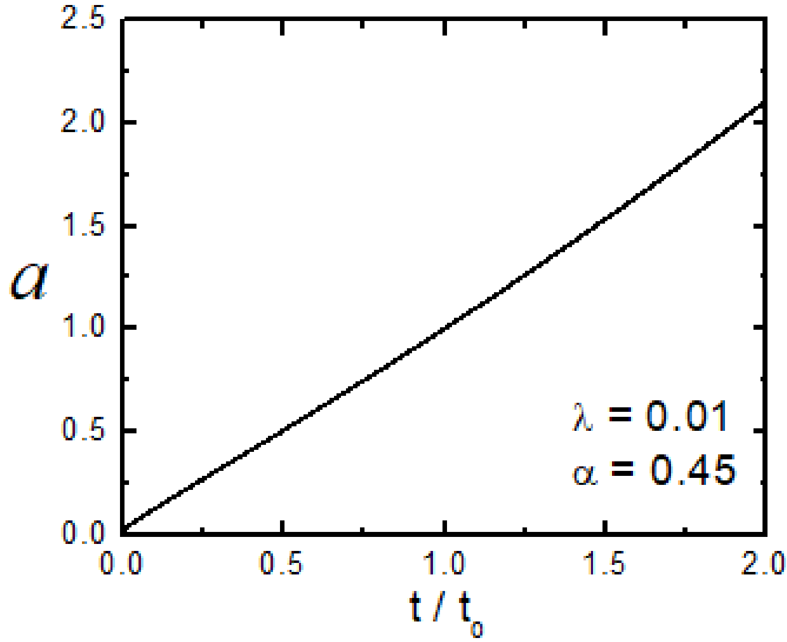

Figure 1. Plot of scale factor (a) versus time, based on Model-1.

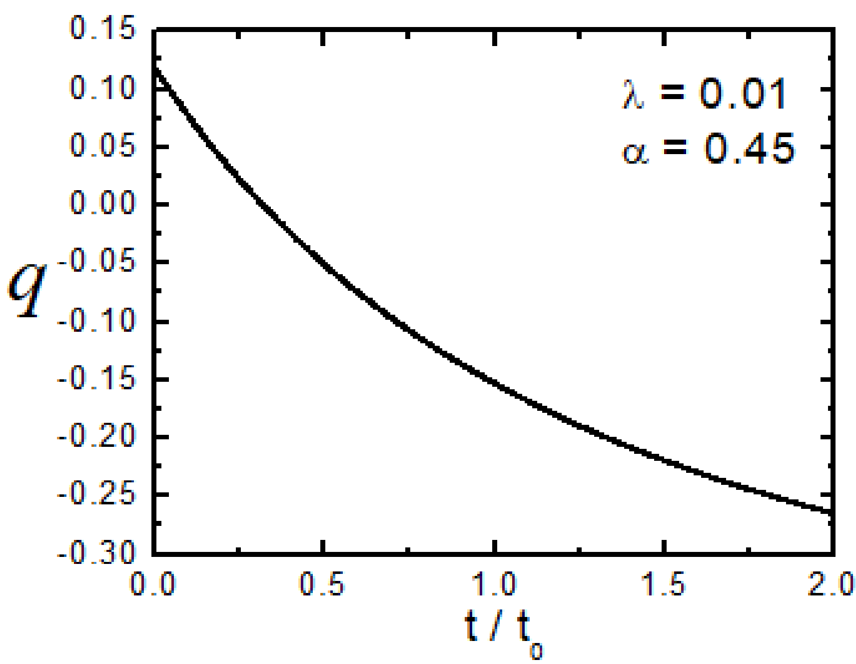

Figure 2. Plot of deceleration parameter (q) versus time, based on Model-1.

FIGURES

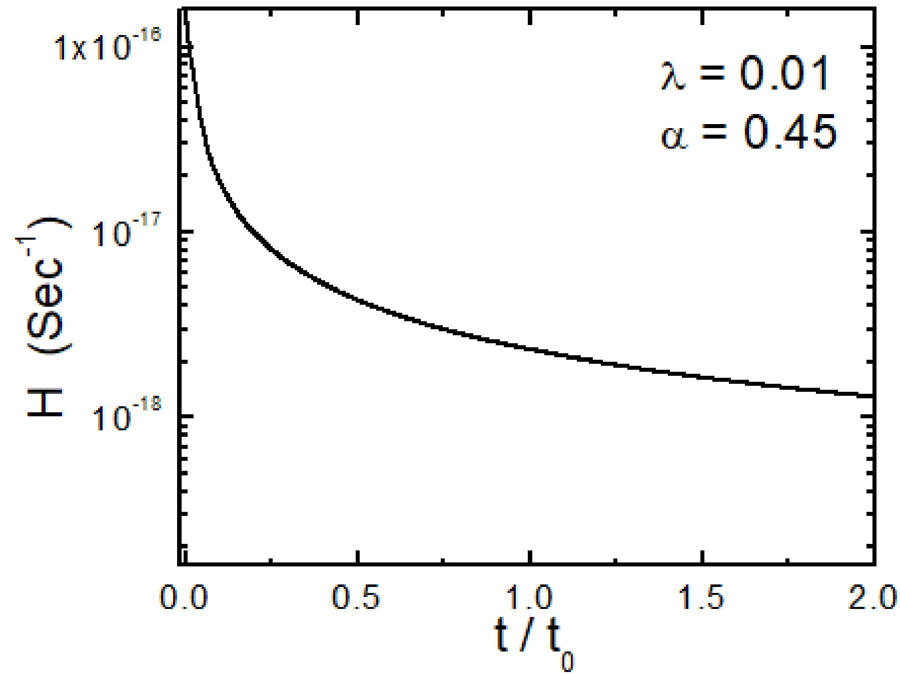

Figure 3. Plot of Hubble parameter (*H*) versus time, based on Model-1.

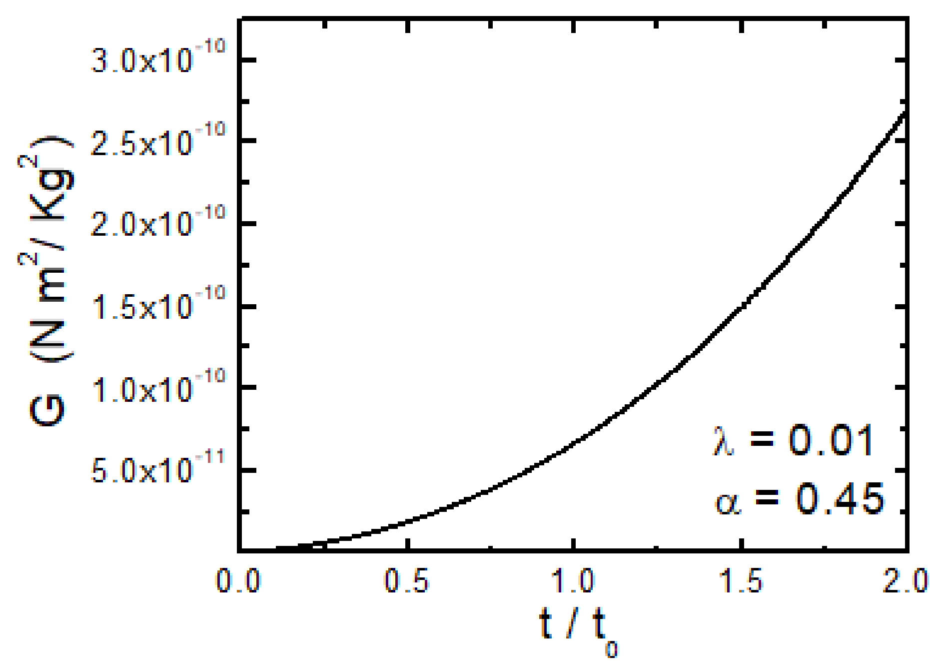

Figure 4. Plot of gravitational constant (*G*) versus time, based on Model-1.

FIGURES

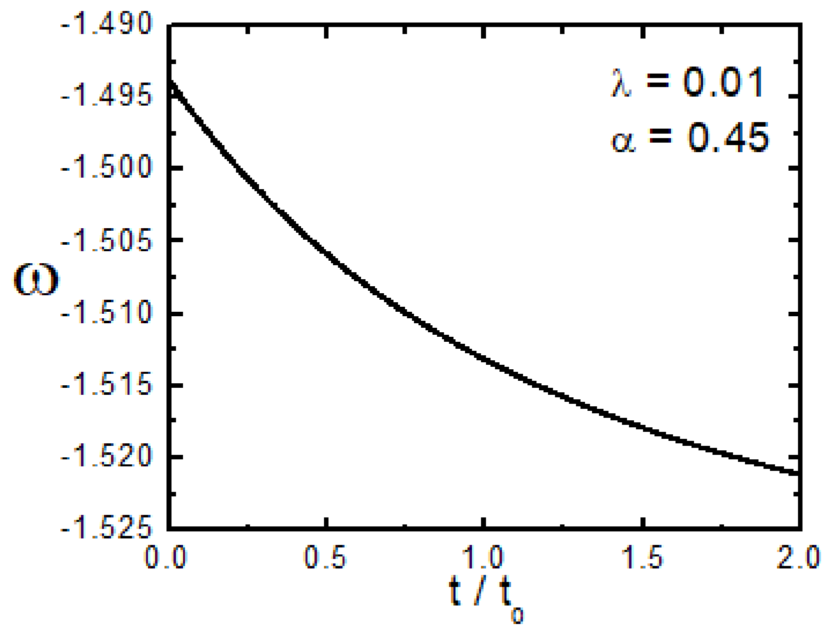

Figure 5. Plot of Brans-Dicke parameter (ω) versus time, based on Model-1.

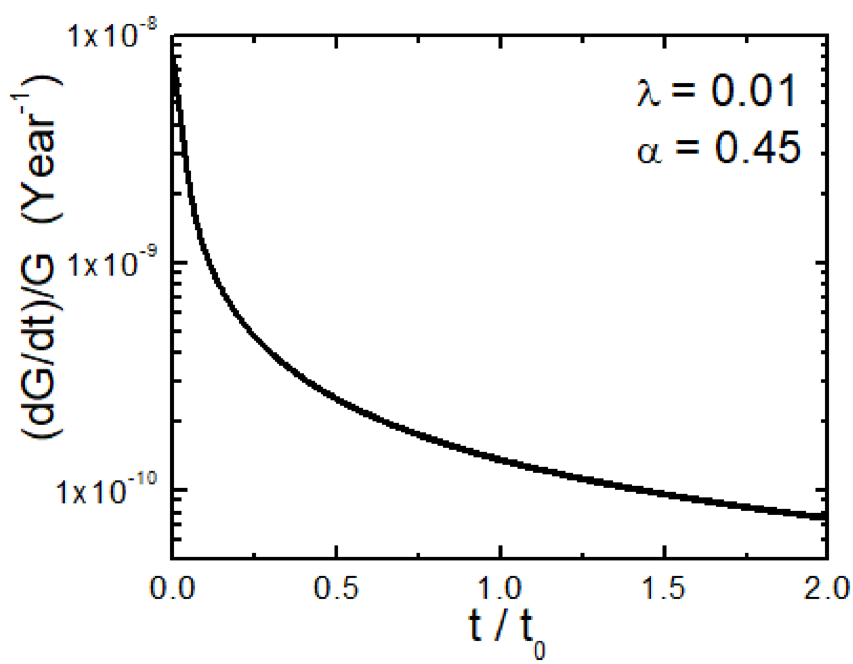

Figure 6. Plot of $\dot{G}/G$ versus time, based on Model-1.

FIGURES

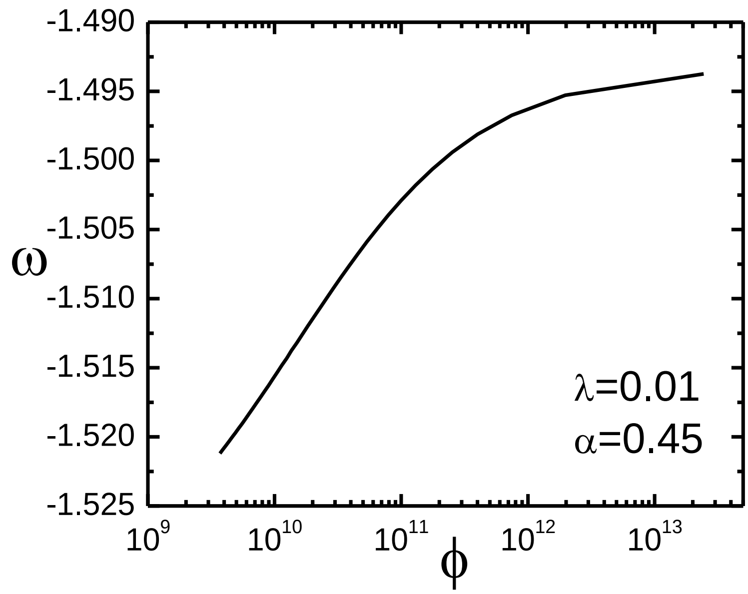

Figure 7. Plot of Brans-Dicke parameter (ω) versus scalar field (ϕ), based on Model-1.

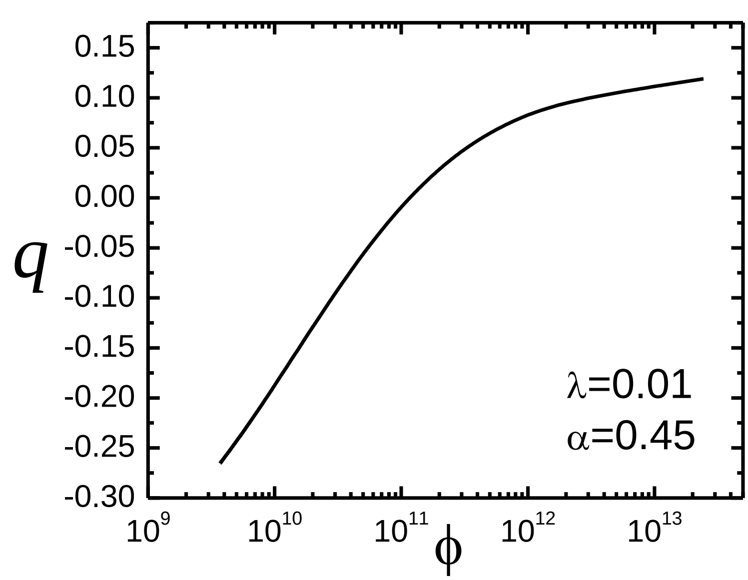

Figure 8. Plot of deceleration parameter (q) versus scalar field (ϕ), based on Model-1.

FIGURES

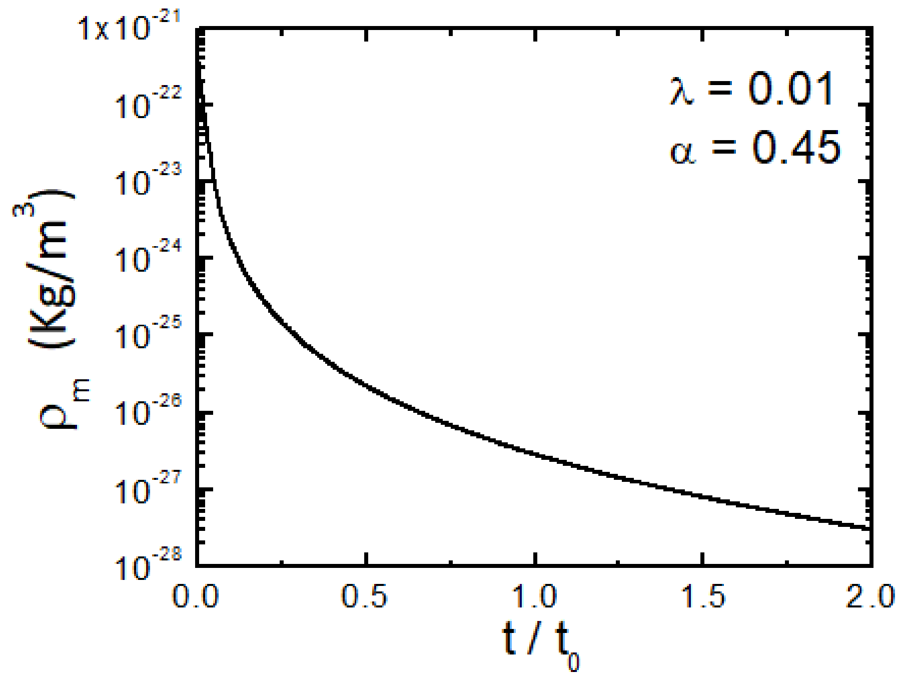

Figure 9. Plot of the density of matter (ρ_m) versus time, based on Model-1.

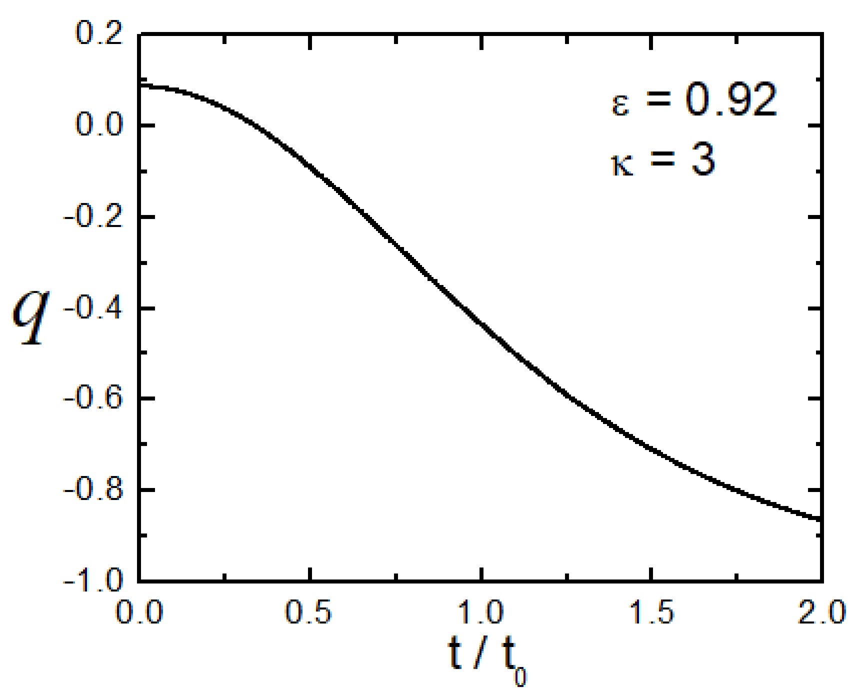

Figure 10. Plot of deceleration parameter (q) versus time, based on Model-2.

FIGURES

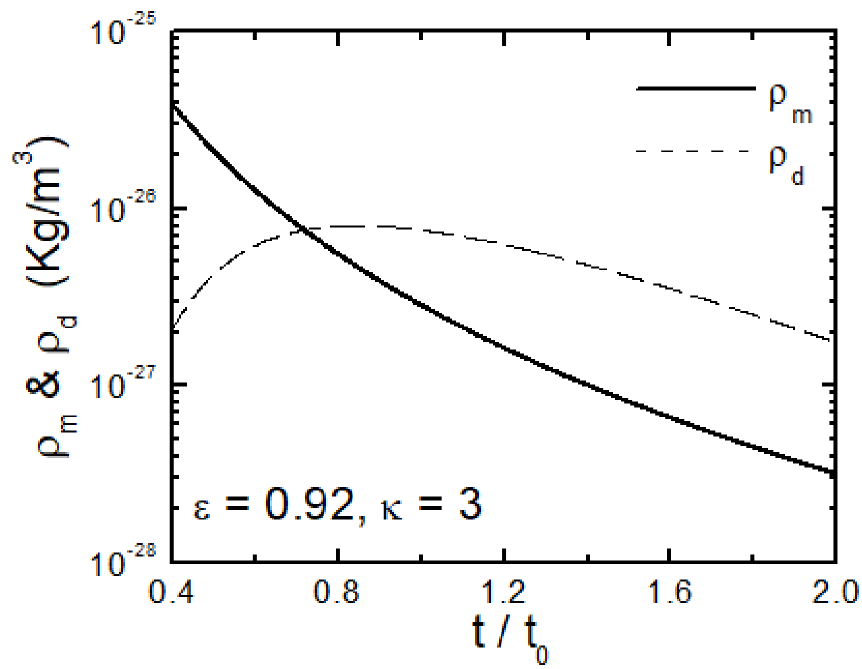

Figure 11. Plots of densities of matter (ρ_m) and dark energy (ρ_D) versus time, based on Model-2.

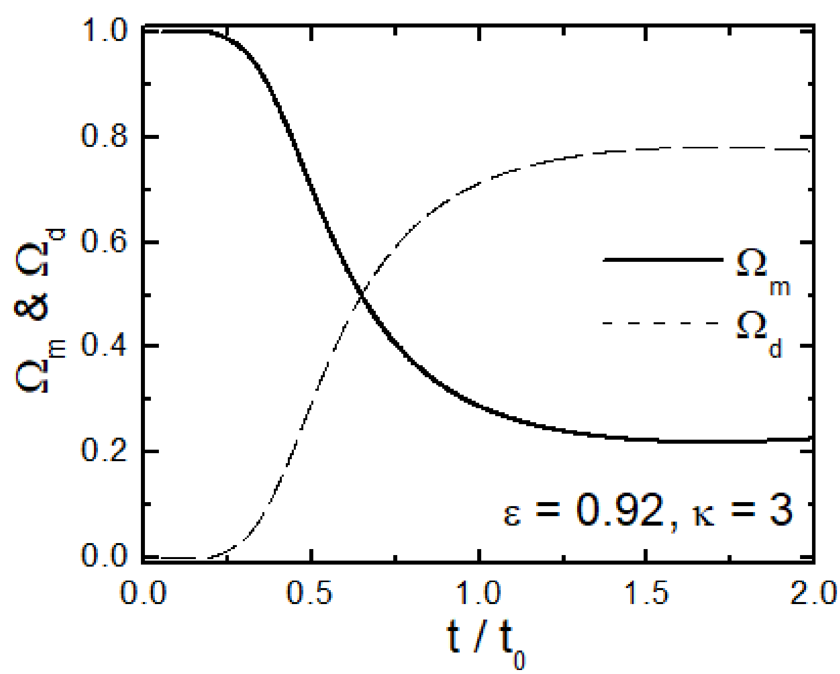

Figure 12. Plots of density parameters versus time, for matter (Ω_m) and dark energy (Ω_D), based on Model-2.