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Rumination on Trigonometric Topological Spaces

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ABSTRACT

The intent of this paper is to deliberate the sine and cosine topologies and some set theory relations. Further we analyse the interior and closure operaters of Sine and Cosine topologies with illustrative examples and discuss about some results on Sine-interior, Sine-Closure, Cos-interior and Cos-closure.

***Keywords*:** Sine-interior, Sine-Closure, Cos-interior, Cos-closure

INTRODUCTION

Trigonometric functions are one of the important group of the elementary functions. All the six trigonometric functions can defined through the sine and cosine functions. Topological ideas are present in almost all areas of todays mathematics. In 1736 the first work which deserves to be considered as the begins of topology is due to Euler. In 1914, Felix Hausdorff coined the term topological space.

Modern topology depends on the ideas of set theory, developed by Georg Cantor in the later part of the 19th century. Quit Recently, the concept of trigonometric topological spaces was coined by S.Malathi and R.Usha Parameswari based on Sine and Cosine topologies. In a bitopological space, we have considered two different topologies but in a trigonometric topological space the two topologies are derived from one topology.

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1. SINE TOPOLOGICAL SPACE

Let S be any non-empty set. The elements of S are taken from $[0, \frac{\pi}{2}]$. Let SinS be the set of all sine values of the corresponding elements of S. Define a function h_{sin} : S → SinS by h_{sin}(s) = Sin(s). Then the function h_{sin} is a bijective function. $h_{sin}(\emptyset) = \emptyset$ and h_{sin}(S) = SinS.

Definition 1.1.

Let (S,τ) be a topological space. τ_{sin} is a topology whose elements are sine values of the corresponding elements of the topology τ. Then τ_{sin} form a toplogy on h_{sin}(S). This topology is known as sine topology of S. The space (h_{sin}(S), τ_{sin}) is called a Sine topological space corresponding to S.

The elements of τ_{sin} are Sin-open sets and that of τ_{sin}^{c} are Sin-closed sets.

Example 1.2.

Let S = $\{0,\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}\}$ with topology τ = $\{\emptyset,\{\frac{\pi}{3}\},\{\frac{\pi}{2}\},\{\frac{\pi}{3},\frac{\pi}{2}\},\{0,\frac{\pi}{2}\},\{0, \frac{\pi}{3}, \frac{\pi}{2}\},S\}$

Then SinS = $\{0, \frac{1}{2},\frac{\sqrt{3}}{2},1\}$

$\tau_{sin} = \{\emptyset,\{\frac{\sqrt{3}}{2}\},\{1\},\{\frac{\sqrt{3}}{2},1\},\{0,1\},\{0,\frac{\sqrt{3}}{2},1\},\text{Sin S}\}$

$\tau_{sin}^{c} = \{\emptyset,\{\frac{1}{2}\},\{\frac{1}{2}, \frac{\sqrt{3}}{2}\},\{0,\frac{1}{2}\},\{0,\frac{1}{2},\frac{\sqrt{3}}{2}\},\{0,\frac{1}{2},1\},\text{Sin S}\}$

Interior and Closure of Sine Topological Space

Definition 1.3.

If (S,τ) is a topological space and J ⊆ SinS. The Sine-interior is defined by the union of all Sin-open sets contained in J. The Sine-interior of J is denoted by int_sJ.

$\text{int}_s J = \cup \{K \subseteq \text{SinS} : K \subseteq J \text{ and K is Sin-open}\}$

Definition 1.4.

If (S,τ) is a topological space and J ⊆ SinS. The Sine-closure is defined by the intersection of all Sin-closed sets containing J. The Sine-closure of J is denoted by cl_sJ.

$\text{cl}_s J = \cap \{K \subseteq \text{SinS} : J \subseteq K \text{ and K is Sin-closed}\}$

Example 1.5.

Let S = $\{\frac{\pi}{6}, \frac{\pi}{3},\frac{\pi}{2}\}$ with topology τ = $\{\emptyset,\{\frac{\pi}{6}\},\{\frac{\pi}{3}\},\{\frac{\pi}{6},\frac{\pi}{3}\},\{\frac{\pi}{3},\frac{\pi}{2}\},S\}$

Then SinS = $\{\frac{1}{2},\frac{\sqrt{3}}{2},1\}$

$\tau_{sin} = \{\emptyset,\{\frac{1}{2}\},\{\frac{\sqrt{3}}{2}\},\{\frac{1}{2},\frac{\sqrt{3}}{2}\},\{\frac{\sqrt{3}}{2},1\},\text{SinS}\}$

$\tau_{sin}^{c} = \{\emptyset,\{\frac{1}{2}\},\{1\},\{\frac{1}{2},1\},\{\frac{\sqrt{3}}{2},1\},\text{SinS}\}$

Let M = $\{\frac{\sqrt{3}}{2}\}$

Then int_sM =$\{\frac{\sqrt{3}}{2}\}$

cl_sM = $\{\frac{\sqrt{3}}{2},1\}$

Theorem 1.6. If (S,τ) is a topological space and J ⊆ SinS. Then int_s(J) is a Sin-open set.

Proof: If (S,τ) is a topological space and J ⊆ SinS.

Lemma: The union of any collection of Sin-open sets is Sin-open.

Proof of lemma:

If $\{V_i\}$ is the family of Sin-open sets.

If W = $\cup V_i$ and x ∈ W Then x ∈ V_i for some i.

Since V_i is sin-open, then there exists a sin-open set V_j of x such that x ∈ $V_j \subset V_i \subseteq \cup V_i$

⇒ x ∈ $\cup V_i$, x is an sin-interior of $\cup V_i$

∴ $\cup V_i$ is sin-open.

∴ W is sin-open.

Now by the definition of int_sJ, and above lemma, int_sJ is a sin-open set.

Theorem 1.7. If (S,τ) is a topological space and J ⊆ SinS. Then int_s(J) ⊆ J, int_s(J) is the largest sin-open set contained in J, J is sin-open if and only if J = int_s(J).

Proof: If (S,τ) is a topological space and J ⊆ SinS.

i) If x ∈ int_s(J)

Then x ∈ K for some sin-open set K ⊆ J

∴ x ∈ J

∴ int_s(J) ⊆ J

ii) If K is any sin-open subset of and let x ∈ K.

By definition, K is an sin-open set containing x and K ⊆ J, is also a sin-open set containing x.

Hence x is an sin-interior of J.

∴ x ∈ int_sJ Since x ∈ K, x ∈ int_sJ

∴ K ⊂ int_sJ

Thus int_sJ contains every sin-open subset K of and hence int_sJ is the largest sin-open set contained in J.

iii) If we assume that J = int_sJ

By theorem 1.6, int_s(J) is an sin-open set.

Since int_s = J, it follows that is also sin-open.

Conversly assume that, J is sin-open.

Since is sin-open, it is identical with largest sin-open subset J.

But int_s(J) is the largest sin-open subset of J.

∴ J = int_s(J)

Theorem 1.8. If (S,τ) is a topological space and J,K are subsets of SinS. Then

i) J ⊆ K ⇒ int_s(J) ⊆ int_s(K)

ii) int_s(J ∩K) = int_s(J)∩int_s(K) and

iii) int_s(J)∪int_s(K)⊆int_s(J ∪K)

Proof: If (S,τ) is a topological space and J,K ⊆ SinS.

i) If J ⊆ K and x ∈ int_s(J)

Then by definition of int_s(J) there exist an sin-open set U such that x ∈ U ⊆ J

⇒ x ∈ U ⊆ K (∵ J ⊆ K)

Then by definition, we have x in int_s(K)

∴ int_s(J) ⊆ int_s(K)

ii) First let us show that int_s(J ∩K) ⊆ int_s(J)∩int_s(K) J ∩K ⊆ J

⇒ int_s(J∩K) ⊆ int_s(J) (by using (i))

J ∩K ⊆ K ⇒ int_s(J ∩K) ⊆ int_s(K) (by using (i))

Thus, int_s(J ∩K) ⊆ int_s(J)∩ int_s(K)

Now, let as show that int_s(J)∩int_s(K) ⊆ int_s(J ∩K)

int_s(J)∩int_s(K)⊂int_s(J) ⊂ J

Also, int_s(J)∩int_s(K) ⊂ int_s(K) ⊂ K But int_s(J)∩int_s(K) is an intersection of two sin-open sets, hence is sin-open.

So by using this result, int_sJ is the largest sin-open subset of J, in the following sense if U ⊂ J and U is sin-open, then U ⊂int_sJ, we get int_s(J)∩ int_s(K) ⊆ int_s(J ∩K)

Thus int_s(J ∩K) = int_s(J)∩int_s(K)

iii) Now, int_s(J) ⊆ J ∪K Also, int_s(K) ⊆ K ⊆ J ∪K

So int_s(J)∪int_s(K) ⊆ J ∪K

Now, int_s(J)∪ int_s(K) is a union of sin-open sets, hence is sin-open.

So by using this result, int_sJ is the largest sin-open subset of J, in the following sense if U ⊂ J and U is sin-open, then U ⊂ int_sJ, we get int_s(J)∪ int_s(K) ⊆ int_s(J ∪K)

Thus int_s(J)∪ int_s(K) ⊆ int_s(J ∪K)

Remark 1.9. $\mathrm{int}_s(J \cup K)$ need not be equal to $\mathrm{int}_s(J) \cup \mathrm{int}_s(K)$

For example, Let S = $\{0,\frac{\pi}{6},\frac{\pi}{4},\frac{\pi}{2}\}$ with topology τ = $\{\emptyset,\{0\},\{0,\frac{\pi}{4}\},\{\frac{\pi}{6},\frac{\pi}{2}\},\{0,\frac{\pi}{6},\frac{\pi}{2}\},S\}$

Then SinS = $\{0,\frac{1}{2},\frac{1}{\sqrt{2}},1\}$

τ_{sin} = $\{\emptyset,\{0\},\{0,\frac{1}{\sqrt{2}}\},\{\frac{1}{2},1\},\{0,\frac{1}{2},1\},\mathrm{SinS}\}$

Let J = $\{\frac{1}{2}\}$ and K = $\{\frac{1}{\sqrt{2}},1\}$

$\mathrm{int}_s J = \emptyset$ and $\mathrm{int}_s K = \emptyset$

Now, $\mathrm{int}_s J \cup \mathrm{int}_s K = \emptyset$

Now, $J \cup K = \{\frac{1}{2},\frac{1}{\sqrt{2}},1\}$, $int_s(J \cup K) = \{\frac{1}{2},1\}$

Here $\mathrm{int}_s(J) \cup \mathrm{int}_s(K) = \emptyset$ But $\mathrm{int}_s(J \cup K) = \{\frac{1}{2},1\}$

∴ $\mathrm{int}_s(J \cup K) \neq \mathrm{int}_s(J) \cup \mathrm{int}_s(K)$

Theorem 1.10. If (S,τ) is a topological space and J ⊆ SinS. Then $\mathrm{cl}_s(J)$ is a Sin-closed set.

Proof: If (S,τ) is a topological space and J ⊆ SinS.

Lemma: intersection of any arbitrary collection of Sin-closed sets is Sin-closed.

Proof of lemma: If $\{C_i\}$ is the arbitrary collection of sin-closed sets.

If $W = \cap C_i$

To show that W is sin-closed. i.e) to show that W^c is sin-open.

Since, Arbitrary union of sin-open sets is sin-open.

∴ $W^c = \cup C_i$ is sin-open.

∴ $W = \cap C_i$ is sin-closed.

Now by definition of $\mathrm{cl}_s(J)$, and above lemma, $\mathrm{cl}_s(J)$ is a sin-closed set.

Theorem 1.11. If (S,τ) is a topological space and J ⊆ SinS. Then $J \subseteq \mathrm{cl}_s(J)$ and $\mathrm{cl}_s(J)$ is the smallest sin-closed set containing J, J is sin-closed if and only if $J = \mathrm{cl}_s J$.

Proof: If (S,τ) is a topological space and J ⊆ SinS.

i) clearly, $J \subseteq \mathrm{cl}_s(J) = \mathrm{cl}_s J$.

ii) If $\{C_i\}$ is the collection of all sin-closed subsets of S containing the set J.

So, that we have $\mathrm{cl}_s J = \cap\{C_i\}$. Since the intersection of an arbitrary collection of sin-closed sets is sin-closed set.

∴ $\mathrm{cl}_s J$ is a sin-closed set.

Also, since $J \subseteq C_i$ for each i, we have $J \subseteq \cap C_i = \mathrm{cl}_s J$

Thus $cl_s J$ is a sin-closed set containing the set J.

Also since $cl_s J = \cap\{C_i\}$, we have $cl_s J \subseteq \{C_i\}$ for each i. Consequently, $cl_s J$ is the smallest sin-closed set containing J.

iii) If is a sin-closed subset of a topological space S, then itself will be the smallest sin-closed superset of J.

So $J = cl_s(J)$

Conversly, let $J = cl_s(J)$

Then J is sin-closed because $cl_s J$ is sin-closed.

Theorem 1.12. If (S,τ) is a topological space and J,K are subsets of Sin S. Then

i) $J \subseteq K \Rightarrow cl_s(J) \subseteq cl_s(K)$

ii) $cl_s(J \cup K) = cl_s(J) \cup cl_s(K)$

iii) $cl_s(J \cap K) \subseteq cl_s(J) \cap cl_s(K)$

Proof: If (S,τ) is a topological space and $J,K \subseteq SinS$.

i) Suppose $J \subseteq K$

Then $J \subseteq K \subseteq cl_s(K)$, $cl_s(K)$ is a sin-closed set contains J.

But $cl_s(J)$ is the smallest sin-closed set that contains J.

$\therefore$ $cl_s(J) \subseteq cl_s(K)$

ii) Always, $J \subseteq J \cup K$ and $K \subseteq J \cup K$

If $J \subseteq K$ then $cl_s(J) \subseteq cl_s(K)$

By using above we get, $cl_s(J) \subseteq cl_s(J \cup K)$ and $cl_s(K) \subseteq cl_s(J \cup K)$

$\therefore$ $cl_s(J) \cup cl_s(K) \subseteq cl_s(J \cup K)$

and, $J \subseteq cl_s(J) \subseteq cl_s(J \cup K)$ and $K \subseteq cl_s(K) \subseteq cl_s(J \cup K)$

So $J \cup K \subseteq cl_s(J) \cup cl_s(K)$

But $cl_s(J) \cup cl_s(K)$ is a union of two sin-closed sets, hence is sin-closed.

Now by using this result, If $cl_s(J)$ is the smallest sin-closed subset of S containing in the following sense : If $J \subset F \subset S$ and F is sin-closed, then $cl_s(J) \subset F$.

We have $cl_s(J \cup K) \subseteq cl_s(J) \cup cl_s(K)$

$\therefore$ $cl_s(J \cup K) = cl_s(J) \cup cl_s(K)$

iii) WKT, $J \cap K \subseteq cl_s(J)$ and $J \cap K \subseteq K \subseteq cl_s(K)$

So, $J \cap K \subset cl_s(J) \cap cl_s(K)$.

$cl_s(J) \cap cl_s(K)$ is an intersection of sin-closed sets, hence is sin-closed.

Now by using this result, If $\mathrm{cl_s}(J)$ is the smallest sin-closed subset of S containing in the following sense: If $J \subset F \subset S$ and F is sin-closed, then $\mathrm{cl_s}(J) \subset F$.

We have $\mathrm{cl_s}(J \cap K) \subseteq \mathrm{cl_s}(J) \cap \mathrm{cl_s}(K)$

Remark 1.13. $\mathrm{cl_s}(J \cap K)$ need not be equal to $\mathrm{cl_s}(J) \cap \mathrm{cl_s}(K)$

For example, Let $S = \{\frac{\pi}{6},\frac{\pi}{4},\frac{\pi}{3},\frac{\pi}{2}\}$ with topology $\tau = \{\emptyset,\{\frac{\pi}{6}\},\{\frac{\pi}{6},\frac{\pi}{4}\},\{\frac{\pi}{6},\frac{\pi}{3}\},\{\frac{\pi}{6},\frac{\pi}{4},\frac{\pi}{3}\},S\}$

Then $\mathrm{SinS} = \{\frac{1}{2},\frac{1}{\sqrt{2}},\frac{\sqrt{3}}{2},1\}$

$\tau_{sin} = \{\emptyset,\{\frac{1}{2}\},\{\frac{1}{2},\frac{1}{\sqrt{2}}\},\{\frac{1}{2},\frac{\sqrt{3}}{2}\},\{\frac{1}{2},\frac{1}{\sqrt{2}},\frac{\sqrt{3}}{2}\},\mathrm{SinS}\}$

$\tau_{sin}^{c} = \{\emptyset,\{1\},\{\frac{1}{\sqrt{2}},1\},\{\frac{\sqrt{3}}{2},1\},\{\frac{1}{\sqrt{2}},\frac{\sqrt{3}}{2},1\},\mathrm{SinS}\}$

Let $= \{\frac{1}{2},1\}$ and $K = \{\frac{1}{\sqrt{2}},1\}$

$J \cap K = \{1\}$ and $\mathrm{cl_s}(J \cap K) = \{1\}$

Now, $\mathrm{cl_s}(J) = \mathrm{SinS}$ and $\mathrm{cl_s}(K) = \{\frac{1}{\sqrt{2}},1\}$

$\mathrm{cl_s}(J) \cap \mathrm{cl_s}(K) = \{\frac{1}{\sqrt{2}},1\}$

$\therefore \mathrm{cl_s}(J \cap K) \neq \mathrm{cl_s}(J) \cap \mathrm{cl_s}(K)$

2. COSINE TOPOLOGICAL SPACE

Let S be any non-empty set. The elements of S are taken from $[0,\frac{\pi}{2}]$. Let CosS be the set of all cosine values of the corresponding elements of S. Define a function h_{cos}: S → CosS by $h_{cos}(s) = \mathrm{Cos}(s)$. Then the function h_{cos} is a bijective function. $h_{cos}(\emptyset) = \emptyset$ and $h_{cos}(S) = \mathrm{CosS}$.

Definition 2.1.

Let (S,τ) be a topological space. τ_{cos} is a topology whose elements are cosine values of the corresponding elements of the topology τ. Then τ_{cos} form a toplogy on $h_{cos}(S)$. This topology is known as cosine topology of S. The space $(h_{cos}(S),\tau_{cos})$ is called a Cosine topological space corresponding to S. The elements of τ_{cos} are Cos-open sets and that of τ_{cos}^{c} are Cos-closed sets.

Example 2.2.

Let $S = \{0,\frac{\pi}{3},\frac{\pi}{4},\frac{\pi}{2}\}$with topology $\tau = \{\emptyset,\{\frac{\pi}{3}\},\{\frac{\pi}{4}\},\{\frac{\pi}{3},\frac{\pi}{4}\},\{0,\frac{\pi}{4}\},\{0,\frac{\pi}{3},\frac{\pi}{4}\},S\}$

Then $\mathrm{CosS} = \{1,\frac{1}{2},\frac{1}{\sqrt{2}},0\}$

$\tau_{cos} = \{\emptyset,\{\frac{1}{2}\},\{\frac{1}{\sqrt{2}}\},\{\frac{1}{2},\frac{1}{\sqrt{2}}\},\{1,\frac{1}{\sqrt{2}}\},\{1,\frac{1}{2},\frac{1}{\sqrt{2}}\},\mathrm{CosS}\}$

$\tau_{cos}^{c} = \{\emptyset,\{0\},\{0,\frac{1}{2}\},\{1,0\},\{1,\frac{1}{2},0\},\{1,\frac{1}{\sqrt{2}},0\},\mathrm{CosS}\}$

Interior and Closure of Cosine Topological Space

Definition 2.3.

If (S,τ) is a topological space and J ⊆ CosS. The union of all Cos-open sets contained in is called a Cos-interior of and is denoted by int_cJ.

int_c= ∪ {K ⊆ Cos S : K ⊆ and K is Cos-open}

Definition 2.4.

If (S,τ) is a topological space and J ⊆ CosS. Then the intersection of all Cos-closed sets containing is called a Cos-closure of and is denoted by cl_c(J).

cl_c(J) = ∩ {K ⊆ Cos S : ⊆ K and K is Cos-closed}

Example 2.5.

Let S = $\{\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\}$ with topology τ = $\{\emptyset,\{\frac{\pi}{3}\},\{\frac{\pi}{4}\},\{\frac{\pi}{4}, \frac{\pi}{3}\},\{\frac{\pi}{4},\frac{\pi}{6}\},\{\frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{6}\},S\}$

Then CosS = $\{\frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}},\frac{1}{2},0\}$

τcos = $\{\emptyset,\{\frac{1}{\sqrt{2}}\},\{\frac{1}{2},\},\{\frac{1}{\sqrt{2}}, \frac{1}{2},\},\{\frac{1}{\sqrt{2}},\frac{\sqrt{3}}{2}\},\{\frac{1}{\sqrt{2}},\frac{\sqrt{3}}{2}, \frac{1}{2}, \},CosS\}$

τccos = $\{\emptyset,\{0\},\{\frac{1}{2},0\},\{\frac{\sqrt{3}}{2},0\},\{\frac{\sqrt{3}}{2},\frac{1}{\sqrt{2}} ,0\},\{\frac{\sqrt{3}}{2}, \frac{1}{2}, ,0\},CosS\}$

Let M = $\{\frac{1}{2},\frac{\sqrt{3}}{2}\}$

Then int_cM = $\{\frac{1}{2},\}$, cl_cM = $\{\frac{\sqrt{3}}{2}, \frac{1}{2},0\}$

Theorem 2.6. If (S,τ) is a topological space and J,K are subsets of CosS. Then

i) int_c(J) is a Cos-open set.

ii) int_c(J) ⊆ J

iii) int_c(J) is the largest cos-open set contained in J.

iv) J is cos-open if and only if = int_c(J)

Proof: The proof of this theorem is similar to the proof of the theorems 1.6 and 1.7

Theorem 2.7. If (S,τ) is a topological space and J,K are subsets of CosS. Then

i) J ⊆ K ⇒ int_c(J) ⊆ int_c(K)

ii) int_c(J ∩K) = int_c(J)∩ int_c(K) and

iii) int_c(J)∪int_c(K) ⊆ int_c(J ∪K)

Proof: The proof of this theorem is similar to theorem 1.8

Remark 2.8. Let (S,τ) is a topological space and J,K ⊆ S. Then int_c(J∪K) need not be equal to int_c(J)∪int_c(K)

For example, Let S = $\{0,\frac{\pi}{6},\frac{\pi}{4},\frac{\pi}{3},\frac{\pi}{2}\}$with topology

$\tau = \{\emptyset,\{0\},\{\frac{\pi}{6},\frac{\pi}{4}\},\{0,\frac{\pi}{3}\},\{\frac{\pi}{3},\frac{\pi}{4},0\},\{0,\frac{\pi}{6},\frac{\pi}{4},\frac{\pi}{3}\},S\}$

Then CosS = $\{1,\frac{\sqrt{3}}{2},\frac{1}{\sqrt{2}},\frac{1}{2},0\}$

$\tau_{\cos} = \{\emptyset,\{1\},\{\frac{\sqrt{3}}{2},\frac{1}{\sqrt{2}}\},\{1,\frac{1}{2}\},\{\frac{\sqrt{3}}{2},\frac{1}{\sqrt{2}},1\},\{1,\frac{\sqrt{3}}{2},\frac{1}{\sqrt{2}},\frac{1}{2}\},CosS\}$

Let J = $\{\frac{1}{\sqrt{2}},\frac{1}{2}\}$ and K = {1}; int_cJ = ∅ and int_cK = {1}

Now, int_cJ∪ int_cK = {1}

Now, J∪K = $\{1,\frac{1}{2},\frac{1}{\sqrt{2}}\}$ int_c(J ∪K) = $\{1,\frac{1}{2}\}$

Here int_c(J)∪int_c(K) = {1} But int_c(J ∪K) = $\{1,\frac{1}{2}\}$

∴ int_c(J ∪K) ≠ int_c(J)∪ int_c(K)

Theorem 2.9. If (S,τ) is a topological space and J,K are subsets of CosS. Then

i) cl_c(J) is a Cos-closed set.

ii) J ⊆ cl_c(J)

iii) cl_c(J) is the smallest cos-closed set containing J.

iv) J is cos-closed if and only if = cl_c(J)

Proof: The proof of this theorem is similar to the proof of the theorems 1.10 and 1.11

Theorem 2.10. If (S,τ) is a topological space and J,K are subsets of CosS. Then

i) J ⊆ K ⇒ cl_c(J) ⊆ cl_c(K)

ii) cl_c(J ∪K) = cl_c(J)∪ cl_c(K) and

iii) cl_c(J ∩K) ⊆ cl_c(J)∩cl_c(K)

Proof: The proof of this theorem is similar to the theorem 1.13

Remark 2.11. Let (S,τ) be a topological space, J,K ⊆ CosS. Then cl_c(J∩K) need not be equal to cl_c(J)∩cl_c(K)

For example, Let S = $\{\frac{\pi}{6},\frac{\pi}{4},\frac{\pi}{3},\frac{\pi}{2}\}$ with topology $\tau = \{\emptyset,\{\frac{\pi}{4}\},\{\frac{\pi}{3}\},\{\frac{\pi}{6},\frac{\pi}{4},\frac{\pi}{3}\},\{\frac{\pi}{4},\frac{\pi}{3},\frac{\pi}{2}\},S\}$

Then CosS = $\{\frac{1}{2},\frac{1}{\sqrt{2}},\frac{\sqrt{3}}{2},1\}$

$\tau_{cos} = \{\emptyset,\{\frac{1}{\sqrt{2}},\frac{\sqrt{3}}{2}\},\{\frac{1}{2},\frac{1}{\sqrt{2}},\frac{\sqrt{3}}{2}\},\{\frac{1}{\sqrt{2}},\frac{\sqrt{3}}{2},1\},CosS\}$

τ^{c}_{cos} = {∅,{$\frac{1}{2}$},{1},{$\frac{1}{2}$,1},CosS}

Let = {$\frac{1}{2}$,1} and K = {$\frac{\sqrt{3}}{2}$,1}

J ∩K = {1} and cl_c(J ∩K) = {1}

Now, cl_c(J) = {$\frac{1}{2}$,1} and cl_c(K) = CosS

cl_c(J)∩cl_c(K) = {$\frac{1}{2}$,1}

∴ cl_c(J ∩K) ≠ cl_c(J)∩cl_c(K)

3. TRIGONOMETRIC TOPOLOGICAL SPACE

Definition 3.1.

Let S be a non-empty set. The elements are S are taken from [0,$\frac{\pi}{2}$]. Define T∪(S) by

$T_\cup$(S) = SinS∪CosS.

Theorem 3.2. If S is a set and J is a subset of S. Then

i) $T_\cup$(S)\(SinJ ∪CosJ) = ($T_\cup$(S)\SinJ)∩($T_\cup$(S)\CosJ)

ii) $T_\cup$(S)\(SinJ ∩CosJ) = ($T_\cup$(S)\SinJ)∪($T_\cup$(S)\CosJ)

Proof:

i) Let p ∈ $T_\cup$(S)\(SinJ ∪CosJ)

⇔ p ∈ $T_\cup$(S) and p∉(SinJ∪CosJ)

⇔ p ∈ $T_\cup$(S) and {p ∉ SinJ and p ∉ CosJ}

⇔ {p ∈ $T_\cup$(S) and p ∉ SinJ} and {p ∈ $T_\cup$(S) and ∉ CosJ}

⇔ {p ∈ $T_\cup$(S)\SinJ} and {p ∈ $T_\cup$(S)\CosJ}

⇔ p ∈ ($T_\cup$(S)\SinJ) ∩ ($T_\cup$(S)\CosJ)

∴ $T_\cup$(S)\(SinJ ∪CosJ) = ($T_\cup$(S)\SinJ)∩($T_\cup$(S)\CosJ)

ii) Let p ∈ $T_\cup$(S)\(SinJ ∩CosJ)

⇔ p ∈ $T_\cup$(S) and p ∉ (SinJ ∩CosJ)

⇔ p ∈ $T_\cup$(S) and {p ∉ SinJ or p ∉ CosJ}

⇔ {p ∈ $T_\cup$(S) and p ∉ SinJ} or {p ∈ $T_\cup$(S) and p ∉ CosJ}

⇔ {p ∈ $T_\cup$(S)\SinJ} or {p ∈ $T_\cup$(S)\CosJ}

⇔ p ∈ ($T_\cup$(S)\SinJ) ∪ ($T_\cup$(S)\CosJ)

∴ $T_\cup$(S)\(SinJ ∩CosJ) = ($T_\cup$(S)\SinJ)∪($T_\cup$(S)\CosJ)

Theorem 3.3. If S is set and J is a subset of S. Then

i) $T_{\cup}(S)\backslash(SinJ \cap CosJ) = (SinS\backslash SinJ) \cup (CosS\backslash CosJ)$

ii) $T_{\cup}(S)\backslash(SinJ \cup CosJ) \subseteq (SinS\backslash SinJ) \cup (CosS\backslash CosJ)$

Proof:

i) Let $p \in T_{\cup}(S)\backslash(SinJ \cap CosJ)$

$\Leftrightarrow p \in (SinS \cup CosS)\backslash(SinJ \cap CosJ)$

$\Leftrightarrow p \in (SinS \cup CosS)$ and $p \notin (SinJ \cap CosJ)$

$\Leftrightarrow p \in (SinS$ or $CosS)$ and $(p \notin SinJ$ or $p \notin CosJ)$

$\Leftrightarrow (p \in SinS$ and $p \notin SinJ)$ or $(p \in CosS$ and $p \notin CosJ)$

$\Leftrightarrow p \in (SinS\backslash SinJ)$ or $p \in (CosS\backslash CosJ)$

$\Leftrightarrow p \in (SinS\backslash SinJ) \cup (CosS\backslash CosJ)$

$\therefore T_{\cup}(S)\backslash(SinJ \cap CosJ) = (SinS\backslash SinJ) \cup (CosS\backslash CosJ)$

ii) Let $p \in T_{\cup}(S)\backslash(SinJ \cup CosJ)$

$\Rightarrow p \in (SinS \cup CosS)\backslash(SinJ \cup CosJ)$

$\Rightarrow p \in (SinS \cup CosS)$ and $p \notin (SinJ \cup CosJ)$

$\Rightarrow p \in (SinS$ or $CosS)$ and $p \notin (SinJ$ and $CosJ)$

$\Rightarrow (p \in SinS$ and $p \notin SinJ)$ or $(p \in CosS$ and $p \notin CosJ)$

$\Rightarrow p \in (SinS\backslash SinJ)$ or $p \in (CosS\backslash CosJ)$

$\Rightarrow p \in (SinS\backslash SinJ) \cap (CosS\backslash CosJ)$

$\therefore T_{\cup}(S)\backslash(SinJ \cup CosJ) \subseteq (SinS\backslash SinJ) \cup (CosS\backslash CosJ)$

Remark 3.4. In the above theorem, the reverse inclusion of (ii) need not be true.

For example, Let $S = \{\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}\}$

Then $SinS = \{\frac{1}{2}, \frac{\sqrt{3}}{2}, 1\}$

$CosS = \{\frac{\sqrt{3}}{2}, \frac{1}{2}, 0\}$

$T_{\cup}(S) = \{0, \frac{1}{2}, \frac{\sqrt{3}}{2}, 1\}$

Now, let $J = \{\frac{\pi}{3}\}$

Then $SinJ = \{\frac{\sqrt{3}}{2}\}$, $CosJ = \{\frac{1}{2}\}$; $SinJ \cup CosJ = \{\frac{\sqrt{3}}{2}, \frac{1}{2}\}$

$SinS\backslash SinJ = \{\frac{1}{2}, \frac{\sqrt{3}}{2}, 1\}\backslash\{\frac{\sqrt{3}}{2}\} = \{\frac{1}{2}, 1\}$

CosS\CosJ = $\{\frac{\sqrt{3}}{2}, \frac{1}{2}, 0\} \setminus \{\frac{1}{2}\} = \{\frac{\sqrt{3}}{2}, 0\}$

Now $T_{\cup}(S) \setminus (SinJ \cup CosJ) = \{0, \frac{1}{2}, \frac{\sqrt{3}}{2}, 1\} \setminus \{\frac{\sqrt{3}}{2}, \frac{1}{2}\} = \{0,1\}$

Now $(SinS \setminus SinJ) \cup (CosS \setminus CosJ) = \{\frac{1}{2}, 1\} \cup \{\frac{\sqrt{3}}{2}, 0\} = \{0, \frac{1}{2}, \frac{\sqrt{3}}{2}, 1\}$

$\therefore (SinS \setminus SinJ) \cup (CosS \setminus CosJ) \nsubseteq T_{\cup}(S) \setminus (SinJ \cup CosJ)$

$\therefore T_{\cup}(S) \setminus (SinJ \cup CosJ) \neq (SinS \setminus SinJ) \cup (CosS \setminus CosJ)$

Result 3.5. If S is a set and J is a subset of S. Then

i) $Sin(S \setminus J) \subseteq T_{\cup}(x) \setminus SinJ$

ii) $Cos(S \setminus J) \subseteq T_{\cup}(x) \setminus CosJ$

Proof:

i) Let $p \in Sin(S \setminus J)$

$\Rightarrow p \in (SinS \setminus SinJ)$

$\Rightarrow p \in SinS$ and $p \notin SinJ$

$\Rightarrow p \in (SinS \cup CosS)$ and $p \notin SinJ$

$\Rightarrow p \in T_{\cup}(S)$ and $p \notin SinJ$

$\Rightarrow p \in T_{\cup}(x) \setminus SinJ$

$\therefore Sin(S \setminus J) \subseteq T_{\cup}(x) \setminus SinJ$

ii) Let $p \in Cos(S \setminus J)$

$\Rightarrow p \in (CosS \setminus CosJ)$

$\Rightarrow p \in CosS$ and $p \notin CosJ$

$\Rightarrow p \in (SinS \cup CosS)$ and $p \notin CosJ$

$\Rightarrow p \in T_{\cup}(S)$ and $p \notin CosJ$

$\Rightarrow p \in T_{\cup}(x) \setminus CosJ$

$\therefore Cos(S \setminus J) \subseteq T_{\cup}(x) \setminus CosJ$

Remark 3.6. If S is a set and J is a subset of S. Then

i) $(T_{\cup}(x) \setminus SinJ)$ need not be equal to $(SinS \setminus SinJ)$

ii) $(T_{\cup}(x) \setminus CosJ)$ need not be equal to $(CosS \setminus CosJ)$

i) For example, Let S = $\{\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{2}\}$

Then SinS = $\{\frac{1}{2}, \frac{1}{\sqrt{2}}, 1\}$

CosS = $\{\frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}, 0\}$

$T_{\cup}(S) = \{0, \frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}, 1\}$

Now, let $J = \{\frac{\pi}{6}, \frac{\pi}{4}\}$

Then $SinJ = \{\frac{1}{2}, \frac{1}{\sqrt{2}}\}$

Now $T_{\cup}(x)\backslash SinJ = \{0, \frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}, 1\}\backslash\{\frac{1}{2}, \frac{1}{\sqrt{2}}\} = \{0, \frac{\sqrt{3}}{2}, 1\}$

Now $SinS\backslash SinJ = \{\frac{1}{2}, \frac{1}{\sqrt{2}}, 1\}\backslash\{\frac{1}{2}, \frac{1}{\sqrt{2}}\} = \{1\}$

$\therefore (T_{\cup}(x)\backslash SinJ) \neq (SinS\backslash SinJ)$

ii) For example, Let $S = \{0, \frac{\pi}{6}, \frac{\pi}{4}\}$

Then $SinS = \{0, \frac{1}{2}, \frac{1}{\sqrt{2}}\}$

$CosS = \{1, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}\}$

$T_{\cup}(S) = \{0, \frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}, 1\}$

Now, let $J = \{0, \frac{\pi}{4}\}$

Then $CosJ = \{1, \frac{1}{\sqrt{2}}\}$

Now $T_{\cup}(x)\backslash CosJ = \{0, \frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}, 1\}\backslash\{1, \frac{1}{\sqrt{2}}\} = \{0, \frac{1}{2}, \frac{\sqrt{3}}{2}\}$

Now $CosS\backslash CosJ = \{1, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}\}\backslash\{1, \frac{1}{\sqrt{2}}\} = \{\frac{\sqrt{3}}{2}\}$

$\therefore (T_{\cup}(x)\backslash CosJ) \neq (CosS\backslash CosJ)$

Theorem 3.7. If S is a set and J is a subset of S. Then

i) $T_{\cup}(S)\backslash SinJ = (SinS\backslash SinJ)\cup(CosS\backslash SinJ)$

ii) $T_{\cup}(S)\backslash CosJ = (SinS\backslash CosJ)\cup(CosS\backslash CosJ)$

Proof:

i) Let $p \in T_{\cup}(S)\backslash SinJ$

$\Leftrightarrow p \in (SinS\cup CosS)\backslash SinJ$

$\Leftrightarrow p \in (SinS \cup CosS)$ and $p \notin SinJ$

$\Leftrightarrow (p \in SinS$ or $p \in CosS)$ and $p \notin SinJ$

$\Leftrightarrow (p \in SinS$ and $p \notin SinJ)$ or $(p \in CosS$ and $p \notin SinJ)$

$\Leftrightarrow (p \in SinS\backslash SinJ)$ or $(p \in CosS\backslash SinJ)$

$\Leftrightarrow p \in (SinS\backslash SinJ)\cup(CosS\backslash SinJ)$

$T_{\cup}(S)\backslash SinJ = (SinS\backslash SinJ)\cup(CosS\backslash SinJ)$

ii) Let $p \in T_{\cup}(S)\backslash CosJ$

$\Leftrightarrow p \in (SinS \cup CosS)\backslash CosJ$

$\Leftrightarrow p \in (SinS \cup CosS)$ and $p \notin CosJ$

$\Leftrightarrow (p \in SinS$ or $p \in CosS)$ and $p \notin CosJ$

$\Leftrightarrow (p \in SinS$ and $p \notin CosJ)$ or $(p \in CosS$ and $p \notin CosJ)$

$\Leftrightarrow (p \in SinS\backslash CosJ)$ or $(p \in CosS\backslash CosJ)$

$\Leftrightarrow p \in (SinS\backslash CosJ)\cup(CosS\backslash CosJ)$

$\therefore T_{\cup}(S)\backslash CosJ = (SinS\backslash CosJ)\cup(CosS\backslash CosJ)$

Definition 3.8.

Let S be a set. The elements of S are taken from $[0,\frac{\pi}{2}]$. Define $T_{\cap}(S)$ by $T_{\cap}(S) = SinS \cap CosS$.

Theorem 3.9. If S is a set and J is a subset of S. Then

i) $T_{\cap}(S)\backslash(SinJ \cup CosJ) = (T_{\cap}(S)\backslash SinJ)\cap(T_{\cap}(S)\backslash CosJ)$

ii) $T_{\cap}(S)\backslash(SinJ\cap CosJ) = (T_{\cap}(S)\backslash SinJ)\cup(T_{\cap}(S)\backslash CosJ)$

Proof: The proof of the theorem is similar to the theorem 3.2

Theorem 3.10. If S is a set and J is a subset of S. Then

i) $T_{\cap}(S)\backslash(SinJ \cup CosJ) = (SinS\backslash SinJ)\cap(CosS\backslash CosJ)$

ii) $T_{\cap}(S)\backslash(SinJ \cap CosJ) \subseteq (SinS\backslash SinJ)\cup(CosS\backslash CosJ)$

Proof:

i) Let $p \in T_{\cap}(S)\backslash(SinJ \cup CosJ)$

$\Leftrightarrow p \in (SinS \cap CosS)\backslash(SinJ \cup CosJ)$

$\Leftrightarrow p \in (SinS \cap CosS)$ and $p \notin (SinJ \cup CosJ)$

$\Leftrightarrow p \in (SinS$ and $CosS)$ and $(p \notin SinJ$ and $p \notin CosJ)$

$\Leftrightarrow (p \in SinS$ and $p \notin SinJ)$ and $(p \in CosS$ and $p \notin CosJ)$

$\Leftrightarrow p \in (SinS\backslash SinJ)$ and $p \in (CosS\backslash CosJ)$

$\Leftrightarrow p \in (SinS\backslash SinJ)\cap(CosS\backslash CosJ)$

$\therefore T_{\cap}(S)\backslash(SinJ \cup CosJ) = (SinS\backslash SinJ)\cap(CosS\backslash CosJ)$

ii) Let $p \in T_{\cap}(S)\backslash(SinJ \cap CosJ)$

$\Rightarrow p \in (SinS \cap CosS)\backslash(SinJ \cap CosJ)$

$\Rightarrow p \in (SinS \cap CosS)$ and $p \notin (SinJ \cap CosJ)$

$\Rightarrow p \in (SinS$ and $CosS)$ and $p \notin (SinJ$ or $CosJ)$

$\Rightarrow$ (p $\in$ SinS and p $\notin$ SinJ) or (p $\in$ CosS and p $\notin$ CosJ)

$\Rightarrow$ p $\in$ (SinS\SinJ) or p $\in$ (CosS\CosJ

$\Rightarrow$ p $\in$ (SinS\SinJ) $\cup$ (CosS\CosJ)

$\therefore$ $T_\cap$(S)\(SinJ $\cap$CosJ) $\subseteq$ (SinS\SinJ)$\cup$(CosS\CosJ)

Remark 3.11. In the above theorem, the reverse inclusion of (ii) need not be true.

For example, Let S = $\{\frac{\pi}{6},\frac{\pi}{3},\frac{\pi}{4},\frac{\pi}{2}\}$

Then SinS = $\{\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{\sqrt{2}},1\}$

CosS = $\{\frac{\sqrt{3}}{2},\frac{1}{2},\frac{1}{\sqrt{2}},0\}$

$T_\cap(S) = \{\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{\sqrt{2}}\}$

Now, let J = $\{\frac{\pi}{6},\frac{\pi}{4}\}$

Then SinJ = $\{\frac{1}{2}\}$ CosJ = $\{\frac{\sqrt{3}}{2},\frac{1}{\sqrt{2}}\}$

SinJ$\cap$CosJ = $\{\frac{1}{\sqrt{2}}\}$

SinS\SinJ = $\{\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{\sqrt{2}},1\}\backslash\{\frac{1}{2},\frac{1}{\sqrt{2}}\} = \{\frac{\sqrt{3}}{2},1\}$

CosS\CosJ = $\{\frac{\sqrt{3}}{2},\frac{1}{2},\frac{1}{\sqrt{2}},0\}\backslash\{\frac{\sqrt{3}}{2},\frac{1}{\sqrt{2}}\} = \{\frac{1}{2},0\}$

Now $T_\cap$(S)\(SinJ $\cap$CosJ) = $\{\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{\sqrt{2}}\}\backslash\{\frac{1}{\sqrt{2}}\} = \{\frac{1}{2},\frac{\sqrt{3}}{2}\}$

Now (SinS\SinJ)$\cup$(CosS\CosJ) = $\{\frac{\sqrt{3}}{2},1\}\cup\{\frac{1}{2},0\} = \{0,\frac{\sqrt{3}}{2},\frac{1}{2},1\}$

$\therefore$ (SinS\SinJ)$\cup$(CosS\CosJ) $\nsubseteq$ $T_\cap$(S)\(SinJ $\cap$CosJ)

$\therefore$ $T_\cap$(S)\(SinJ $\cap$CosJ) $\neq$ (SinS\SinJ)$\cup$(CosS\CosJ)

Result 3.12. If S is a set and J is a subset of S. Then

i) $T_\cap$(x)\SinJ $\subseteq$ Sin(S\J)

ii) $T_\cap$(x)\CosJ $\subseteq$ Cos(S\J)

Proof:

i) Let p $\in$ $T_\cap$(x)\SinJ

$\Rightarrow$ p $\in$ (SinS $\cap$CosS)\SinJ

$\Rightarrow$ p $\in$ (SinS $\cap$CosS) and p $\notin$ SinJ

$\Rightarrow$ p $\in$ (SinS and CosS) and p $\notin$ SinJ

$\Rightarrow$ (p $\in$ SinS and p $\notin$ SinJ) and (p $\in$ CosS and p $\notin$ SinJ)

$\Rightarrow$ p $\in$ (SinS\SinJ) and p $\in$ (CosS\SinJ)

$\therefore$ p $\in$ (SinS\SinJ)

$\therefore$ $T_\cap$(x)\SinJ $\subseteq$ Sin(S\J)

ii) Let p $\in$ $T_\cap$(x)\CosJ

$\Rightarrow$ p $\in$ (SinS $\cap$CosS)\CosJ

$\Rightarrow$ p $\in$ (SinS $\cap$CosS) and p $\notin$ CosJ

$\Rightarrow$ p $\in$ (SinS and CosS) and p $\notin$ CosJ

$\Rightarrow$ (p $\in$ SinS and p $\notin$ CosJ) and (p $\in$ CosS and p $\notin$ CosJ)

$\Rightarrow$ p $\in$ (SinS\CosJ) and p $\in$ (CosS\CosJ)

$\therefore$ p $\in$ (CosS\CosJ)

$\therefore$ $T_\cap$(x)\CosJ $\subseteq$ Cos(S\J)

Remark 3.13. If S is a set and J is a subset of S. Then

i) ($T_\cap$(x)\SinJ) need not be equal to (SinS\SinJ)

ii) ($T_\cap$(x)\CosJ) need not be equal to (CosS\CosJ)

i) For example, Let S = $\{0, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{2}\}$

Then SinS = $\{0, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}, 1\}$ CosS = $\{1, \frac{1}{2}, \frac{1}{\sqrt{2}}, 0\}$

$T_\cap(S) = \{0, \frac{1}{\sqrt{2}}, 1\}$

Now, let J = $\{\frac{\pi}{4}\}$

Then SinJ = $\{\frac{1}{\sqrt{2}}\}$

Now $T_\cap$(x)\SinJ = $\{0, \frac{1}{\sqrt{2}}, 1\} \setminus \{\frac{1}{\sqrt{2}}\} = \{0,1\}$

Now SinS\SinJ = $\{0, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}, 1\} \setminus \{\frac{1}{\sqrt{2}}\} = \{0, \frac{\sqrt{3}}{2}, 1\}$

$\therefore$ ($T_\cap$(x)\SinJ) $\neq$ (SinS\SinJ)

ii) For example, Let S = $\{0, \frac{\pi}{6}, \frac{\pi}{3}\}$

Then SinS = $\{0, \frac{1}{2}, \frac{\sqrt{3}}{2}\}$ CosS = $\{1, \frac{\sqrt{3}}{2}, \frac{1}{2}\}$

$T_\cap(S) = \{\frac{\sqrt{3}}{2}, \frac{1}{2}\}$

Now, let J = $\{\frac{\pi}{6}\}$

Then CosJ = $\{\frac{\sqrt{3}}{2}\}$

Now $T_\cap(x)\backslash CosJ = \{\frac{\sqrt{3}}{2}, \frac{1}{2}\}\backslash\{\frac{\sqrt{3}}{2}\} = \{\frac{1}{2}\}$

Now $CosS\backslash CosJ = \{1, \frac{\sqrt{3}}{2}, \frac{1}{2}\}\backslash\{\frac{\sqrt{3}}{2}\} = \{1, \frac{1}{2}\}$

$\therefore (T_\cap(x)\backslash CosJ) \neq (CosS\backslash CosJ)$

Theorem 3.14. If S is a set and J is a subset of S. Then

i) $T_\cap(S)\backslash SinJ = (SinS\backslash SinJ)\cap(CosS\backslash SinJ)$

ii) $T_\cap(S)\backslash CosJ = (SinS\backslash CosJ)\cap(CosS\backslash CosJ)$

Proof: The proof of the theorem is similar to the theorem 3.7

Definition 3.15.

Let S be a set. The elements of S are taken from $[0,\frac{\pi}{2}]$ and τ be a topology on S. We define a set $\mathcal{T} = \{\emptyset, J \cup N \cup T_\cap(S) : J \in \tau_{sin} \text{ and } N \in \tau_{cos}\}$. Then $\mathcal{T}$ form a topology on $T_\cup(S)$.

This topology is known as trigonometric topology on $T_\cup(S)$. The space $(T_\cup(S), \mathcal{T})$ is said to be a trigonometric topological space.

The elements of $\mathcal{T}$ are trigonometric open sets and that of $\mathcal{T}^c$ are trigonometric closed sets.

Example 3.16.

Let $S = \{0, \frac{\pi}{4}, \frac{\pi}{3}\}$ with topology $\tau = \{\emptyset, \{\frac{\pi}{3}\}, \{0, \frac{\pi}{4}\}, S\}$

Then SinS = {0, 1 √2, √3 2 } CosS = {1, 1 √2, 1 2}

$T_\cup(x)$ = {0, 1 √2, √3 2 , 1 2,1} $T_\cap(x)$ = { 1 √2}

τ_{sin} = {∅,{ √3 2 },{0, 1 √2},SinS};

τ_{cos} = {∅,{1 2},{1, 1 √2},CosS}

$\mathcal{T}$ = {∅,$T_\cap(x)$,{1 2, 1 √2},{1, 1 √2},CosS,{ √3 2 , 1 √2},{ √3 2 , 1 2, 1 √2},{√3 2 ,1, 1 √2}, {√3 2 ,1, 1 √2, 1 2},{0, 1 √2},{0, 1 2, 1 √2},{0,1, 1 √2},{0, 1 √2,1, 1 2},SinS,{0, 1 √2,√3 2 , 1 2},{0, 1 √2,√3 2 ,1},$T_\cup(S)$}

CONCLUSION

In this project, we discuss the concepts of Sine and Cosine topological spaces and their basic properties. Also we talked out the properties of interior and closure operators in Sine and Cosine topologies as mentioned in [2] with illustrative counter examples.

The main goal of our future work is to analyze the role of Trigonometric topological spaces in various field of Mathematics.

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