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Even Vertex Tetrahedral Mean Graphs

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ABSTRACT

The n^{th} tetrahedral number is denoted by T_n and is of the form $T_n = \frac{1}{6}n\,(n+1)\,(n+2)$. A graph G with p vertices and q edges is said to have an even vertex tetrahedral mean labeling if there exists an injective function f: V(G) → $\{0, 2, 4, \ldots, 2T_q\text{-}2\,,\, 2T_q\}$ such that the induced edge function f^*: E(G) → $\{T_1, T_2, \ldots, T_q\}$ defined by $f^*(uv) = \frac{f(u)+f(v)}{2}$ $\forall\ e = uv \in E(G)$ is a bijection. A graph which admits even vertex tetrahedral mean labeling is called an even vertex tetrahedral mean graph. In this paper, we introduce even vertex tetrahedral mean labeling and we prove that path, star, bistar, coconut tree, caterpillar, shrub, P_m @ P_n, banana tree, Y- tree and F-tree are even vertex tetrahedral mean graphs.

***Keywords*:** Tetrahedral number, even vertex tetrahedral mean labeling, even vertex tetrahedral mean graph

1. INTRODUCTION AND DEFINITIONS

Throughout this paper, by a graph, we mean a finite, undirected and simple graph. Let G (V, E) be a graph with p vertices and q edges. For notation and terminology, we follow [1, 3, 6] and [13].

Graph labeling is one of the fascinating areas of graph theory with wide ranging applications. Graph labeling was first introduced in 1960's. A graph labeling is an assignment

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of integers to the vertices or edges or both subject to certain conditions. If the domain of the mapping is the set of vertices (edges / both) then the labeling is called the vertex (edge / total) labeling. Most popular graph labeling trace their origin to one introduced by Rosa [16]. Rosa called a function (labeling) f a β-valuation of a graph in the year 1966 and Golomb [5] called it as graceful labeling. There are several types [8, 9, 12, 15, 17, 21, 25-42] of graph labeling and a detailed survey is found in [4].

The concept of mean labeling was introduced and studied by Somasundaram and Ponraj [18, 19]. Further some more results on mean graphs are discussed in [7, 10, 11, 14, 20, 22, 23] A graph G with p vertices and q edges is said to have an even vertex odd mean labeling if there exists an injective function f: V(G) → {0, 2, 4, . . . , 2q-2, 2q} such that the induced edge function f^*: E(G) → $\{1, 3, \ldots, 2q-1\}$ defined by $f^*(uv) = \frac{f(u)+f(v)}{2}$ $\forall$ e = uv ∈ E(G) is a bijection . A graph which admits an even vertex odd mean labeling is called an even vertex odd mean graph [2, 24]. The following definitions are necessary for present study.

Definition 1.1: A graph G is a finite non-empty set of objects called vertices together with a set of unordered pairs of distinct vertices of G called edges. The vertex set and the edge set of G are denoted by V(G) and E(G) respectively. The number of elements of V(G) = p is called the order of G and the number of elements of E(G) = q is called the size of G. A graph of order p and size q is called a (p, q) - graph. If e = *uv* is an edges of G, we say that *u* and *v* are adjacent and that *u* and *v* are incident with e.

Definition 1.2: The degree of a vertex *v* in a graph G is defined to be the number of edges incident on v and is denoted by deg(v). A graph is called r-regular if deg(*v*) = r for each *v* ∈ V(G). The minimum of {deg(*v*) : *v* ∈V(G)} is denoted by δ and maximum of {deg(*v*) : *v* ∈ V(G) } is denoted by △. A vertex of degree 0 is called an isolated vertex, a vertex of degree is called a pendant vertex or an end vertex.

Definition 1.3: A connected acyclic graph is called a tree.

Definition 1.4: A graph in which any two distinct points are adjacent is called a complete graph. The complete graph with n points is denoted by K_n.

Definition 1.5: A path P_n is obtained by joining u_i to the consecutive vertices u_{i+1} for $1 \leq i \leq$ n-1.

Definition 1.6: The complete bipartite graph $K_{1,n}$ is called a star graph.

Definition 1.7: The bistar $B_{m,n}$ is a graph obtained from K_2 by joining *m* pendant edges to one end of K_2 and n pendant edges to the other end of K_2.

Definition 1.8: The Coconut tree CT (*n*, *m*) is a graph which is obtained by identifying the central vertex of the star $K_{1,m}$ with a pendent vertex of a path P_n.

Definition 1.9: A caterpillar is a tree with a path P_m: $v_1, v_2, \ldots, v_m$, called spine with leaves(pendant vertices) known as feet attached to the vertices of the spine by edges known as legs. If every spine vertex v_i is attached with n_i number of leaves then the caterpillar is denoted by S(n_1, n_2,…. , n_m).

Definition 1.10: Shrub St(n_1, n_2,...., n_m) is a graph obtained by connecting a vertex v_0 to the central vertex of each of m number of stars.

Definition 1.11: The graph $P_m@P_n$ is obtained from P_m and m copies of P_n by identifying one pendant vertex of the i^{th} copy of P_n with i^{th} vertex of P_m where P_m is a path of length of m−1.

Definition 1.12: Banana tree Bt(n_1, n_2,...., n_m) is a graph obtained by connecting a vertex v_0 to one leaf of each of m number of stars.

Definition 1.13: The Y- Tree is a graph obtained from path by appending an edge to a vertex of a path adjacent to an end point and it is denoted by Y_n where n is the number of vertices in the tree.

Definition 1.14: F- Tree on n+2 vertices denoted by $F(P_n)$, is obtained from a path P_n by attaching exactly two pendant vertices to the n-1 and n^{th} vertex of P_n.

Definition 1.15: A tetrahedral number, or triangular pyramidal number, is a figurate number that represents a pyramid with a triangular base and three sides, called a tetrahedron. The n^{th} tetrahedral number is denoted by T_n, then $T_n = \frac{1}{6} n\,(n+1)\,(n+2)$. The first few tetrahedral numbers are 1, 4, 10, 20, 35, 56, 84, 120, 165, 220, ……

2. MAIN RESULTS

Definition 2.1: A graph G with p vertices and q edges is said to have an even vertex tetrahedral mean labeling if there exists an injective function f: V(G) → $\{0, 2, 4, \ldots, 2T_q\text{-}2\ , 2T_q\}$ such that the induced edge function f^*: E(G) → $\{T_1, T_2, \ldots, T_q\}$ defined by $f^*(uv) = \frac{f(u)+f(v)}{2}$ ∀ e = uv ∈ E(G) is a bijection. A graph which admits even vertex tetrahedral mean labeling is called an even vertex tetrahedral mean graph.

Theorem 2.2: The path $P_n (n \geq 2)$ is an even vertex tetrahedral mean graph.

Proof: Let $v_1, v_{2,\ldots\ldots} v_n$ be the vertices of the path P_n.

Let $e_i = v_i v_{i+1}$ $(1 \leq i \leq n-1)$ be the edges of P_n. Here P_n has n-1 edges.

Define f: V(P_n)→ $\{0,2,4, \ldots ,2T_{n-1}\text{-}2\ , 2T_{n-1}\}$ as follows:

$f(v_1) = 0$

$f(v_j) = 2(T_{j-1} - T_{j-2} + T_{j-3} - \ldots\ldots\ldots + (-1)^j T_1)$ for $2 \leq j \leq n$.

Clearly f is injective and for each vertex label f, the induced edge label f^* is defined to be

$f^*(e_i) = T_i$ for $1 \leq i \leq n-1$.

Then the induced edge labels are the tetrahedral numbers $T_1, T_{2,}, \dots, T_{n-1}$. Hence f is an even vertex tetrahedral mean labeling. Hence P_n is an even vertex tetrahedral mean graph.

Example 2.3: The even vertex tetrahedral mean labeling of P_7 is given in Figure 1.

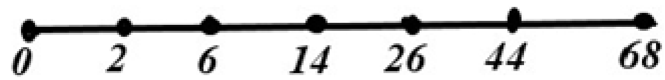

Fig. 1

Theorem 2.4: The star graph $K_{1,n}(n \geq 1)$ is an even vertex tetrahedral mean graph.

Proof: Let v be the apex vertex and let $v_1, v_{2,} \dots, v_n$ be the pendant vertices of the star $K_{1,n}$ the star $K_{1,n}$ has n edges. Define f : V($K_{1,n}$) → {0,2,4, . . . ,2T_n-2 , 2T_n} as follows.

f(v) = 0,

$f(v_j)$ = 2T_j for 1≤ $j \leq n$.

Clearly f is injective and for each vertex label f, the induced edge label f^* is defined to be

$f(vv_j) = T_j$ for 1≤ j ≤ n.

Then the induced edge labels are the tetrahedral numbers $T_1, T_{2,}, \dots . T_n$. Hence f is an even vertex tetrahedral mean labeling. Thus $K_{1,n}$ is an even vertex tetrahedral mean graph.

Example 2.5: The even vertex tetrahedral means labeling of $K_{1,9}$ is shown in Figure 2.

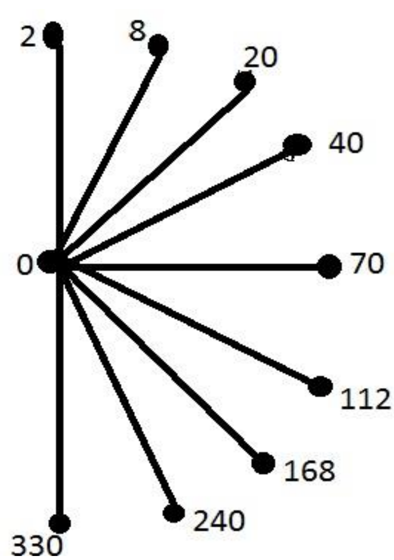

Fig. 2

Theorem 2.6: The bistar $B_{m,n}$ $(m \geq 1, n \geq 1)$ is an even vertex tetrahedral mean graph.

Proof: Let V($B_{m,n}$) = $\{u, v, u_i, v_j : 1 \leq i \leq m, \leq 1 \leq j \leq n\}$ and

E($B_{m,n}$) = $\{uv, uu_i, vv_j : 1 \leq i \leq m, 1 \leq j \leq n\}$. Here $B_{m,n}$ has m+n+2 vertices and m+n+1 dges. Define f :V($B_{m,n}$) → $\{0,2,4, \ldots, 2T_{m+n+1}\}$ as follows:

$f(u) = 0$,

$f(v) = 2$

$f(u_i) = 2T_{i+1}$ for $1 \leq i \leq m$

$f(v_j) = 2(T_{m+j+1} - 1)$ for $1 \leq j \leq n$

Clearly f is injective and for each vertex label f, the induced edge label f^* is defined as follows.

$f^*(uv) = T_1$

$f^*(uu_i) = T_{i+1}$ for $1 \leq i \leq m$

$f^*(vv_j) = T_{m+j+1}$ for $1 \leq j \leq n$

Then the induced edge labels are the first m+n+1 tetrahedral numbers. Hence f is an even vertex tetrahedral mean labeling. Thus $B_{m,n}$ is an even vertex tetrahedral mean graph.

Example 2.7: The even vertex tetrahedral mean labeling of $B_{5,6}$ is given in Figure 3.

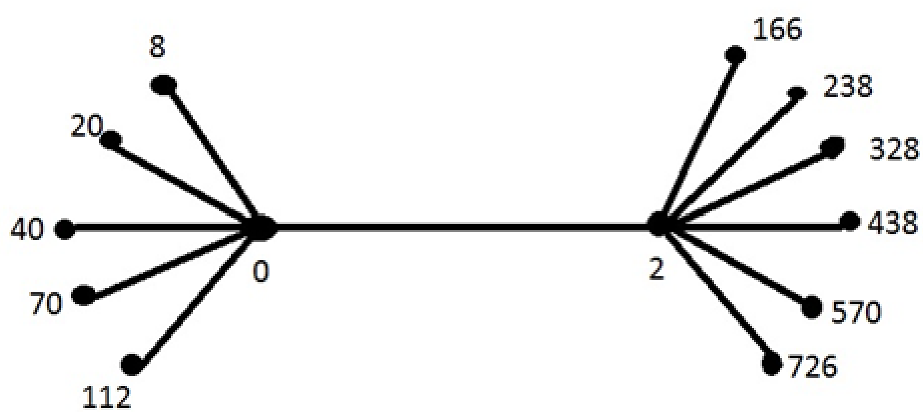

Fig. 3

Theorem 2.8: The coconut tree CT (n, m), obtained by identifying the central vertex of the star $K_{1,m}$ with a pendent vertex of a path P_n , is a even vertex tetrahedral mean graph.

Proof: Let $u_1, \ldots, u_n$ be the vertices of a path on n *vertices* ($n \geq 2$) and $v_1, v_2, \ldots, v_m$ be the pendant vertices being adjacent with u.

Define f: V(CT(n, m)) → $\{0,2,4, \ldots, 2T_{m+n-1} - 2, 2T_{m+n-1}\}$ as follows:

$f(v_i) = 2T_i$ for $1 \le i \le m$

$f(u_1) = 0$

$f(u_j) = 2T_{m+j-1}$ - f (u_{j-1}) for $2 \le j \le n$.

Clearly f is injective and for each vertex label f, the induced edge label f^* is defined as follows:

$f^*(u_1v_i) = T_i$ for $1 \le i \le m$,

$f^*(u_ju_{j+1}) = T_{m+j}$ for $1 \le j \le n-1$.

Then the induced edge labels are the first m+ n-1 tetrahedral numbers. Hence f is an even vertex tetrahedral mean labeling. Thus Coconut tree is an even vertex tetrahedral mean graph.

Example 2.9: The even vertex tetrahedral mean labeling of CT(4,8) is given in Figure 4.

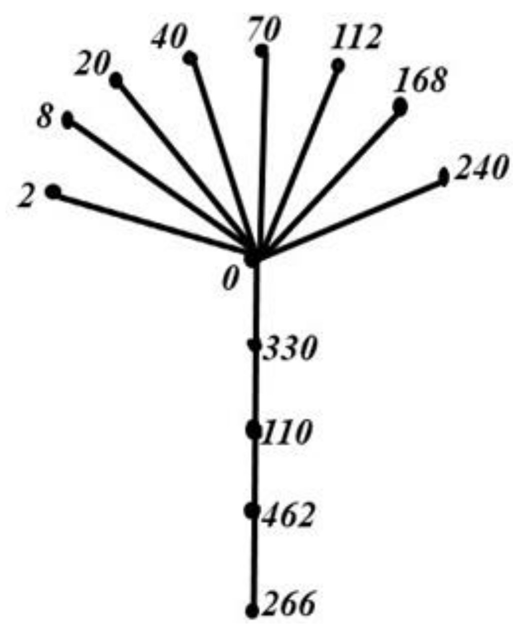

Fig. 4

Theorem 2.10: The caterpillar S($n_1, n_2, \ldots, n_m$) is an even vertex tetrahedral mean graph.

Proof: Let $v_1, v_2, \ldots, v_m$ be the vertices of the path P_m and v_i^j $(1 \le i \le n_j, 1 \le j \le m)$ be the pendant vertices joined with v_j $(1 \le j \le m)$ by an edge . Then

$V(S(n_1, n_2, \ldots, n_m)) = \{v_j, v_i^j : 1 \le i \le n_j, 1 \le j \le m)\}$

$E(S(n_1, n_2, \ldots, n_m)) = \{v_t v_{t+1}, v_j v_i^j : 1 \le t \le m-1, 1 \le i \le n_j, 1 \le j \le m\}$.

We define f: $V(S(n_1, n_2, \ldots, n_m)) \rightarrow \{0, 2, 4, \ldots, 2T_{m-1+n_1+n_2+\cdots+n_m}\}$ as follows:

$f(v_1) = 0$,

f(v_j) = $2(T_{j-1} - T_{j-2} + T_{j-3} - \ldots\ldots\ldots\ldots + (-1)^j T_1)$ for $2 \le j \le m$,

$f(v_i^1) = 2T_{m-1+i}$ for $1 \le i \le n_1$.

$f(v_i^j) = 2T_{m-1+n_1+n_2+\cdots\ldots+n_{j-1}+i} + (-1)^{j-1} 2(T_1 - T_2 + T_3 - \ldots\ldots\ldots + (-1)^j T_{j-1})$ for $1 \le i \le n_j$ and $2 \le j \le m$. Clearly f is injective and for the each vertex label *f*, the induced edge label f^* is defined as follows:

$f^*(v_j v_{j+1}) = T_j$, $1 \le j \le m-1$

$f^*(e_i^1) = T_{m-1+i}$, $1 \le i \le n_1$.

$f^*(e_i^j) = T_{m-1+n_1+n_2+\cdots\ldots+n_{j-1}+i}$ for $1 \le j \le n_j$ and $2 \le j \le m$.

Then the edge labels are the tetrahedral numbers

$T_1, T_2, \ldots\ldots\ldots, T_{m-1}, T_m, \ldots\ldots, T_{m-1+n_1+n_2+\ldots\ldots+n_m}$. Hence f is an even vertex tetrahedral mean labeling. Thus S($n_1, n_2, \ldots\ldots, n_m$) is an even vertex tetrahedral mean graph.

Example 2.11: The even vertex tetrahedral mean labeling of S(4,6,5,7) is shown in Figure 5.

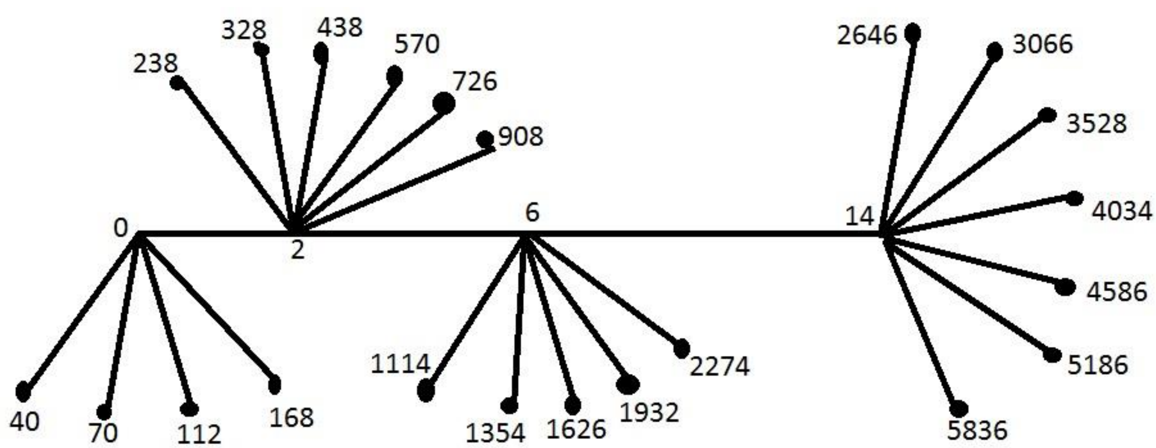

Fig. 5

Theorem 2.12: The shrub St($n_1, n_2, \ldots\ldots, n_m$) is an even vertex tetrahedral mean graph.

Proof: Let v_0, v_j, u_i^j $(1 \le j \le m, 1 \le i \le n_j)$ be the vertices of St($n_1, n_2, \ldots\ldots, n_m$).

Then E(St($n_1, n_2, \ldots\ldots, n_m$)) = $\{v_0 v_j, v_j u_i^j$ for $1 \le i \le n_j$ and $1 \le j \le m\}$.

Define*f*: V (St($n_1, n_2, \ldots\ldots, n_m$)) → $\{0, 2, 4, \ldots, 2(T_{m+n_1+n_2+\cdots\ldots+n_m})\}$ as follows

$f(v_0) = 0$

$f(v_j) = 2T_j$ for $1 \le j \le m$.

$f(u_i^j) = 2(T_{m+n_1+n_2+\cdots\ldots+n_{j-1}+i} - T_j)$ for $1 \le j \le m$ and $1 \le i \le n_j$

Let $e_i^j = v_j u_i^j$ for $1 \le i \le n_j$ and $1 \le j \le m$. Clearly f is injective and for each vertex label f, the induced edge label f^* is defined as follows:

$f^*(v_0 v_j) = T_j$ for $1 \le j \le m$.

$(e_i^j) = T_{m+n_1+n_2+\cdots\ldots+n_{j-1}+i}$ for $1 \le j \le m$ and $1 \le i \le n_j$.

Clearly f^* is bijection. Then f is an even vertex tetrahedral mean labeling of St($n_1, n_2, \ldots, n_m$). Thus St($n_1, n_2, \ldots, n_m$) is an even vertex tetrahedral mean graph.

Example 2.13: The even vertex tetrahedral mean labeling of St(5,6,5) is shown in Figure 6.

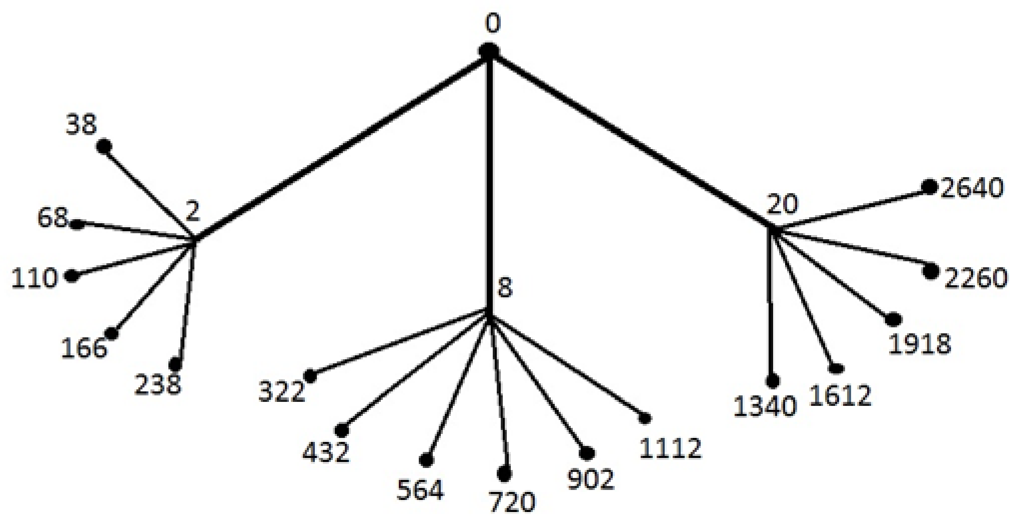

Fig. 6

Theorem 2.14: The graph $P_m @ P_n$ is an even vertex tetrahedral mean graph.

Proof: Let $\{v_j, u_j^i, 1 \le i \le n, 1 \le j \le m\}$ be the vertices of $P_n @ P_m$ with $v_j = u_j^1$, $(1 \le j \le m)$. Then E($P_n @ P_m$) = $\{v_j, v_{j+1}, u_j^i u_j^{i+1} : 1 \le j \le m-1, 1 \le i \le n-1\}$.

Define f : V($P_n @ P_m$) → {0,2,4, . . . , T_{mn-1}} as follows:

$f(u_1^1) = 0$,

$f(u_j^1) = 2(T_{j-1} - T_{j-2} + T_{j-3} - \ldots + (-1)^j T_1)$ for $2 \le j \le m$.

$f(u_1^2) = 2T_m$,

$f(u_j^2) = 2(T_{m+j-1} - T_{j-1} + T_{j-2} - \ldots\ldots + (-1)^{j-1} T_1)$ for $2 \le j \le m$,

$f(u_j^i) = 2(T_{(i-1)m+j-1} - T_{(i-2)m+j-1} + T_{(i-3)m+j-1} - \ldots\ldots + (-1)^{i-1} T_{m+j-1} + (-1)^{i-1}\, 2(T_{j-1} - T_{j-2} + T_{j-3} - \ldots + (-1)^j T_1)$ for $1 \le j \le m, 3 \le i \le n$.

For each vertex label f, the induced edge f^*, is defined as follows:

$f^*(v_jv_{j+1})$ = $T_j$for 1≤ $j \leq m-1$,

$f^*(u_j^1u_j^2)$ = T_{m+j-1} for $1 \leq j \leq m$

$f^*(u_j^iu_j^{i+1})$ = T_{mi+j-1} for $1 \leq j \leq m$ and 2≤ $i \leq n-1$.

Clearly f^* is bijection. Therefore f is an even vertex tetrahedral mean Labeling of P_n @ P_m .

Thus P_m @ P_n is an even vertex tetrahedral mean graph.

Example 2.15: The even vertex tetrahedral mean labeling of P_5 @ P_5 is shown in Figure 7.

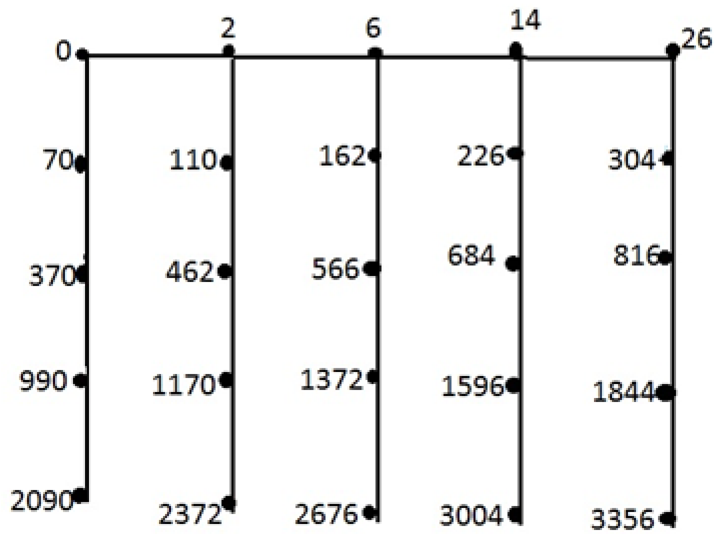

Fig. 7

Theorem 2.16: The banana tree Bt(n, n,……., n)(m times) is an even vertex tetrahedral mean graph.

Proof: Let v_0,v_i,w_i, w_{ij} ($1 \leq i \leq m,\ 2 \leq i \leq n$) be the vertices of Bt(n,n,……,n)(m times).

Then E(Bt(n,n,……,n)) = $\{v_0v_i$,v_iw_i, w_iw_{ij} for $1 \leq i \leq m, 2 \leq j \leq n\}$.

Define$f$:V (Bt(n,n,……,n))→ {0,2,4, . . . , $2T_{nm+m}$} as follows:

$f(v_0)$ = 0 ,

$f(v_i)$ = $2T_i$, for 1≤ $i \leq m$

$f(w_i)$ = 2(T_{m+j}–Te_i) for $1 \leq i \leq m$.

$f(w_{ij})$ = 2($T_{2m-3+2i+j}$)-$f(w_i)$ for 1≤ $i \leq m$ *and* $2 \leq j \leq n$.

Clearly f is injective and for each vertex label f, the induced edge label f^* is defined as follows:

$f^*(\{v_0v_i)$ = $T_i$for $1 \leq i \leq m$,

$f^*(v_iw_i)$ = T_{m+i} for $1 \leq i \leq m$

$f^*(w_i w_{ij}) = T_{2m-3+2i+j}$for $1 \le i \le m$ and $2 \le j \le n.$

Clearly f^* is bijection. Therefore f is an even vertex tetrahedral mean labeling of Bt(n, ,……..,n). Hence the banana tree Bt(n, n,……., n)(m times) is an even vertex tetrahedral mean graph.

Example 2.17: The even vertex tetrahedral mean labeling of Bt(4, 4,……., 4)(4 times) is shown in Figure 8.

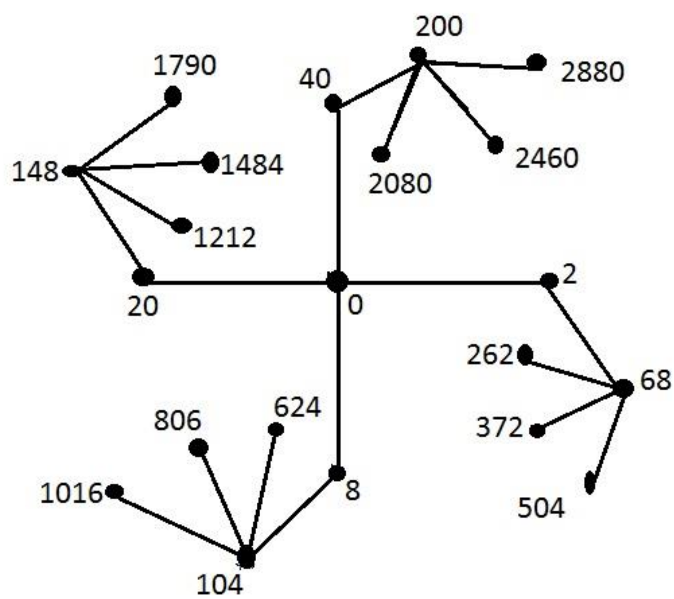

Fig. 8

Theorem 2.18: Any Y-tree Y_n is an even vertex tetrahedral mean graph.

Proof: Let v_1 , $v_2, \dots , v_n$ be the vertices of Y_n. Let $e_i = v_i v_{i+1}$ $(1 \le i \le n-2)$ and $v_{n-2}v_n$be the edges of Y_n. Y tree has n vertices and n-1 edges. Define f:V(Y_n)→ $\{0,2,4, \dots , 2T_{n-1}\}$ as follows:

$f(v_1) = 0$

For 2≤ $i \le n-1$,$f(v_i) = 2\,(T_{i-1})$-$f(v_{i-1})$

$f(V_n) = 2(\,T_{n-1})$-$f\,(v_{n-2})$

Clearly f is injective and for each vertex label f, the induced edge label f^* is defined to be

$f^*(e_i) = T_i$ for (1≤ $i \le n-2$) and

$f^*(v_{n-2}v_n) = T_{n-1}$

Clearly f^* is bijection. Hence f is an even vertex tetrahedral mean labeling.

Thus Y_n is an even vertex tetrahedral mean graph.

Example 2.19: The even vertex tetrahedral mean labeling of Y_8 is given in Figure 9.

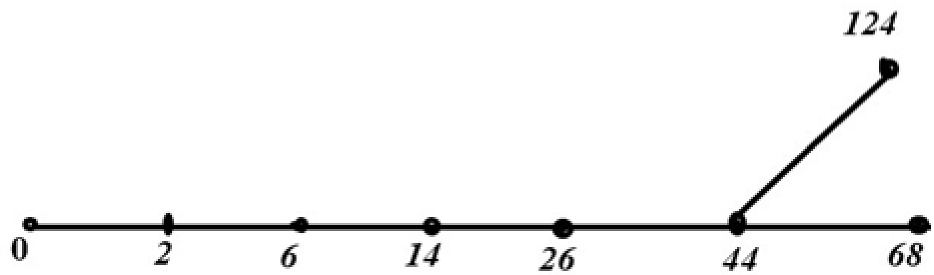

Fig. 9

Theorem 2.20: The F tree, $F(P_n), n \geq 3$ is an even vertex tetrahedral mean graph.

Proof: Let u, v, v_i be the vertices of $F(P_n), n \geq 3$. Let $e_i = v_i v_{i+1}$ $(1 \leq i \leq n-1)$ and $v_{n-1}u$ and $v_n v$ be the edges of $F(P_n)$. Here F tree has n+2 vertices and n+1 edges.

Define f :V($F(P_n)$)→ {0,2,4, . . . , $2T_{n+1}$} as follows:

$f(v_1) = 0$

For $2 \leq i \leq n$, $f(v_i) = 2\,(T_{i-1}) - f(v_{i-1})$

$f(u) = 2(T_n) - f(v_{n-2})$ and

$f(v) = 2(T_{n+1}) - f(v_n)$

Clearly f is injective and for each vertex label f, the induced edge label f^* is defined to be

$f^*(e_i) = T_i$ for $(1 \leq i \leq n-1)$

$f^*(v_{n-1}u) = T_n$ and

$f^*(v_n v) = T_{n+1}$

Clearly f^* is bijection. Hence f is an even vertex tetrahedral mean labeling.

Thus $F(P_n)$ is an even vertex tetrahedral mean graph.

Example 2.21: The even vertextetrahedral mean labeling of $F(P_6)$ is given in Figure 10.

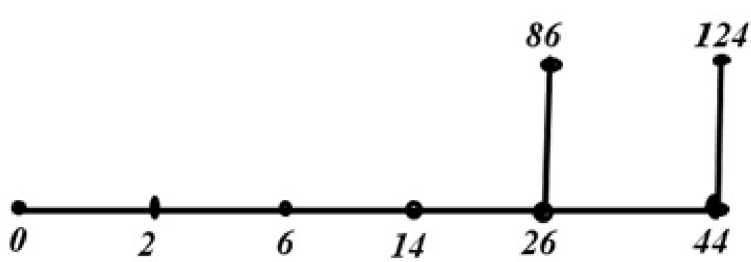

Fig. 10

3. CONCLUSION

In this paper, we have studied the even vertex tetrahedral mean labeling of some tree related graphs. This work contributes several new results to the theory of graph labeling.

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