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Higher order triangular graceful labeling of some graphs

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ABSTRACT

A (p, q) graph G is said to admit higher order triangular graceful labeling if its vertices can be labeled by the integers from 0 to q^{th} higher order triangular numbers such that the induced edge labels obtained by the absolute difference of the labels of end vertices are the first q higher order triangular numbers. A graph G which admits higher order triangular graceful labeling is called a higher order triangular graceful graph. In this paper, third order, fourth order, fifth order triangular graceful labeling are introduced and third order, fourth order, fifth order triangular graceful labeling of star graph, subdivision of star, nK_2, path, comb, bistar, coconut tree, n$K_{1,3}$ are studied.

***Keywords*:** third order, fourth order, fifth order, fifth order triangular numbers, fifth order triangular graceful labeling, fifth order triangular graceful graph

1. INTRODUCTION

The graph considered in this paper are finite, undirected and without loops or multiple edges. Let G = (V, E) be a graph with p vertices and q edges. Terms not defined here are used in the sense of Harary [1], Parthasarathy [2]. For number theoretic terminology, we refer to [3, 4] and [5].

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Graph labeling is one of the fascinating areas of graph theory with wide ranging applications. Graph labeling was first introduced in 1960's. A graph labeling is an assignment of integers to the vertices (edges / both) subject to certain conditions. If the domain of the mapping is the set of vertices (edges / both) then the labeling is called the vertex (edge / total) labeling.

Most popular graph labeling trace their origin to one introduced by Rosa [6]. Rosa called a function (labeling) f a β-valuation of a graph G with q edges if f is an injection from the vertices of G to the set $\{0,1,2,\dots,q\}$ such that each edge xy is assigned the label $|f(x)-f(y)|$, the resulting edge labels are distinct and Golomb [7] called it as graceful labeling. Acharya [8] constructed certain infinite families of graceful graphs. There are several types [8-11] of graph labeling and a detailed survey is found in [12]. The concept of polygonal graceful labeling was introduced by D.S.T. Ramesh and M. P. Syed Ali Nisaya [16, 17, 19]. For more information related to graph labeling and its applications, see [13-15, 18, 20-38].

2. PRELIMINARIES

The following definitions are necessary for present study.

Definition 2.1: A graph G is a finite non-empty set of objects called vertices together with a set of unordered pairs of distinct vertices of G called edges. The vertex set and the edge set of G are denoted by V(G) and E(G) repectively. The number of elements of V(G) = p is called the order of G and the number of elements of E(G) = q is called the size of G. A graph of order p and size q is called a (p,q) - graph. If e = *uv* is an edges of G, we say that *u* and *v* are adjacent and that *u* and *v* are incident with e.

Definition 2.2: The degree of a vertex *v* in a graph G is defined to be the number of edges incident on v and is denoted by deg(v). A graph is called r-regular if deg(*v*) = r for each *v*∈ V(G). The minimum of {deg *v* : *v*∈ V(G)} is denoted by δ and maximum of {deg *v* : *v* ∈ V(G)} is denoted by △. A vertex of degree 0 is called an isolated vertex, a vertex of degree is called a pendant vertex or an end vertex.

Definition 2.3: The complete bipartite graph $K_{1,n}$ is called a Star graph

Definition 2.4: A graph, which can be formed from a given graph G by breaking up each edge into exactly two segments by inserting intermediate vertices between its two ends is called a sub division graph. It is denoted by S(G).

Definition 2.5: nG is a graph which contains n copies of the graph G. That is, $nG = \bigcup_{i=1}^{n} G_i$ where each $G_i = G$.

Definition 2.6: A path P_n is obtained by joining u_i to the consecutive vertices u_{i+1} for $1 \le i \le n-1$.

Definition 2.7: The graph obtained by joining a single pendant edge to each vertex of a path P_n is called a Comb graph. It is denoted by $P_n \odot K_1$

Definition 2.8: The bistar B(m, n) is the graph obtained from K_2 by joining m pendant edges to one end of K_2 and n pendant edges to the other end of K_2. The edge of K_2 is called the central edge of B(m, n) and the vertices of K_2 are called the central vertices of B(m, n).

Definition 2.9: A closed trail whose origin and internal vertices are distinct is called a Cycle. A cycle of length n is called n-cycle. It is denoted by C_n.

Definition 2.10: A connected acyclic graph is called a tree

Definition 2.11: A coconut tree CT(m, n) is the graph obtained from the path P_m by appending n new pendant edges at an end vertex of P_m.

Definition 2.12: A graph in which any two distinct points are adjacent is called a complete graph. The complete graph with n points is denoted by K_n.

Definition 2.13: A third order triangular number is a number obtained by adding all the cubes of positive integers less than or equal to a given positive integer n. If the n^{th} third order triangular number is denoted by $C_n,\ then\ C_n = 1^3 + 2^3 +, \dots, +n^3 = \frac{1}{4}n^2(n+1)^2$. The third order triangular numbers are $1, 9, 36, 100, 225, 441, 784, 1296, 2025, 3025, 4356, 6084, \dots$

Definition 2.14: A fourth order triangular number is a number obtained by adding all the fourth powers of positive integers less than or equal to a given positive integer n. If the n^{th} fourth order triangular number is denoted by $D_n,\ then\ D_n = 1^4 + 2^4 +, \dots, +n^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$. The fourth order triangular numbers are $1,\ 17,\ 98,\ 354,\ 979,\ 2275,\ 4676,\ 8772,\ 15333,\ 25333,\ 39974,\ 60710, \dots$

Definition 2.15: A fifth order triangular number is a number obtained by adding all the fifth powers of positive integers less than or equal to a given positive integer n. If the n^{th} fifth order triangular number is denoted by $E_n,\ then\ E_n = 1^5 + 2^5 +, \dots, +n^5 = \frac{1}{12}n^2(n+1)^2(2n^2+2n-1)$. The fifth order triangular numbers are $1,\ 33,\ 276,\ 1300,\ 4425,\ 12201,\ 29008,\ 61776,\ 120825,\ 220825, \dots$

3. MAIN RESULTS

Definition 3.1: A third order triangular graceful labeling of a graph G is an one to one function $\varphi: V(G) \to \{0,1,2, \dots, C_q\}$ that induces a bijection $\varphi^*: E(G) \to \{C_1, C_2, \dots, C_q\}$ of the edges of G defined by $\varphi^*(uv) = |\varphi(u) - \varphi(v)|\ \forall\ e = uv \epsilon E(G)$. The graph which admits such labeling is called a third order triangular graceful graph.

Definition 3.2: A fourth order triangular graceful labeling of a graph G is an one to one function $\varphi: V(G) \to \{0,1,2, \dots, D_q\}$ that induces a bijection $\varphi^*: E(G) \to \{D_1, D_2, \dots, D_q\}$ of the edges of G defined by $\varphi^*(uv) = |\varphi(u) - \varphi(v)|\ \forall\ e = uv \epsilon E(G)$. The graph which admits such labeling is called a fourth order triangular graceful graph.

Definition 3.3: A fifth order triangular graceful labeling of a graph G is an one to one function $\varphi: V(G) \to \{0,1,2,\dots,E_q\}$ that induces a bijection $\varphi^*: E(G) \to \{E_1, E_2, \dots, E_q\}$ of the edges of G defined by $\varphi^*(uv) = |\varphi(u) - \varphi(v)|\ \forall\ e = uv\epsilon E(G)$. The graph which admits such labeling is called a fifth order triangular graceful graph.

Theorem 3.4: The star $K_{1,n}$ is a third order triangular graceful graph for all $n \geq 1$.

Proof: Let G be a star graph $K_{1,n}$ for all $n \geq 1$. Let v be the unique vertex in one partition of G and $v_1, v_2, \dots, v_n$ be the n vertices in the other. Hence G has $(n+1)$ vertices and n edges. Define $\varphi: V(G) \to \{0,1,2,\dots,C_n\}$ by $\varphi(v) = 0$ and $\varphi(v_i) = C_i$ where $1 \leq i \leq n$. Clearly φ is one to one. The induced edge function $\varphi^*: E(G) \to \{C_1, C_2, \dots, C_n\}$ is defined as $\varphi^*(e_i) = C_i$ where $1 \leq i \leq n$. Clearly φ^* is a bijection and $\varphi^*\big(E(G)\big) = \{C_1, C_2, \dots, C_n\}$. Thus G admits third order triangular graceful labeling. Hence the star $K_{1,n}$ is a third order triangular graceful graph for all $n \geq 1$.

Example 3.5: The third order triangular graceful labeling of $K_{1,5}$ is shown in Figure 1.

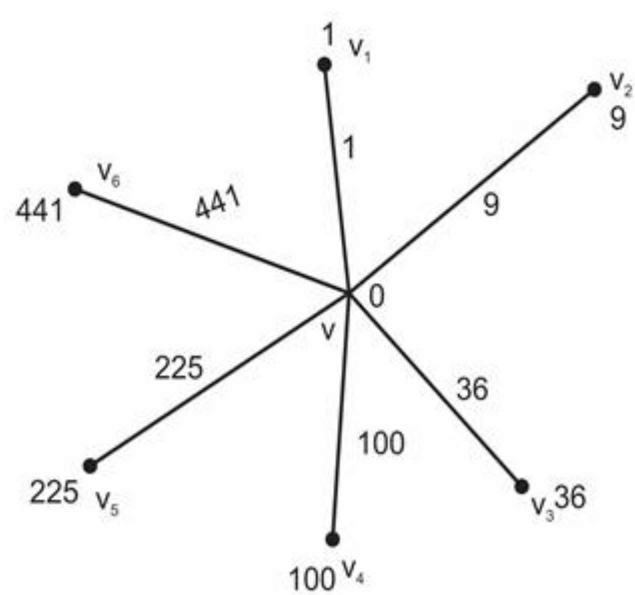

Figure 1

Theorem 3.6: $S(K_{1,n})$, the subdivision of the star $K_{1,n}$ is a third order triangular graceful graph for all $n \geq 1$.

Proof: Let G be a subdivision graph of the star $K_{1,n}$ for all $n \geq 1$.

Let $V(G) = \{v, v_i, u_i: 1 \leq i \leq n\}$ and $E(G) = \{vv_i, v_iu_i: 1 \leq i \leq n\}$

Then G has $2n+1$ vertices and $2n$ edges. Define $\varphi: V(G) \to \{0,1,2,\dots,C_{2n}\}$ as follows.

$\varphi(v) = 0$

$\varphi(v_i) = C_{2n-(i-1)}$ where $1 \le i \le n$

$\varphi(u_i) = C_{2n-(i-1)} - C_i$ where $1 \le i \le n$

Clearly φ is one to one. The induced edge function $\varphi^*: E(G) \to \{C_1, C_2, \dots, C_{2n}\}$ is defined as follows.

$\varphi^*(vv_i) = C_{2n-(i-1)}$ where $1 \le i \le n$

$\varphi^*(v_i u_i) = C_i$ where $1 \le i \le n$

Clearly φ^* is a bijection and $\varphi^*\big(E(G)\big) = \{C_1, C_2, \dots, C_{2n}\}$. Therefore G admits third order triangular graceful labeling. Hence the graph $S\big(K_{1,n}\big)$ for all $n \ge 1$ is a third order triangular graceful graph.

Example 3.7: The third order triangular graceful labeling of $S(K_{1,5})$ is shown in Figure 2.

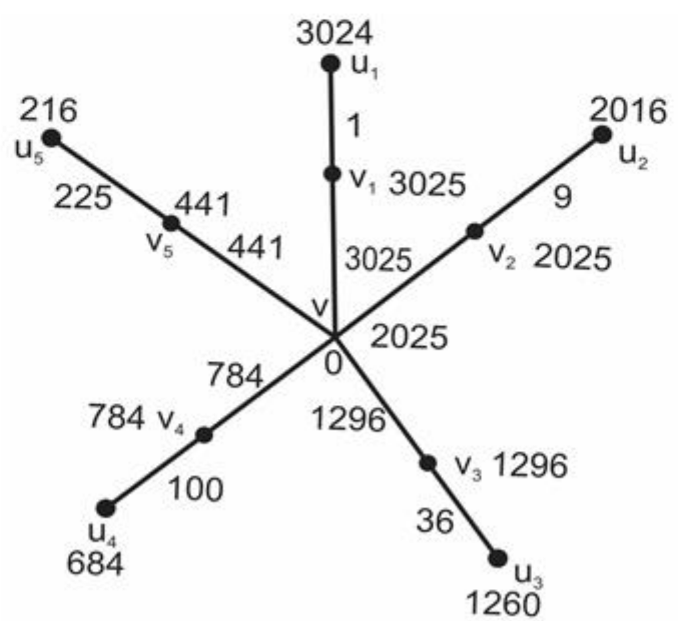

Figure 2

Theorem 2.8: nK_2 is a third order triangular graceful graph for all $n \ge 1$.

Proof: Let G be a graph which contains n copies of K_2.

Let $V(G) = \{v_{i1}, v_{i2} : 1 \le i \le n\}$ and $E(G) = \{v_{i1}v_{i2} : 1 \le i \le n\}$.

Hence G has $2n$ vertices and n edges.

Define $\varphi: V(G) \to \{0,1,2, \dots, C_n\}$ as follows.

$\varphi(v_{11}) = 0$

$\varphi(v_{12}) = C_n$

$$\varphi(v_{i1}) = \sum_{j=1}^{i-1}(n-j) \ where \ 2 \le i \le n$$

$$\varphi(v_{i2}) = C_{n-(i-1)} + \varphi(v_{i1}) \ where \ 2 \le i \le n$$

Clearly φ is one to one. The induced edge function $\varphi^*: E(G) \to \{C_1, C_2, \dots, C_n\}$ is defined as follows.

$$\varphi^*(v_{i1}v_{i2}) = C_{n-(i-1)} \ , where \ 1 \le i \le n$$

Clearly φ^* is a bijection and $\varphi^*\big(E(G)\big) = \{C_1, C_2, \dots, C_n\}$. Therefore G admits third order triangular graceful labeling. Hence the graph nK_2 for all $n \ge 1$ is a third order triangular graceful graph.

Theorem 3.9: The path P_n on n vertices is a third order triangular graceful graph for all $n \ge 2$.

Proof: Let G be a path P_n on n vertices where $n \ge 2$. Let $V(G) = \{v_1, v_2, \dots, v_n\}$ and $E(G) = \{v_i v_{i+1}: 1 \le i \le n-1\}$. Then G has n vertices and $n-1$ edges. Let $s = n-1$.

Define $\varphi: V(G) \to \{0, 1, 2, \dots, C_s\}$ as follows.

$$\varphi(v_1) = 0$$

$$\varphi(v_i) = \begin{cases} \varphi(v_{i-1}) - C_{s-(i-2)} & if \ i \ is \ odd \ 2 \le i \le n \\ \varphi(v_{i-1}) + C_{s-(i-2)} & if \ i \ is \ even \ 2 \le i \le n \end{cases}$$

Clearly φ is one to one. The induced edge function $\varphi^*: E(G) \to \{C_1, C_2, \dots, C_s\}$ is defined as $\varphi^*(v_i v_{i+1}) = C_{n-i}, 1 \le i \le n-1$.

Clearly φ^* is a bijection and $\varphi^*\big(E(G)\big) = \{C_1, C_2, \dots, C_{n-1}\}$. Therefore G admits third order triangular graceful labeling. Hence the path P_n on n vertices is a third order triangular graceful graph for all $n \ge 2$.

Example 3.10: The third order triangular graceful labeling of P_7 is shown in Figure 3.

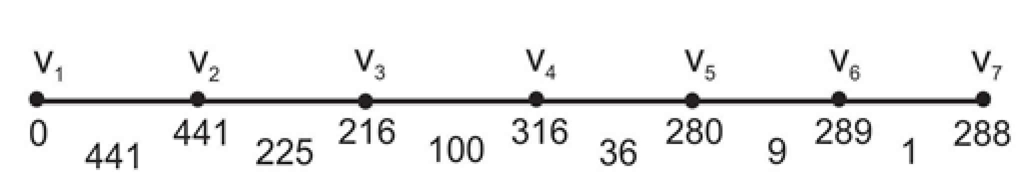

Figure 3

Theorem 3.11: The comb graph $P_n \odot K_1$ is a third order triangular graceful graph for all $n \ge 2$.

Proof: Let G be a comb graph $P_n \odot K_1$.Then $V(G) = \{u_i, w_i : where \ 1 \le i \le n\}$

$E(G) = \{u_i u_{i+1} : where \ 1 \le i \le n-1\} \cup \{u_i w_i : where \ 1 \le i \le n\}$

Hence G has $2n$ vertices and $2n-1$ edges. Let $s = 2n-1$.

Define $\varphi : V(G) \to \{0,1,2,\dots,C_s\}$ as follows.

$\varphi(u_1) = 0$

$$\varphi(u_i) = \begin{cases} \varphi(u_{i-1}) - C_{s-(i-2)} \text{ if } i \text{ is odd } 2 \le i \le n \\ \varphi(u_{i-1}) + C_{s-(i-2)} \text{ if } i \text{ is even } 2 \le i \le n \end{cases}$$

$\varphi(w_1) = C_{2s+1}$

$\varphi(w_i) = \varphi(u_i) + C_{s+(i-1)}, 2 \le i \le n.$

Clearly φ is one to one. The induced edge function $\varphi^* : E(G) \to \{C_1, C_2, \dots, C_{2n-1}\}$ is defined as follows.

$\varphi^*(u_i u_{i+1}) = C_{n-i}, 1 \le i \le n-1$

$\varphi(u_1 w_1) = C_{2s+1}$

$\varphi(u_i w_i) = C_{s+(i-1),}\ 2 \le i \le n.$

Clearly φ^* is a bijection and $\varphi^*(E(G)) = \{C_1, C_2, \dots, C_{2n-1}\}$.

Therefore G admits third order triangular graceful labeling.

Hence the comb $P_n \odot K_1$ is a third order triangular graceful graph for all $n \ge 2$.

Example 3.12: The third order triangular graceful labeling of $P_5 \odot K_1$ is shown in Figure 4.

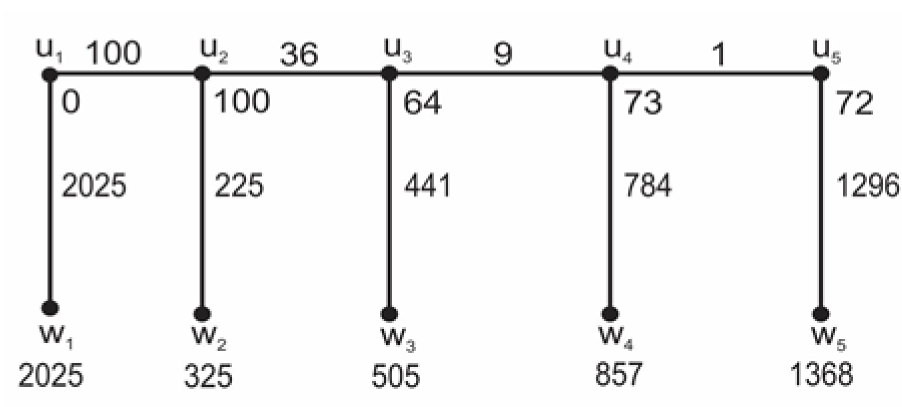

Figure 4

Theorem 3.13: The bistar $B(m,n)$ is a third order triangular graceful graph for all $m,n \ge 1$.

Proof: Let G be a bistar $B(m,n)$. Let $V(G) = \{u, v, u_i, v_j : 1 \le i \le m\,; 1 \le j \le n\}$ and

$E(G) = \{uv, uu_i, vv_j : 1 \le i \le m\,; 1 \le j \le n\}$

Hence G has $m+n+2$ vertices and $m+n+1$ edges.

Define $\varphi : V(G) \to \{0,1,2, \dots, C_{m+n+1}\}$ as follows.

$\varphi(u) = 0$

$\varphi(v) = C_{m+n+1}$

$\varphi(u_i) = C_{m+n+1-i}$ where $1 \leq i \leq m$.

$\varphi(v_j) = C_{m+n+1-i} - C_j$ where $1 \leq j \leq n$.

Clearly φ is one to one. The induced edge function $\varphi^* : E(G) \to \{C_1, C_2, \dots, C_{m+n+1}\}$ is defined as follows.

$\varphi^*(uv) = C_{m+n+1}$

$\varphi^*(uu_i) = C_{m+n+1-i}$ where $1 \leq i \leq m$

$\varphi^*(vv_j) = C_j$ where $1 \leq j \leq n$.

Clearly φ^* is a bijection and $\varphi^*(E(G)) = \{C_1, C_2, \dots, C_{m+n+1}\}$. Therefore G admits third order triangular graceful labeling. Hence the graph $C(m, n)$ for all $m, n \geq 1$ is a third order triangular graceful graph.

Example 3.14: The third order triangular graceful labeling of $CT(5,4)$ is shown in Figure 5.

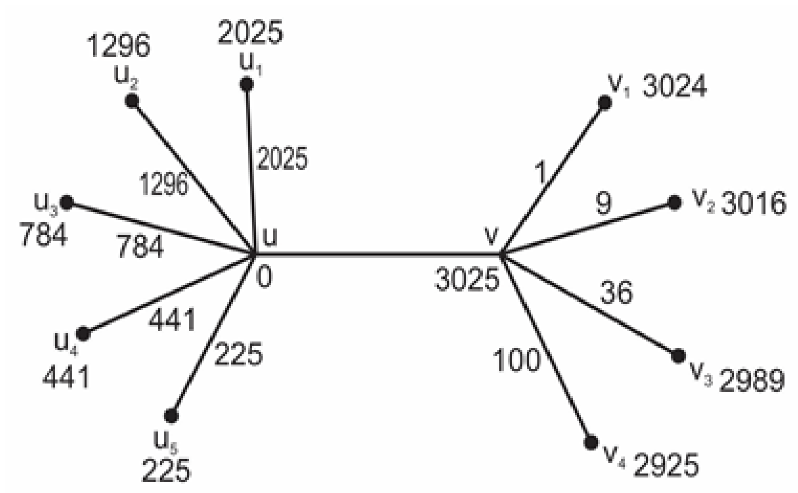

Figure 5

Theorem 3.15: Coconut tree $CT(m, n)$ is a third order triangular graceful graph for all $m, n \geq 1$.

Proof: Let G be a coconut tree $CT(m, n)$. Then $V(G) = \{w_j, v_i\ 1 \leq j \leq m\ ;\ 1 \leq i \leq n\}$ and $E(G) = \{v_1 w_j, v_i v_{i+1} : 1 \leq j \leq m\ ; 1 \leq i \leq n-1\}$. Hence G has $m+n$ vertices and $m+n-1$ edges. Let $s = m+n$. Define $\varphi : V(G) \to \{0,1,2, \dots, C_s\}$ as follows.

$\varphi(v_1) = 0$

$$\varphi(v_i) = \begin{cases} \varphi(v_{i-1}) - C_{n-(i-2)} \text{ if } i \text{ is odd } 2\le i\le n \\ \varphi(v_{i-1}) + C_{n-(i-2)} \text{ if } i \text{ is even } 2\le i\le n \end{cases}$$

$\varphi(w_i) = C_{s-(j-1)}$; $1 \le j \le m$.

Clearly φ is one to one. The induced edge function $\varphi^* : E(G) \to \{C_1, C_2, \dots, C_{m+n-1}\}$ is defined as follows.

$\varphi^*(v_i v_{i+1}) = C_{n-i}$ where $1 \le i \le n-1$

$\varphi(v_1 w_j) = C_{s-(j-1)}$ where $1 \le j \le m$ and $s = m + n$

Clearly φ^* is a bijection and $\varphi^*(E(G)) = \{C_1, C_2, \dots, C_{m+n-1}\}$.

Therefore G admits third order triangular graceful labeling.

Hence the graph $CT(m, n)$ is a third order triangular graceful graph.

Theorem 3.16: $nK_{1,3}$ is a third order triangular graceful graph for all $n \ge 1$.

Proof: Let G be a graph which contains n copies of $K_{1,3}$.

Let $V(G) = \{x_i, u_i, v_i, w_i : where\ 1 \le i \le n\}\ and\ E(G) = \{x_i u_i, x_i v_i, x_i w_i : where\ 1 \le i \le n\}$.

Hence G has $4n$ vertices and $3n$ edges.

Define $\varphi : V(G) \to \{0,1,2, \dots, C_{3n}\}$ as follows.

$$\varphi(x_i) = \begin{cases} C_{3n} - 2(n-i) & if\ 1 \le i < n \\ 0 & if\ i = n \end{cases}$$

$$\varphi(u_i) = \begin{cases} \varphi(x_i) - C_{3i-2} & if\ 1 \le i < n \\ C_{3i-2} & if\ i = n \end{cases}$$

$$\varphi(v_i) = \begin{cases} \varphi(x_i) - C_{3i-1} & if\ 1 \le i < n \\ C_{3i-1} & if\ i = n \end{cases}$$

$$\varphi(w_i) = \begin{cases} \varphi(x_i) - C_{3i} & if\ 1 \le i < n \\ C_{3i} & if\ i = n \end{cases}$$

Clearly φ is one to one. The induced edge function $\varphi^* : E(G) \to \{C_1, C_2, \dots C_{3n}\}$ is defined as follows.

$$\varphi^*(x_i u_i) = \begin{cases} C_1\ if\ i = 1 \\ C_4\ if\ i = 2 \\ . \\ . \\ . \\ C_{3n-2}\ if\ i = n \end{cases}$$

ie, $\varphi^*(x_i u_i) = C_{3i-2}\ where\ 1 \le i \le n$.

$$\varphi^*(x_i v_i) = \begin{cases} C_2 \; if \; i = 1 \\ C_5 \; if \; i = 2 \\ \quad . \\ \quad . \\ \quad . \\ C_{3n-1} \; if \; i = n \end{cases}$$

ie, $\varphi^*(x_i v_i) = C_{3i-1} \; where \; 1 \le i \le n.$

and $$\varphi^*(x_i w_i) = \begin{cases} C_3 \; if \; i = 1 \\ C_6 \; if \; i = 2 \\ \quad . \\ \quad . \\ \quad . \\ C_{3n} \; if \; i = n \end{cases}$$

ie, $\varphi^*(x_i w_i) = C_{3i} \; where \; 1 \le i \le n.$

Clearly φ^* is a bijection and $\varphi^*\big(E(G)\big) = \{C_1, C_2, \dots C_{3n}\}$. Therefore G admits third order triangular graceful labeling. Hence the graph $nK_{1,3}$ for all $n \ge 1$ is a third order triangular graceful graph.

Theorem 3.17: The star $K_{1,n}$ is a fourth order triangular graceful graph for all $n \ge 1$.

Proof: Let G be a star graph $K_{1,n}$ for all $n \ge 1$.

Let v be the unique vertex in one partition of G and $v_1, v_2, \dots, v_n$ be the n vertices in the other.

Hence G has $(n+1)$ vertices and n edges.

Define $\varphi: V(G) \to \{0,1,2, \dots, D_n\}$ by $\varphi(v) = 0$ and $\varphi(v_i) = D_i$ where $1 \le i \le n$.

Clearly φ is one to one. The induced edge function $\varphi^*: E(G) \to \{D_1, D_2, \dots, D_n\}$ is defined as $\varphi^*(e_i) = D_i$ where $1 \le i \le n$.

Clearly φ^* is a bijection and $\varphi^*\big(E(G)\big) = \{D_1, D_2, \dots, D_n\}$. Thus G admits fourth order triangular graceful labeling. Hence the star $K_{1,n}$ is a fourth order triangular graceful graph for all $n \ge 1$.

Theorem 3.18: $S\big(K_{1,n}\big)$, the subdivision of the star $K_{1,n}$ is a fourth order triangular graceful graph for all $n \ge 1$.

Proof: Let G be a subdivision graph of the star $K_{1,n}$ for all $n \ge 1$.

Let $V(G) = \{v, v_i, u_i : 1 \le i \le n\}$ and

$E(G) = \{vv_i, v_i u_i : 1 \le i \le n\}$

Then G has $2n+1$ vertices and $2n$ edges. Define $\varphi: V(G) \to \{0,1,2, \dots, D_{2n}\}$ as follows.

$\varphi(v) = 0$

$\varphi(v_i) = D_{2n-(i-1)}$ where $1 \leq i \leq n$

$\varphi(u_i) = D_{2n-(i-1)} - D_i$ where $1 \leq i \leq n$

Clearly φ is one to one. The induced edge function $\varphi^*: E(G) \to \{D_1, D_2, \dots, D_{2n}\}$ is defined as follows.

$\varphi^*(vv_i) = D_{2n-(i-1)}$ where $1 \leq i \leq n$

$\varphi^*(v_i u_i) = D_i$ where $1 \leq i \leq n$

Clearly φ^* is a bijection and $\varphi^*\big(E(G)\big) = \{D_1, D_2, \dots, D_{2n}\}$.

Therefore G admits fourth order triangular graceful labeling.

Hence the graph $S\big(K_{1,n}\big)$, for all $n \geq 1$ is a fourth order triangular graceful graph.

Theorem 3.19: nK_2 is a fourth order triangular graceful graph for all $n \geq 1$.

Proof: Let G be a graph which contains a n copies of K_2.

Let $V(G) = \{v_{i1}, v_{i2} : 1 \leq i \leq n\}$ and $E(G) = \{v_{i1}v_{i2} : 1 \leq i \leq n\}$.

Hence G has $2n$ vertices and n edges. Define $\varphi: V(G) \to \{0,1,2, \dots, D_n\}$ as follows.

$\varphi(v_{11}) = 0$

$\varphi(v_{12}) = D_n$

$$\varphi(v_{i1}) = \sum_{j=1}^{i-1}(n-j) \ where \ 2 \leq i \leq n.$$

$\varphi(v_{i2}) = D_{n-(i-1)} + \varphi(v_{i1})$ *where* $2 \leq i \leq n$.

Clearly φ is one to one. The induced edge function $\varphi^*: E(G) \to \{D_1, D_2, \dots, D_n\}$ is defined as follows.

$\varphi^*(v_{i1}v_{i2}) = D_{n-(i-1)}$, *where* $1 \leq i \leq n$

Clearly φ^* is a bijection and $\varphi^*\big(E(G)\big) = \{D_1, D_2, \dots, D_n\}$.

Therefore G admits fourth order triangular graceful labeling.

Hence the graph nK_2 for all $n \geq 1$ is a fourth order triangular graceful graph.

Theorem 3.20: The path P_n on n vertices is a fourth order triangular graceful graph for all $n \geq 2$.

Proof: Let G be a path P_n on n vertices where $n \geq 2$. Let $V(G) = \{v_1, v_2, \dots, v_n\}$ and

$\mathrm{E}(G) = \{v_i v_{i+1}: 1 \leq i \leq n-1\}$

Then G has n vertices and $n-1$ edges.

Let $s = n - 1$.

Define $\varphi: V(G) \to \{0, 1, 2, \dots, D_s\}$ as follows.

$\varphi(v_1) = 0$

$$\varphi(v_i) = \begin{cases} \varphi(v_{i-1}) - D_{s-(i-2)} & \text{if } i \text{ is odd } 2 \le i \le n \\ \varphi(v_{i-1}) + D_{s-(i-2)} & \text{if } i \text{ is even } 2 \le i \le n \end{cases}$$

Clearly φ is one to one. The induced edge function $\varphi^*: E(G) \to \{D_1, D_2, \dots, D_s\}$ is defined as $\varphi^*(v_i v_{i+1}) = D_{n-i}, 1 \le i \le n-1$.

Clearly φ^* is a bijection and $\varphi^*\big(E(G)\big) = \{D_1, D_2, \dots, D_{n-1}\}$.

Therefore G admits fourth order triangular graceful labeling. Hence the path P_n on n vertices is a fourth order triangular graceful graph for all $n \ge 2$.

Theorem 3.21: The comb graph $P_n \odot K_1$ is a fourth order triangular graceful graph for all $n \ge 2$.

Proof: Let G be a comb graph $P_n \odot K_1$. Then $V(G) = \{u_i, w_i : 1 \le i \le n\}$ and

$E(G) = \{u_i u_{i+1} : 1 \le i \le n-1\} \cup \{u_i w_i : 1 \le i \le n\}$

Hence G has $2n$ vertices and $2n-1$ edges. Let $s = 2n-1$.

Define $\varphi : V(G) \to \{0,1,2, \dots, D_s\}$ as follows.

$\varphi(u_1) = 0$

$$\varphi(u_i) = \begin{cases} \varphi(u_{i-1}) - D_{s-(i-2)} & \text{if } i \text{ is odd } 2 \le i \le n \\ \varphi(u_{i-1}) + D_{s-(i-2)} & \text{if } i \text{ is even } 2 \le i \le n \end{cases}$$

$\varphi(w_1) = D_{2s+1}$

$\varphi(w_i) = \varphi(u_i) + D_{s+(i-1)}, 2 \le i \le n.$

Clearly φ is one to one. The induced edge function $\varphi^* : E(G) \to \{D_1, D_2, \dots, D_{2n-1}\}$ is defined as follows.

$\varphi^*(u_i u_{i+1}) = D_{n-i}, 1 \le i \le n-1$

$\varphi(u_1 w_1) = D_{2s+1}$

$\varphi(u_i w_i) = D_{s+(i-1),}\ 2 \le i \le n.$

Clearly φ^* is a bijection and $\varphi^*\big(E(G)\big) = \{D_1, D_2, \dots, D_{2n-1}\}$. Therefore G admits fourth order triangular graceful labeling. Hence the comb $P_n \odot K_1$ is a fourth order triangular graceful graph for all $n \ge 2$.

Theorem 3.22: The bistar $B(m, n)$ is a fourth order triangular graceful graph for all $m, n \ge 1$.

Proof: Let G be a bistar $B(m, n)$. Let $V(G) = \{u, v, u_i, v_j : 1 \le i \le m\,; 1 \le j \le n\}$ and

$E(G) = \{uv, uu_i, vv_j : 1 \le i \le m\,; 1 \le j \le n\}$. Hence G has $m + n + 2$ vertices and $m + n + 1$ edges. Define $\varphi : V(G) \to \{0,1,2, \dots, D_{m+n+1}\}$ as follows.

$\varphi(u) = 0$

$\varphi(v) = D_{m+n+1}$

$\varphi(u_i) = D_{m+n+1-i}$ where $1 \leq i \leq m$

$\varphi(v_j) = D_{m+n+1-i} - D_j$ where $1 \leq j \leq n$

Clearly φ is one to one. The induced edge function $\varphi^* : E(G) \to \{D_1, D_2, \ldots, D_{m+n+1}\}$ is defined as follows.

$\varphi^*(uv) = D_{m+n+1}$

$\varphi^*(uu_i) = D_{m+n+1-i}$ where $1 \leq i \leq m$

$\varphi^*(vv_j) = D_j$ where $1 \leq j \leq n$

Clearly φ^* is a bijection and $\varphi^*(E(G)) = \{D_1, D_2, \ldots, D_{m+n+1}\}$.

Therefore G admits fourth order triangular graceful labeling.

Hence the graph $B(m,n)$ for all $m, n \geq 1$ is a fourth order triangular graceful graph.

Theorem 3.23: Coconut tree $CT(m,n)$ is a fourth order triangular graceful graph for all $m, n \geq 1$.

Proof: Let G be a coconut tree $CT(m,n)$. Then $V(G) = \{w_j, v_i\ 1 \leq j \leq m, 1 \leq i \leq n\}$ and

$E(G) = \{v_1 w_j, v_i v_{i+1} : 1 \leq j \leq m\,; 1 \leq i \leq n-1\}$.

Hence G has $m+n$ vertices and $m+n+1$ edges. Let $s = m+n$

Define $\varphi : V(G) \to \{0,1,2, \ldots, D_s\}$ as follows.

$\varphi(v_1) = 0$

$$\varphi(v_i) = \begin{cases} \varphi(v_{i-1}) - D_{n-(i-2)\ if\ i\ is\ odd\ 2 \leq i \leq n} \\ \varphi(v_{i-1}) + D_{n-(i-2)\ if\ i\ is\ even\ 2 \leq i \leq n} \end{cases}$$

$\varphi(w_i) = D_{s-(j-1)}\,;\ 1 \leq j \leq m.$

Clearly φ is one to one. The induced edge function $\varphi^* : E(G) \to \{D_1, D_2, \ldots, D_{m+n-1}\}$ is defined as follows.

$\varphi^*(v_i v_{i+1}) = D_{n-i}$ where $1 \leq i \leq n-1$

$\varphi(v_1 w_j) = D_{s-(j-1)}$ where $1 \leq j \leq m$ and $s = m+n$

Clearly φ^* is a bijection and $\varphi^*(E(G)) = \{D_1, D_2, \ldots, D_{m+n-1}\}$.

Therefore G admits fourth order triangular graceful labeling.

Hence the graph $CT(m,n)$ is a fourth order triangular graceful graph.

Theorem 3.24: $nK_{1,3}$ is a fourth order triangular graceful graph for all $n \geq 1$.

Proof: Let G be a graph which contains n copies of $K_{1,3}$.

Let $V(G) = \{x_i, u_i, v_i, w_i : where\ 1 \le i \le n\}\ and$

$E(G) = \{x_i u_i, x_i v_i, x_i w_i : where\ 1 \le i \le n\}$.

Hence G has $4n$ vertices and $3n$ edges. Define $\varphi : V(G) \to \{0,1,2,\dots,D_{3n}\}$ as follows.

$$\varphi(x_i) = \begin{cases} D_{3n} - 2(n-i) & if\ 1 \le i < n \\ 0 & if\ \ i = n \end{cases}$$

$$\varphi(u_i) = \begin{cases} \varphi(x_i) - D_{3i-2} & if\ 1 \le i < n \\ D_{3i-2} & if\ \ i = n \end{cases}$$

$$\varphi(v_i) = \begin{cases} \varphi(x_i) - D_{3i-1} & if\ 1 \le i < n \\ D_{3i-1} & if\ \ i = n \end{cases}$$

$$\varphi(w_i) = \begin{cases} \varphi(x_i) - D_{3i} & if\ 1 \le i < n \\ D_{3i} & if\ \ i = n \end{cases}$$

Clearly φ is one to one. The induced edge function $\varphi^* : E(G) \to \{D_1, D_2, \dots\ D_{3n}\}$ is defined as follows.

$$\varphi^*(x_i u_i) = \begin{cases} D_1\ if\ i = 1 \\ D_4\ if\ i = 2 \\ \quad \vdots \\ D_{3n-2}\ if\ i = n \end{cases}$$

ie, $\varphi^*(x_i u_i) = D_{3i-2}\ where\ 1 \le i \le n$.

$$\varphi^*(x_i v_i) = \begin{cases} D_2\ if\ i = 1 \\ D_5\ if\ i = 2 \\ \quad \vdots \\ D_{3n-1}\ if\ i = n \end{cases}$$

ie, $\varphi^*(x_i v_i) = D_{3i-1}\ where\ 1 \le i \le n$.

And $\varphi^*(x_i w_i) = \begin{cases} D_3\ if\ i = 1 \\ D_6\ if\ i = 2 \\ \quad \vdots \\ D_{3n}\ if\ i = n \end{cases}$

ie, $\varphi^*(x_i w_i) = D_{3i}\ where\ 1 \le i \le n$. Clearly φ^* is a bijection and $\varphi^*\big(E(G)\big) = \{D_1, D_2, \dots\ D_{3n}\}$. Therefore G admits fourth order triangular graceful labeling. Hence the graph $nK_{1,3}$ for all $n \ge 1$ is a fourth order triangular graceful graph.

Theorem 3.25: The star $K_{1,n}$ is a fifth order triangular graceful graph for all $n \ge 1$.

Proof: Let G be a star graph $K_{1,n}$ for all $n \geq 1$. Let v be the unique vertex in one partition of G and $v_1, v_2, \dots, v_n$ be the n vertices in the other. Hence G has $(n+1)$ vertices and n edges.

Define $\varphi : V(G) \to \{0,1,2,\dots,E_n\}$ by $\varphi(v) = 0$ and $\varphi(v_i) = E_i$ where $1 \leq i \leq n$.

Clearly φ is one to one. The induced edge function $\varphi^*: E(G) \to \{E_1, E_2, \dots, E_n\}$ is defined as $\varphi^*(e_i) = E_i$ where $1 \leq i \leq n$. Clearly φ^* is a bijection and $\varphi^*\big(E(G)\big) = \{E_1, E_2, \dots, E_n\}$. Thus G admits fifth order triangular graceful labeling. Hence the star $K_{1,n}$ is a fifth order triangular graceful graph for all $n \geq 1$.

Theorem 3.26: $S\big(K_{1,n}\big)$, the subdivision of the star $K_{1,n}$ is a fifth order triangular graceful graph for all $n \geq 1$.

Proof: Let G be a subdivision graph of the star $K_{1,n}$ for all $n \geq 1$.
Let $V(G) = \{v, v_i, u_i : 1 \leq i \leq n\}$ and $E(G) = \{vv_i, v_iu_i : 1 \leq i \leq n\}$. Then G has $2n+1$ vertices and $2n$ edges.

Define $\varphi : V(G) \to \{0,1,2,\dots,E_{2n}\}$ as follows.

$\varphi(v) = 0$

$\varphi(v_i) = E_{2n-(i-1)}$ where $1 \leq i \leq n$

$\varphi(u_i) = E_{2n-(i-1)} - E_i$ where $1 \leq i \leq n$

Clearly φ is one to one. The induced edge function $\varphi^*: E(G) \to \{E_1, E_2, \dots, E_{2n}\}$ is defined as follows. $\varphi^*(vv_i) = E_{2n-(i-1)}$ where $1 \leq i \leq n$

$\varphi^*(v_iu_i) = E_i$ where $1 \leq i \leq n$

Clearly φ^* is a bijection and $\varphi^*\big(E(G)\big) = \{E_1, E_2, \dots, E_{2n}\}$.

Therefore G admits fifth order triangular graceful labeling.

Hence the graph $S\big(K_{1,n}\big)$, for all $n \geq 1$is a fifth order triangular graceful graph.

Theorem 3.27: nK_2 is a fifth order triangular graceful graph for all $n \geq 1$.

Proof: Let G be a graph which contains a n copies of K_2.

Let $V(G) = \{v_{i1}, v_{i2} : 1 \leq i \leq n\}$ and

$E(G) = \{v_{i1}v_{i2} : 1 \leq i \leq n\}$.

Hence G has $2n$ vertices and n edges.

Define $\varphi: V(G) \to \{0,1,2,\dots,E_n\}$ as follows.

$\varphi(v_{11}) = 0$

$\varphi(v_{12}) = E_n$

$$\varphi(v_{i1}) = \sum_{j=1}^{i-1}(n-j) \ where \ 2 \le i \le n$$

$$\varphi(v_{i2}) = E_{n-(i-1)} + \varphi(v_{i1}) \ where \ 2 \le i \le n$$

Clearly φ is one to one. The induced edge function $\varphi^*: E(G) \to \{E_1, E_2, \dots, E_n\}$ is defined as follows. $\varphi^*(v_{i1}v_{i2}) = E_{n-(i-1)}$, $where$ $1 \le i \le n$. Clearly φ^* is a bijection and $\varphi^*(E(G)) = \{E_1, E_2, \dots, E_n\}$. Therefore G admits fifth order triangular graceful labeling. Hence the graph nK_2 for all $n \ge 1$ is a fifth order triangular graceful graph.

Theorem 3.28: The path P_n on n vertices is a fifth order triangular graceful graph for all $n \ge 2$.

Proof: Let G be a path P_n on n vertices where $n \ge 2$. Let $V(G) = \{v_1, v_2, \dots, v_n\}$ and

$E(G) = \{v_i v_{i+1}: 1 \le i \le n-1\}$

Then G has n vertices and $n-1$ edges.

Let $s = n - 1$.

Define $\varphi: V(G) \to \{0, 1, 2, \dots, E_s\}$ as follows.

$\varphi(v_1) = 0$

$$\varphi(v_i) = \begin{cases} \varphi(v_{i-1}) - E_{s-(i-2)} & if \ i \ is \ odd \ 2 \le i \le n \\ \varphi(v_{i-1}) + E_{s-(i-2)} & if \ i \ is \ even \ 2 \le i \le n \end{cases}$$

Clearly φ is one to one. The induced edge function $\varphi^*: E(G) \to \{E_1, E_2, \dots, E_s\}$ is defined as $\varphi^*(v_i v_{i+1}) = E_{n-i}$, $1 \le i \le n-1$. Clearly φ^* is a bijection and $\varphi^*(E(G)) = \{E_1, E_2, \dots, E_{n-1}\}$. Therefore G admits fifth order triangular graceful labeling. Hence the path P_n on n vertices is a fifh order triangular graceful graph for all $n \ge 2$.

Theorem 3.29: The comb graph $P_n \odot K_1$ is a fifth order triangular graceful graph for all $n \ge 2$.

Proof: Let G be a comb graph $P_n \odot K_1$. Then $V(G) = \{u_i, w_i : 1 \le i \le n\}$ and

$E(G) = \{u_i, u_{i+1} : 1 \le i \le n-1\} \cup \{u_i w_i : 1 \le i \le n\}$. Hence G has $2n$ vertices and $2n - 1$ edges. Let $s = 2n - 1$. Define $\varphi : V(G) \to \{0,1,2, \dots, E_s\}$ as follows.

$\varphi(u_1) = 0$

$$\varphi(u_i) = \begin{cases} \varphi(u_{i-1}) - E_{s-(i-2)} \ if \ i \ is \ odd \ 2 \le i \le n \\ \varphi(u_{i-1}) + E_{s-(i-2)} \ if \ i \ is \ even \ 2 \le i \le n \end{cases}$$

$\varphi(w_1) = E_{2s+1}$

$\varphi(w_i) = \varphi(u_i) + E_{s+(i-1)}, 2 \le i \le n.$

Clearly φ is one to one. The induced edge function $\varphi^* : E(G) \to \{E_1, E_2, \dots, E_{2n-1}\}$ is defined as $\varphi^*(u_i u_{i+1}) = E_{n-i}, 1 \le i \le n-1$

$\varphi(u_1 w_1) = E_{2s+1}$

$\varphi(u_i w_i) = E_{s+(i-1),}\ 2 \leq i \leq n.$

Clearly φ^* is a bijection and $\varphi^*(E(G)) = \{E_1, E_2, \dots, E_{2n-1}\}$.

Therefore G admits fifth order triangular graceful labeling.

Hence the comb $P_n \odot K_1$ is a fifth order triangular graceful graph for all $n \geq 2$.

Theorem 3.30: The bistar $B(m,n)$ is a fifth order triangular graceful graph for all $m,n \geq 1$.

Proof: Let G be a bistar $B(m,n)$. Let $V(G) = \{u, v, u_i, v_j : 1 \leq i \leq m \,; 1 \leq j \leq n\}$ and

$E(G) = \{uv, uu_i, vv_j : 1 \leq i \leq m \,; 1 \leq j \leq n\}$.

Hence G has $m+n+2$ vertices and $m+n+1$ edges.

Define $\varphi : V(G) \to \{0,1,2, \dots, E_{m+n+1}\}$ as follows.

$\varphi(u) = 0$

$\varphi(v) = E_{m+n+1}$

$\varphi(u_i) = E_{m+n+1-i}$ where $1 \leq i \leq m$

$\varphi(v_j) = E_{m+n+1-i} - E_j$ where $1 \leq j \leq n$

Clearly φ is one to one. The induced edge function $\varphi^* : E(G) \to \{E_1, E_2, \dots, E_{m+n+1}\}$ is defined as follows.

$\varphi^*(uv) = E_{m+n+1}$

$\varphi^*(uu_i) = E_{m+n+1-i}$ where $1 \leq i \leq m$

$\varphi^*(vv_j) = E_j$ where $1 \leq j \leq n$

Clearly φ^* is a bijection and $\varphi^*(E(G)) = \{E_1, E_2, \dots, E_{m+n+1}\}$.

Therefore G admits fifth order triangular graceful labeling.

Hence the graph $B(m,n)$ for all $m,n \geq 1$ is a fifth order triangular graceful graph.

Theorem 3.31: Coconut tree $CT(m,n)$ is a fifth order triangular graceful graph for all $m,n \geq 1$.

Proof: Let G be a coconut tree $CT(m,n)$.

Then $V(G) = \{w_j, v_i\ 1 \leq j \leq m, 1 \leq i \leq n\}$ and

$E(G) = \{v_1 w_j, v_i v_{i+1} : 1 \leq j \leq m \,; 1 \leq i \leq n-1\}$.

Hence G has $m+n$ vertices and $m+n+1$ edges.

Let $s = m+n$

Define $\varphi : V(G) \to \{0,1,2, \dots, E_s\}$ as follows.

$\varphi(v_1) = 0$

$$\varphi(v_i) = \begin{cases} \varphi(v_{i-1}) - E_{n-(i-2)} \text{ if } i \text{ is odd } 2 \le i \le n \\ \varphi(v_{i-1}) + E_{n-(i-2)} \text{ if } i \text{ is even } 2 \le i \le n \end{cases}$$

$\varphi(w_i) = E_{s-(j-1)}, 1 \le j \le m.$

Clearly φ is one to one. The induced edge function $\varphi^* : E(G) \to \{E_1, E_2, \dots, E_{m+n-1}\}$ is defined as follows.

$\varphi^*(v_i v_{i+1}) = E_{n-i}$ where $1 \le i \le n-1$

$\varphi(v_1 w_j) = E_{s-(j-1)}$ where $1 \le j \le m$ and $s = m + n$

Clearly φ^* is a bijection and $\varphi^*(E(G)) = \{E_1, E_2, \dots, E_{m+n-1}\}$

Therefore G admits fifth order triangular graceful labeling.

Hence the graph $CT(m, n)$ is a fifth order triangular graceful graph.

Theorem 3.32: $nK_{1,3}$ is a fifth order triangular graceful graph for all $n \ge 1$.

Proof: Let G be a graph which contains n copies of $K_{1,3}$.

Let $V(G) = \{x_i, u_i, v_i, w_i : where\ 1 \le i \le n\}\ and$

$E(G) = \{x_i u_i, x_i v_i, x_i w_i : where\ 1 \le i \le n\}.$

Hence G has $4n$ vertices and $3n$ edges. Define $\varphi : V(G) \to \{0, 1, 2, \dots, E_{3n}\}$ as follows.

$$\varphi(x_i) = \begin{cases} E_{3n} - 2(n-i) & if\ 1 \le i < n \\ 0 & if\ i = n \end{cases}$$

$$\varphi(u_i) = \begin{cases} \varphi(x_i) - E_{3i-2} & if\ 1 \le i < n \\ E_{3i-2} & if\ i = n \end{cases}$$

$$\varphi(v_i) = \begin{cases} \varphi(x_i) - E_{3i-1} & if\ 1 \le i < n \\ E_{3i-1} & if\ i = n \end{cases}$$

$$\varphi(w_i) = \begin{cases} \varphi(x_i) - E_{3i} & if\ 1 \le i < n \\ E_{3i} & if\ i = n \end{cases}$$

Clearly φ is one to one. The induced edge function $\varphi^* : E(G) \to \{E_1, E_2, \dots\ E_{3n}\}$ is defined as follows.

$$\varphi^*(x_i u_i) = \begin{cases} E_1\ if\ i = 1 \\ E_4\ if\ i = 2 \\ \quad . \\ \quad . \\ \quad . \\ E_{3n-2}\ if\ i = n \end{cases}$$

ie, $\varphi^*(x_i u_i) = E_{3i-2}\ where\ 1 \le i \le n.$

$$\varphi^*(x_i v_i) = \begin{cases} E_2 \; if \; i = 1 \\ E_5 \; if \; i = 2 \\ \quad \vdots \\ E_{3n-1} \; if \; i = n \end{cases}$$

ie, $\varphi^*(x_i v_i) = E_{3i-1} \; where \; 1 \le i \le n.$

And $\varphi^*(x_i w_i) = \begin{cases} E_3 \; if \; i = 1 \\ E_6 \; if \; i = 2 \\ \quad \vdots \\ E_{3n} \; if \; i = n \end{cases}$

ie, $\varphi^*(x_i w_i) = E_{3i} \; where \; 1 \le i \le n.$

Clearly φ^* is a bijection and $\varphi^*\big(E(G)\big) = \{E_1, E_2, \dots \; E_{3n}\}$.

Therefore G admits fifth order triangular graceful labeling. Hence the graph $nK_{1,3}$ for all $n \ge 1$ is a fifth order triangular graceful graph.

4. CONCLUSIONS

In this paper, the third order, fourth order and fifth order triangular graceful labeling are introduced. Also the third order, fourth order and fifth order triangular graceful labeling of star graph, subdivision of path, comb, bistar, coconut tree, n$K_{1,3}$ are studied. This work contributes several new results to the theory of graph labeling.

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