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## Some Special Graceful Labeling Results of Pentagonal Pyramidal Graceful Graphs

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### ABSTRACT

Numbers of the form  $\frac{n^2(n+1)}{2}$  for all  $n \geq 1$  are called pentagonal pyramidal numbers. Let  $G$  be a graph with  $p$  vertices and  $q$  edges. Let  $\Psi : V(G) \rightarrow \{0, 1, 2, \dots, M_r\}$  where  $M_r$  is the  $r^{th}$  pentagonal pyramidal number be an injective function. Define the function  $\Psi^* : E(G) \rightarrow \{1, 6, 18, \dots, M_r\}$  such that  $\Psi^*(uv) = |\Psi(u) - \Psi(v)|$  for all edges  $uv \in E(G)$ . If  $\Psi^*(E(G))$  is a sequence of distinct consecutive pentagonal pyramidal numbers  $\{M_1, M_2, \dots, M_r\}$ , then the function  $\Psi$  is said to be pentagonal pyramidal graceful labeling and the graph which admits such a labeling is called a pentagonal pyramidal graceful graph. In this paper, some special graceful labeling results of pentagonal pyramidal graceful graphs is studied.

**Keywords:** Pentagonal pyramidal graceful number, pentagonal pyramidal graceful labeling, pentagonal pyramidal graceful graphs

### 1. INTRODUCTION AND DEFINITIONS

Graphs considered in this paper are finite, undirected and (simple) without loops or multiple edges. Let  $G = (V, E)$  be a graph with  $p$  vertices and  $q$  edges. Graph labeling is one of the fascinating areas of graph theory with wide ranging applications. Graph labeling was first introduced in 1960's. A graph labeling is an assignment of integers to the vertices (edges / both)

subject to certain conditions. If the domain of the mapping is the set of vertices (edges / both) then the labeling is called the vertex (edge / total) labeling. For number theoretic terminology, we refer to [1] and [2]. Terms not defined here are used in the sense of Parthasarathy [3] and Bondy and B. R. Murthy [4].

Most popular graph labeling trace their origin to one introduced by Rosa [5]. Rosa called a function (labeling)  $f$  a  $\beta$ -valuation of a graph  $G$  with  $q$  edges if  $f$  is an injection from the vertices of  $G$  to the set  $\{0,1,2, \dots, q\}$  such that each edge  $xy$  in  $G$  is assigned the label  $|f(x) - f(y)|$ , the resulting edge labels are distinct consecutive numbers and Golomb [6] called it as graceful labeling. Acharya [7] constructed certain infinite families of graceful graphs.

Labeled graphs are becoming an increasing useful family of mathematical models for a broad range of application like designing X-Ray crystallography, formulating a communication network addressing system, determining an optimal circuit layouts, problems in additive number theory etc. For more information related to graph labeling and its applications, see [8-49]. There are several types of graph labeling and a detailed survey is found in [50]. The following definitions are necessary for present study.

**Definition 1.1:** Let  $G$  be a  $(p, q)$  graph. A one to one function  $f: V(G) \rightarrow \{0,1,2,\dots,q\}$  is called a graceful labeling of  $G$  if the induced edge labeling  $f': E(G) \rightarrow \{1,2,\dots,q\}$  defined by  $f'(e) = |f(u)-f(v)|$  for each  $e = uv$  of  $G$  is also one to one. The graph  $G$  graceful labeling is called graceful graph.

**Definition 1.2:** Bistar is the graph obtaining by joining the apex vertices of two copies of star  $K_{1,n}$ .

**Definition 1.3:** Let  $v_1, v_2, \dots, v_n$  be the  $n$  vertices of a path  $P_n$ . From each vertex  $v_i, i = 1, 2, \dots, n$  there are  $m_i, i = 1, 2, \dots, n$  pendent vertices say  $v_{i1}, v_{i2}, \dots, v_{im_i}$ . The result graph is a caterpillar and is denoted by  $B(m_1, m_2, \dots, m_n)$ .

**Definition 1.4:** A coconut tree  $CT(n, m)$  is the graph obtained from the path  $P_m$  by appending  $n$  new pendant edges at an end vertex of  $P_m$ .

**Definition 1.5:** A path  $P_n$  is obtained by joining  $u_i$  to the consecutive vertices  $u_{i+1}$  for  $1 \leq i \leq n - 1$ .

**Definition 1.6:** Let  $G$  be a graph with  $p$  vertices and  $q$  edges. Let  $\Psi : V(G) \rightarrow \{0, 1, 2, \dots, M_r\}$  where  $M_r$  is the  $r^{th}$  pentagonal pyramidal number be an injective function. Define the function  $\Psi^* : E(G) \rightarrow \{1, 6, 18, \dots, M_r\}$  such that  $\Psi^*(uv) = |\Psi(u) - \Psi(v)|$  for all edges  $uv \in E(G)$ . If  $\Psi^*(E(G))$  is a sequence of distinct consecutive pentagonal pyramidal numbers  $\{M_1, M_2, \dots, M_r\}$ , then the function  $\Psi$  is said to be pentagonal pyramidal graceful labeling and the graph which admits such a labeling is called a pentagonal pyramidal graceful graph.

**Definition 1.7:** A graph  $G$  is a finite non-empty set of objects called vertices together with a set of unordered pairs of distinct vertices of  $G$  called edges. The vertex set and the edge set of  $G$  are denoted by  $V(G)$  and  $E(G)$  respectively. The number of elements of  $V(G) = p$  is called the order of  $G$  and the number of elements of  $E(G) = q$  is called the size of  $G$ . A graph of order  $p$

and size  $q$  is called a  $(p,q)$  - graph. If  $e = uv$  is an edges of  $G$ , we say that  $u$  and  $v$  are adjacent and that  $u$  and  $v$  are incident with  $e$ .

**Definition 1.8:** The degree of a vertex  $v$  in a graph  $G$  is defined to be the number of edges incident on  $v$  and is denoted by  $\deg(v)$ . A graph is called  $r$ -regular if  $\deg(v) = r$  for each  $v \in V(G)$ . The minimum of  $\{\deg v : v \in V(G)\}$  is denoted by  $\delta$  and maximum of  $\{\deg v : v \in V(G)\}$  is denoted by  $\Delta$ . A vertex of degree 0 is called an isolated vertex, a vertex of degree is called a pendant vertex or an end vertex.

**Definition 1.9:** A connected acyclic graph is called a tree.

**Definition 1.10:** A graph in which any two distinct points are adjacent is called a complete graph. The complete graph with  $n$  points is denoted by  $K_n$ .

**Definition 1.11:** A graph  $G$  is said to be connected if for every pair  $u,v$  of vertices a  $u-v$  path. Otherwise  $G$  is disconnected.

**Definition 1.12:** A graph that has neither selfloop nor parallel edges is called a simple graph.

**Definition 1.13:** The complete bipartite graph  $K_{1,n}$  is called a Star and it has  $n + 1$  vertices and  $n$  edges.

## 2. RESULTS

**Theorem 2.1:** Let  $G$  be a path with  $m$  vertices. Then  $G$  is pentagonal pyramidal graceful for all  $m \geq 3$ .

**Proof:** Let  $G$  be a path with  $m$  vertices.

Let  $V(G) = \{v_i : 1 \leq i \leq m\}$  be the vertex set of  $G$  and

$E(G) = \{v_i v_{i+1} : 1 \leq i \leq m-1\}$  be the edge set of  $G$ .

Hence  $G$  has  $m$  vertices and  $m-1$  edges.

Let  $r = m-1$ .

Define a function  $\Psi: V(G) \rightarrow \{0,1,2,\dots,M_r\}$  as follows

$$\Psi(v_1) = 0$$

$$\Psi(v_2) = M_r$$

$$\Psi(v_i) = \Psi(v_{i-1}) - M_{r-(i-2)} \text{ if } i \text{ is odd and } 3 \leq i \leq m.$$

$$= \Psi(v_{i-1}) + M_{r-(i-2)} \text{ if } i \text{ is even and } 3 \leq i \leq m.$$

Let  $\Psi^*$  be the induced edge labeling of  $f$ .

$$\text{Then } \Psi^*(v_1 v_2) = M_r$$

$$\Psi^*(v_i v_{i+1}) = M_{r-(i-1)}; 2 \leq i \leq m-1.$$

The induced edge labels  $M_1, M_2, \dots, M_r$  are distinct and consecutive pentagonal pyramidal numbers.

Hence the graph  $G$  is a pentagonal pyramidal graceful.

**Example 2.2:** Pentagonal pyramidal graceful labeling of  $P_5$  is given in Fig. 1.

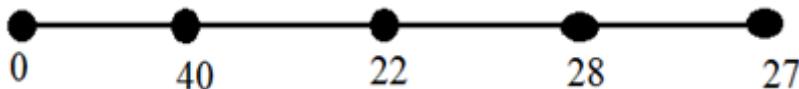


Fig. 1

**Theorem 2.3:** Caterpillars are pentagonal pyramidal graceful.

**Proof:** Let  $v_1, v_2, \dots, v_m$  be the  $m$  vertices of the path  $P_m$ .

From each vertex  $v_i, i=1,2,\dots,m$ , there are  $n_i, i = 1,2,\dots,m$ , pendent vertices say  $v_{i1}, v_{i2}, \dots, v_{ini}$ .

The resultant graph is a caterpillar and is denoted as  $B(n_1, n_2, \dots, n_m)$ .

Assume  $m \geq 3$ .

Clearly  $B(n_1, n_2, \dots, n_m)$  has  $n_1 + n_2 + \dots + n_m + (m-1)$  edges

Let  $r = n_1 + n_2 + \dots + n_m + (m-1)$

Defined  $\Psi: V(B(n_1, n_2, \dots, n_m)) \rightarrow \{0, 1, 2, \dots, M_r\}$  as follows.

$$\Psi(v_1) = 0$$

$$\Psi(v_{1i}) = M_{r-(i-1)}, \text{ where } i=1,2,\dots,n$$

$$\Psi(v_2) = \Psi(v_1) + M_{r-n_1},$$

$$\Psi(v_{2i}) = \Psi(v_2) - M_{r-n_1-i}, \text{ where } i = 1,2,\dots, n_2.$$

$$\Psi(v_3) = \Psi(v_2) - M_{r-n_1-n_2-1}.$$

$$\Psi(v_{3i}) = \Psi(v_3) + M_{r-n_1-n_2-1-i}, \text{ where } i=1,2,\dots, n_3.$$

$$\Psi(v_4) = \Psi(v_3) + M_{r-n_1-n_2-n_3-2}.$$

$$\Psi(v_{4i}) = \Psi(v_4) - M_{r-n_1-n_2-n_3-2-i} \text{ where } i = 1,2,\dots, n_4 \text{ and so on.}$$

$$\Psi(v_m) = \Psi(v_{m-1}) - M_{r-n_1-n_2-L-n_{m-1}-(m-2)} \text{ if } m \text{ is odd}$$

$$\Psi(v_m) = \Psi(v_{m-1}) + M_{r-n_1-n_2-L-n_{m-1}-(m-2)} \text{ if } m \text{ is even}$$

$$\Psi(v_{mi}) = \Psi(v_m) - M_{r-n_1-n_2-L-n_{m-1}-(m-2)-i} \text{ if } m \text{ is even and } 1 \leq i \leq n_m$$

$$\Psi(v_{mi}) = \Psi(v_m) + M_{r-n_1-n_2-L-n_{m-1}-(m-2)-i} \text{ if } m \text{ is odd and } 1 \leq i \leq n_m$$

$$\begin{aligned} \text{For } i = n_m, \Psi(v_{mn_m}) &= \Psi(v_m) \pm M_{r-n_1-n_2-L-n_{m-1}-(m-2)-n_m} \\ &= \Psi(v_m) \pm M_{r-n_1-n_2-L-n_{m-1}-(m-2)} = \Psi(v_m) \pm M_1 \end{aligned}$$

Clearly the vertex labels are distinct and the resulting edge labels are of the form  $\{M_1, M_2, \dots, M_r\}$ .

Thus caterpillar are pentagonal pyramidal graceful.

**Example 2.4:** The pentagonal pyramidal graceful labeling of a graph B(1,3,2) in Fig. 2.

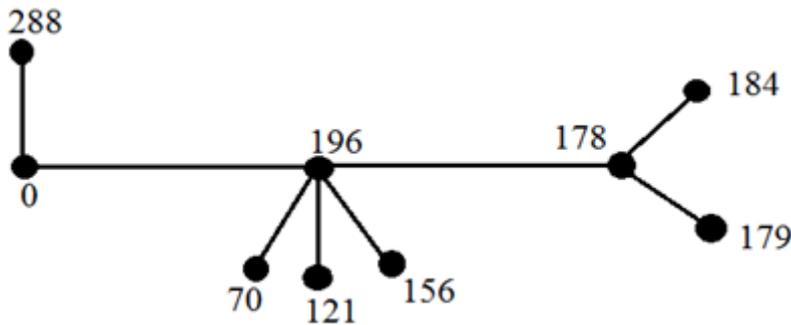


Fig. 2

**Theorem 2.5:** Coconut tree  $CT(n,m)$  is pentagonal pyramidal graceful for all  $n \geq 1, m \geq 2$ .

**Proof:** Let  $G$  be the graph  $CT(n,m)$ .

Let  $V(G) = \{v, v_i, u_j : 1 \leq i \leq n, 1 \leq j \leq m-1\}$  and  $E(G) = \{vv_i, vu_1, u_j u_{j+1} : 1 \leq i \leq n, 1 \leq j \leq m-1\}$ .

$G$  has  $n + m$  vertices and  $n + m - 1$  edges.

Let  $r = n + m - 1$ .

Let  $\Psi : V(G) \rightarrow \{0, 1, 2, \dots, M_r\}$  be defined as follows

$$\Psi(v) = 0$$

$$\Psi(v_i) = M_{r-i+1}; 1 \leq i \leq n$$

$$\Psi(u_1) = M_{r-n}$$

$$\Psi(u_j) = \Psi(u_{j-1}) + M_{r-n-(j-1)} \text{ if } j \text{ is odd and } 2 \leq j \leq m-1$$

$$= \Psi(u_{j-1}) - M_{r-n-(j-1)} \text{ if } j \text{ is even and } 2 \leq j \leq m-1$$

Let  $\Psi^*$  be the induced edge labeling of  $\Psi$ .

$$\text{Then } \Psi^*(vv_i) = M_{r-i+1}; 1 \leq i \leq n.$$

$$\Psi^*(vu_1) = M_{r-n}.$$

$$\Psi^*(u_j u_{j+1}) = M_{r-n-j} ; 1 \leq j \leq m-2.$$

The induced edge labels  $M_1, M_2, \dots, M_r$  are distinct and consecutive pentagonal pyramidal numbers.

Hence Coconut tree is pentagonal pyramidal graceful.

**Example 2.6:** Pentagonal pyramidal graceful labeling of  $CT(5,4)$  is given in Fig. 3.

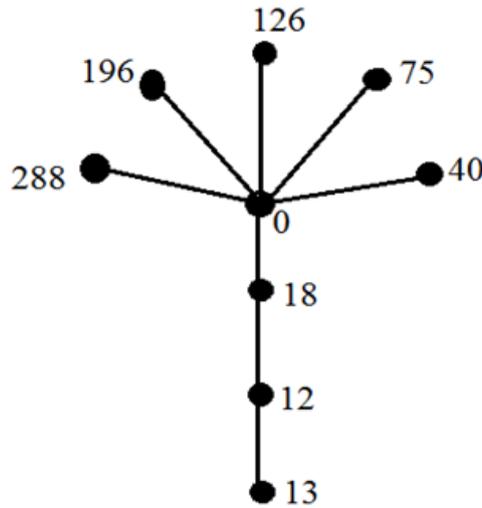


Fig. 3

**Theorem 2.7:** The bistar  $B(n_1, n_2)$  where  $n_1 \geq 1$  and  $n_2 \geq 1$  is pentagonal pyramidal graceful.

**Proof:** Let  $P_2$  be a path on two vertices and let  $v_1$  and  $v_2$  be the vertices of  $P_2$ .

From  $v_1$  there are  $n_1$  pendent vertices say  $v_{11}, v_{12}, \dots, v_{1n}$  and from  $v_2$ , there are  $n_2$  pendent vertices say  $v_{21}, v_{22}, \dots, v_{2n_2}$ .

The resulting graph is a bistar  $B(n_1, n_2)$ .

Let  $G = (V, E)$  be the bistar  $B(n_1, n_2)$ .

Let  $V(G) = \{v_i : i=1,2\} \cup \{v_{1j} : 1 \leq j \leq n_1\} \cup \{v_{2j} : 1 \leq j \leq n_2\}$  and

$E(G) = \{v_1 v_2\} \cup \{v_1 v_{1j} : 1 \leq j \leq n_1\} \cup \{v_2 v_{2j} : 1 \leq j \leq n_2\}$ .

Then  $G$  has  $n_1 + n_2 + 2$  vertices and  $n_1 + n_2 + 1$  edges.

Let  $n_1 + n_2 + 1 = r$  (say)

Now label the vertices  $v_1, v_2$  of  $P_2$  as 0 and 1.

Then label the  $n_1$  vertices adjacent to  $v_1$  other than  $v_2$  as  $M_r, M_{r-1}, M_{r-2}, \dots, M_{r-n_1+1}$ .

Next label the  $n_2$  vertices adjacent to  $v_2$  other than  $v_1$  as  $M_{r-n_1+1}, \dots, M_{r-n_1-n_2+1}+1$ .

We shall prove that  $G$  admits pentagonal pyramidal graceful labeling.

From the definition, it is clear that  $\max \Psi(v) \in \{0, 1, 2, \dots, M_r\}$  for all  $v \in V(G)$

Also from the definition, all the vertices of  $G$  have different labeling.

Hence  $\Psi$  is one to one.

It remains to show that the edges values are of the form  $\{M_1, M_2, \dots, M_r\}$ .

The induced edges function  $\Psi^*: E(G) \rightarrow \{1, 6, \dots, M_r\}$  is defined as follows

$$\Psi^*(v_i v_{ij}) = M_{r-(j-1)} \text{ if } i = 1 \text{ and } 1 \leq j \leq n_1$$

$$\Psi^*(v_i v_{ij}) = M_{r-(n_1+j-1)} \text{ if } i = 2 \text{ and } 1 \leq j \leq n_2.$$

$$\text{And } \Psi^*(v_1 v_2) = M_1.$$

Clearly  $\Psi^*$  is one to one and  $\Psi^*(E(G)) = \{M_1, M_2, \dots, M_r\}$ .

Therefore  $G$  admits pentagonal pyramidal graceful labeling.

Hence the graph  $B(n_1, n_2)$  is pentagonal pyramidal graceful.

**Example 2.8:** The pentagonal pyramidal graceful labeling of  $B(2,3)$  is given in Fig. 4.



Fig. 4

**Theorem 2.9:** The caterpillar  $B(n_1, 0, n_2)$  is pentagonal pyramidal graceful for all  $n_1, n_2 \geq 1$ .

**Proof:** Let  $v_1, v_2, v_3$  be the three vertices of  $P_3$ .

From  $v_1$  there are  $n_1$  pendent vertices say  $u_1, u_2, \dots, u_{n_1}$  and from  $v_3$ , there are  $n_2$  pendent vertices say  $w_1, w_2, \dots, w_{n_2}$ .

The resulting graph is denoted as  $B(n_1, 0, n_2)$ .

Let it be  $G = (V, E)$ .

Then  $G$  has  $n_1 + n_2 + 3$  vertices and  $n_1 + n_2 + 2$  edges.

Let  $r = n_1 + n_2 + 2$ .

Define  $\Psi: V(G) \rightarrow \{0, 1, 2, \dots, M_r\}$  as follows.

$$\Psi(v_1) = M_r$$

$$\Psi(v_2) = 0$$

$$\Psi(v_3) = M_{r-n_1-1}$$

$$\Psi(u_i) = M_r - M_{r-i} \text{ where } 1 \leq i \leq n_1,$$

$$\Psi(w_j) = M_{r-n_1-1} + M_j, \text{ where } 1 \leq j \leq n_2.$$

We shall prove that G admits pentagonal pyramidal graceful labeling.

From the definition, it is clear that  $\max \Psi(v) = M_r$  for all  $v \in V(G)$  and  $\Psi(v) \in \{0, 1, 2, \dots, M_r\}$ .

Also from the definition, all the vertices of G have different labeling.

Hence  $\Psi$  is one to one.

It remains to show that the edge values are of the form  $\{M_1, M_2, \dots, M_r\}$ .

The induced edge function  $\Psi^*: E(G) \rightarrow \{1, 2, \dots, M_r\}$  is defined as follows  $\Psi^*(v_1v_2) = M_r$ .

$$\Psi^*(v_2v_3) = M_{r-n_1-1}$$

$$\Psi^*(v_1u_i) = M_{r-i} \text{ where } 1 \leq i \leq n_1.$$

$$\Psi^*(v_3w_j) = M_j, \text{ where } 1 \leq j \leq n_2.$$

Clearly  $\Psi^*$  is one to one and  $\Psi^*(E(G)) = \{M_1, M_2, \dots, M_r\}$ .

Therefore G admits pentagonal pyramidal graceful labeling.

Hence the graph  $B(n_1, 0, n_2)$  is pentagonal pyramidal graceful for all  $n_1, n_2 \geq 1$ .

**Example 2.10:** The pentagonal pyramidal graceful labeling of  $B(2,0,3)$  is given in Fig. 5.

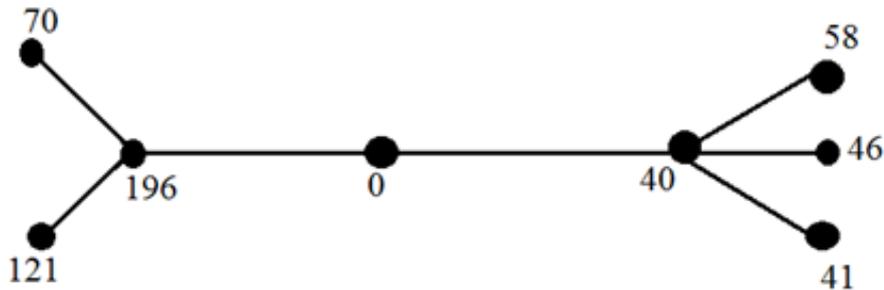


Fig. 5

**Corollary 2.11:** The caterpillar  $B(n_1, 1, n_2)$  is pentagonal pyramidal graceful for all  $n_1, n_2 \geq 1$ .

**Example 2.12:** The pentagonal pyramidal graceful labeling of  $B(2,1,3)$  is given in Fig. 6.

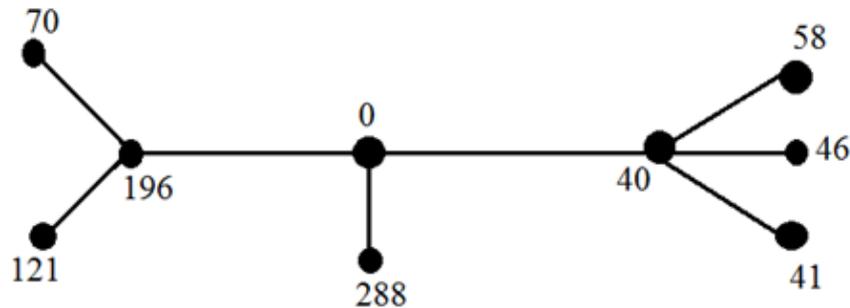


Fig. 6

**Theorem 2.13:** Olive trees are pentagonal pyramidal graceful.

**Proof:** Let  $u_0$  be the root of the Olive tree  $O(T_n)$ .

Let  $u_{11}, u_{12}, \dots, u_{1n}$  be the vertices in the first level.

Then there are  $n$  edges in the first level.

Also, let  $u_{22}, u_{23}, \dots, u_{2n}$  be the vertices in the second level.

Hence  $u_{1i}u_{2i}, i = 2,3,4, \dots, n$  be the  $n - 1$  edges in the second level.

Let  $u_{33}, u_{34}, \dots, u_{3n}$  be the vertices in the third level.

Thus  $u_{2j}u_{3j}, j = 3,4, \dots, n$  be the  $n - 2$  edges in the third level.

Proceeding like this  $u_{nn}$  be the unique vertex in the  $n^{th}$  level and the corresponding edge will be  $u_{n-1n}u_{nn}$ .

Now the total number of edges in  $O(T_n)$  is  $r$ (say).

Consider the vertex function  $\Psi : V(O(T_n)) \rightarrow \{0,1,2, \dots, M_r\}$ . Label the vertices as  $\Psi(u_0) = 0$  and  $\Psi(u_{1i}) = M_{r-(i-1)}, 1 \leq i \leq n$  so as the edge values are  $M_r, M_{r-1}, \dots, M_{r-(n-1)}$ .

$\Psi(u_{2i}), i = 2,3, \dots, n$  are obtained by  $\Psi(u_{1i}) - x$  ( $x$  must be distinct and suitably chosen for each  $i$ ) so as the edge values are  $M_{r-n}, M_{r-(n+1)}, \dots, M_{r-(2n-2)}$ .

$\Psi(u_{3i}), i = 3,4, \dots, n$  are obtained by  $\Psi(u_{2i}) + y, i = 3,4, \dots, n$  ( $y$ 's are distinct and suitably chosen for each  $i$ ) so as the edge values are  $M_{r-(2n-1)}, M_{r-(2n)}, \dots, M_{r-(3n-4)}$ .

Proceeding like this (that is, alternatively subtracting and adding suitable, distinct positive quantities with the  $\Psi$  values of the previous level), in the last level  $\Psi(u_{nn})$  will obtain the value from  $\Psi(u_{n-1n})$  in such a way that the values of the edge must be  $1 = M_1$ .

Then the edge values of the olive tree are  $\{M_1, M_2, \dots, M_r\}$ .

For the above process, we shall give the following algorithm also.

Define  $\Psi : V(T) \rightarrow \{0,1,2, \dots, M_r\}$  as follows.

$$\Psi(u_0) = 0$$

$$\Psi(u_{1i}) = M_{r-(i-1)}, \quad 1 \leq i \leq n$$

For  $2 \leq t \leq n$  and  $t \leq i \leq n$ ,

$$\Psi(u_{ti}) = \begin{cases} \Psi(u_{t-1i}) - M_{(K_{t-1}-(i-t+1))} & \text{if } t \text{ is even} \\ \Psi(u_{t-1i}) + M_{(K_{t-1}-(i-t+1))} & \text{if } t \text{ is odd} \end{cases}$$

where  $K_i = m - 2(n - 1) - \sum_{j=1}^{i-2} (n - (j + 1))$ ,  $i \geq 3$  and

$$K_i = m - i(n - 1), \quad i = 1, 2.$$

Clearly  $\Psi$  is injective and the set of edge labels which are absolute differences of the labels of the adjacent vertices are  $M_1, M_2, \dots, M_r$ .

Hence olive trees are pentagonal pyramidal graceful labeling.

**Example 2.14:** The pentagonal pyramidal graceful labeling of the olive tree  $O(T_4)$  is displaced in Fig. 7.

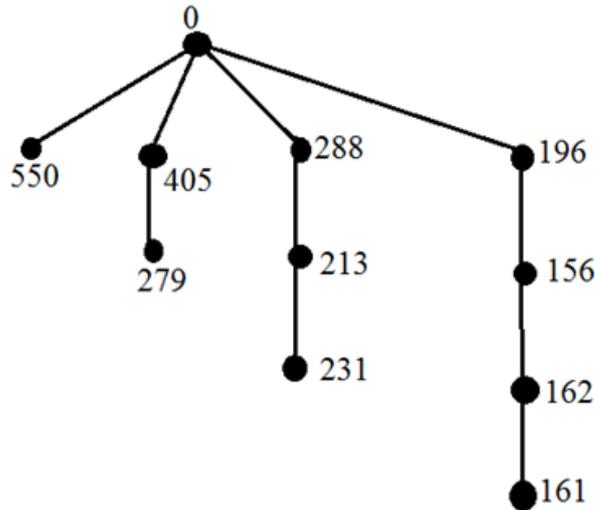


Fig. 7

### 3. CONCLUSIONS

In this paper, we have introduced pentagonal pyramidal graceful labeling graphs and studied graceful labeling of some graphs. This work contributes several new results to the theory of graph labeling. The pentagonal pyramidal graceful can be verified for many other graphs.

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