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Adams operations Ψ^k on $bu_{p*}(B\mathbb{Z}/p)^{\wedge n}$

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ABSTRACT

We explain the action of the Adams operation Ψ^k, for an odd integer k, on $bu_{p*}(B\mathbb{Z}/p)$ mainly by using the injectivity of $bu_{p*}(B\mathbb{Z}/p)$ into the periodic case $K_*(B\mathbb{Z}/p)$, where the analogous results for the periodic case when p = 2 are given in [2, p.123]. After that we extend this description to $bu_{p*}(B\mathbb{Z}/p)^{\wedge n}$ for $n \in N_0$ using the decomposition of $bu_{p*}(B\mathbb{Z}/p)^{\wedge n}$, see [6].

***Keywords*:** The classifying space of the cyclic group of prime order $B\mathbb{Z}/p$, The connective unitary K-theory, Adams operations Ψ^k

1. INTRODUCTION

Let bu_* denote connective unitary K-homology on the stable homotopy category of CW spectra [1] so that if X is a space without a basepoint its unreduced bu-homology is $bu_*(\Sigma^\infty X^+)$, the homology of the suspension spectrum of the disjoint union of X with a base-point. In particular $bu_*(\Sigma^\infty S^0) = \mathbb{Z}[u]$ where deg(u) = 2.

For a prime number p, we have bu_p, the connective unitary K-theory with p-adic integer coefficients $\mathbb{Z}_p$, Where $bu_p \simeq \vee_{i=1}^{p-1} \Sigma^{2i-2} lu$, lu is the Adams summand such that $bu_{p*}(S^0) \cong \oplus_{i=1}^{p-1} lu_{*-2i+2}(S^0)$, $lu_*(S^0) \cong \mathbb{Z}_p[u^{p-1}] \cong \mathbb{Z}_p[v]$ and deg (v) = 2(p-1).

Let B\Z/p be the classifying space of the cyclic group of prime order p. In 1972, Holzsager [4] split the space $\Sigma B\mathbb{Z}/p$ with p-adic coefficients into the wedge of p-1 spaces B_i, where B_i

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has homology only in dimensions 2k (p-1)+2i, for all natural numbers k. So the spectrum $\sum^{\infty} B\mathbb{Z}/p$ splits as $\sum^{\infty} B\mathbb{Z}/p \simeq \bigvee_{i=1}^{p-1} \sum^{\infty} B_i$, see also [6]. Here the spectrum B_i has stable cells in dimension $2k(p-1)+2i-\epsilon$, for $\epsilon = 0,1$ such that $2k(p-1)+2i-\epsilon \geq 0$. The splitting of $B\mathbb{Z}/p$ as a spectrum is also written as $B\mathbb{Z}/p \simeq \bigvee_{i=1}^{p-1} B_i$.

By [5], for the case E = lu the Adams summand and X = B$\mathbb{Z}$/p, and by using the Thom isomorphism, we have the following homotopy equivalence $lu \wedge \sum^2 B_i \simeq lu \wedge B_{i+1}$ for $1 \leq i < p-1$. Inductively on i, we get $lu \wedge \sum^{2(i-1)} B_1 \simeq lu \wedge B_i$.

The main aim of this paper is to explain how the Adams operations Ψ^k act on $bu_{p*}(B\mathbb{Z}/p)^{\wedge n}$ for any prime number p.

Briefly, in the first section we will introduce some concepts, which related to the topic and fix some notations that will support our results in this paper.

In §2, we explain the action of Ψ^k on $bu_{2*}(B\mathbb{Z}/2)$ by using the action of Ψ^k on the periodic case $K_*(B\mathbb{Z}/2)$ which is explained in [2]. In [6] we decomposed $bu_{2*}(B\mathbb{Z}/2)^{\wedge n}$ as a direct sum of some graded groups, see also [7], in this section also we explain the action of Ψ^k on each of these summands to deduce the action of Ψ^k on $bu_{2*}(B\mathbb{Z}/2)^{\wedge n}$.

Similarly, in §3, we explain the action of the Adams operation Ψ^k on $lu_*(B_1)$ and on $lu_*(B_1)^{\wedge n}$ for any prime p and n > 1. The splitting $bu_p \simeq \bigvee_{i=1}^{p-1} \sum^{2i-2} lu$, the Holzsager splitting $B\mathbb{Z}/p \simeq \bigvee_{i=1}^{p-1} B_i$, the homotopy equivalence $lu \wedge \sum^{2(i-1)} B_1 \simeq lu \wedge B_i$ and the decomposition of $bu_{p*}(B\mathbb{Z}/p)^{\wedge n}$ from $lu_*(B_1)$ all of these yields the main purpose of this paper [21-28].

2. PRELIMINARY NOTIONS

When we consider operations for complex K-theory, rather than ordinary cohomology, we get unstable operations Ψ^K, for $k \in \mathbb{Z}$, which are called the Adams operations. These were originally introduced unstably by Adams in [20] to count the number of linearly independent vector fields on S^{n-1}, for $n = (2a+1)2^b$ and a, b $\in \mathbb{Z}$. We exploit the periodicity of K to describe Adams operations just of bidegree (0,0) as elements in $K^0(K)$ with the naturality axiom of cohomology operations. In [15] it is introduced the construction for Adams operations for a compact Hausdorff space X in terms of the exterior power operations λ^k, and extend that to any space X, after that stabilise them by introducing suitable coefficients, to invert some elements of $\mathbb{Z}$.

These operations have the following properties.

Proposition 2.1. For any x, y ∈ K(X) and for k ∈ $\mathbb{Z}$, there is a ring homomorphism

$$\Psi^k : K(X) \to K(X)$$

which satisfies the following,

(i) $\Psi^k f^* = f^* \Psi^k$, for all maps f: X → Y, (naturality).

(ii) $\Psi^k(x+y) = \Psi^k(x) + \Psi^k(y)$, (additive).

(iii) $\Psi^k(xy) = \Psi^k(x)\Psi^k(y)$, (multiplicative).

(iv) $\Psi^k\left(\Psi^j(x)\right) = \Psi^k(x)$, for j ≥ 0.

(v) If L is a line bundle, $\Psi^k(L) = L^k$.

(vi) If $u \in \widetilde{K}(S^2 n)$, then $\Psi^k(u) = k^n u$

(vii) If p is prime, then $\Psi^k(x) \equiv x^p \bmod p$.

Proposition 2.2. By the Atiyah-Hirzebruch spectral sequences in homology for the space $X = B\mathbb{Z}/2$ and the spectra E = bu and E = KU, see [3], and by using the Mapping Lemma and the Comparison Theorem for two specific spectral sequences we have the following:

Proposition 2.3. For $X = B\mathbb{Z}/2$, the map

$$l_*: bu_{2^*}(X) \to K_{2^*}(X)$$

is injective.

For more explanations see [9], [10], [2], [11] and [12].

Mainly on Adams operations Ψ^k we need this map to produce the action of Ψ^k on $bu_{2*}(B\mathbb{Z}/2)$ such that Ψ^k is compatible with ι_*. However, all we know at the moment about this map ι_* is that it is injective. In order to get more properties we need to introduce another spectrum, called the Milnor spectrum or the Thom spectrum MU, which is constructed also in terms of BU, see [2], [18] and [19].

In this paper, we use this spectrum just in one place to produce a commutative diagram connecting this spectrum with the spectra bu and K and to exploit information about its Adams operations.

The corresponding homology and cohomology theories for this spectrum, respectively, are called complex bordism and complex cobordism.

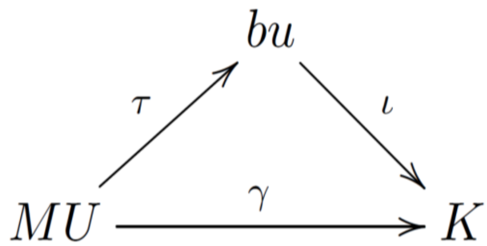
Here $MU^*(X)$ is an algebra over $MU^*(pt)$ and $MU_*(X)$ is a module over $MU_*(pt)$ for any X, where $MU_*(pt)= \mathbb{Z}[x_2, x_4, ...]$, a polynomial algebra on even degree generators. By [2, section 6.1], for $X = \mathbb{R}P^\infty$ we have

$$MU_*(\mathbb{R}P^\infty) = \mathbb{Z}\langle 1\rangle \oplus \frac{MU_*(pt)(\beta_1, \beta_3, \dots)}{(2\beta_{2m+1} + a_{1,1}\beta_{2m-1} + \Sigma_{i,j\geq 1, i+j\geq 3} a_{i,j}\beta_{2m+3-2i-2j})}$$

where: $\deg(\beta_i) = i, m \geq 0$ and $a_{i,j} \in MU_{2i+2j-2}(pt)$; a precise description for $a_{i,j}$ can be found in [15, p.56,57].

By [2, p. 121], there is a natural transformation of rings $\gamma^*: MU^*(X) \to K^*(X)$ called The Conner-Floyd map. The restriction of this map to a point sends $a_{1,1}$ to the Bott element u and the other $a_{i,j}$ to zero.

Remark 2.4. By the connectivity of MU and bu, the canonical map of ring spectra γ: MU → K can be lifted to a map τ : MU → bu to make the following diagram



commute.

In the section on Adams operations we apply $\pi_*(B\mathbb{Z}/2\wedge-)$ to this diagram, we will make these maps explicit in order to produce an action of these operations on $bu_*(B\mathbb{Z}/2)$ where the action of the operations Ψ^k on $MU_*(\mathbb{R}P^2 n)$ are explained in [2, Lemma 2.5] as the following:

Lemma 2.5. Let k be an odd integer and consider

$$\Psi^K: MU_*\left(\mathbb{R}P^{2n};\mathbb{Z}_{(2)}\right) \to MU_*\left(\mathbb{R}P^{2n};\mathbb{Z}_{(2)}\right).$$

Then, for $1 \le j \le n$,

$$\Psi^K\left(\beta_{2n+1-2j}\right) = k^{n+1-j}\beta_{2n+1-2j} \in MU_{2n+1-2j}\left(\mathbb{R}P^{2n};\mathbb{Z}_{(2)}\right).$$

Notation 2.6.

- For $n \ge 1$, in §2, we write P_n for $(B\mathbb{Z}/2)^{\wedge n}$, the n−fold smash product of $B\mathbb{Z}/2$. In particular, $P_1 = B\mathbb{Z}/2$, whereas in §3, we write P_n for $(B\mathbb{Z}/p)^{\wedge n}$.
- we write A_* for $bu_{p*}(P_1)$.
- For a $\mathbb{Z}$-graded group B_* we write $B_*[n]$ for the graded group with $B_j[n] = B_{j+n}$, so that $bu_*(X)[-1] = bu_{*-1}(X)$.

Deftnition 2.7. [6]

Let X be a graded group, and $r \ge 0$. We define $T^r(X)_*$ as

$$T^r(X)_* = T(T^{r-1}(X)_*)_*$$

where $T^0(X)_* = X$ and $T^1(X)_* = T(X)_* = Tor^1_{\mathbb{Z}_p[u]}(A_*, X)[-1]$.

From this definition we can deduce that:

(1) $T^r(X)_* = T^m(T^k(X)_*)_*$, for $m + k = r$.

(2) We have $T^r(A_*)_* = \overbrace{T(T(\ldots T(A_*)_*\ldots)_*)_*}^{r-\text{times}}$, where, by [8] §2.7 when p = 2, $T(A_*)_*$ is non-zero just in degrees $2t + 1 \ge 3$. Then, by applying $T(A_*)_* \otimes_{\mathbb{Z}_2[u]}$ − instead of $A_* \otimes_{\mathbb{Z}_2[u]}$ − to the free resolution of A_*, which is described in [8] Example 2.9, with shifting by (−1) and by using induction on r, we can calculate the graded group $T^r(A_*)_*$ This is non-zero just in degrees $2t + 1$ for $t \ge r$.

Notation 2.8. For the rest of this paper, we will write:

- A_*^r for $A_*^{\otimes^r}$, the tensor of A_* with itself over $\mathbb{Z}_p[u]$ r-times,
- $A_* \otimes B_*$ for $A_* \otimes_{\mathbb{Z}_2[u]} B_*$, for a $\mathbb{Z}_p[u]$ -module B_*, and
- $T_*^{j_r, j_{r-1}, \ldots j_1}$ for $T^{j_r}(A_* \otimes T^{j_{r-1}}(A_* \otimes T^{j_{r-2}}(\ldots T^{j_2}(A_* \otimes T^{j_1}(A_*)_*)_* \ldots)_*)_*)_*$, where $j_i \in \mathbb{N}_0$.

Definition 2.9. [6] Let $0 \leq k \leq n-1$, we define the weight k iterated T as

$$W_n^k = \bigoplus_{\sum j_i = k} T_*^{j_{n-k}, j_{n-k-1}, \ldots j_1}$$

where: $j_i \in \mathbb{N}_0$.

It is easy to check that:

(i) $W_n^k = 0$, for $k \geq n$,

(ii) $W_n^{n-1} = T_*^{n-1}$, $W_n^0 = A_*^n$, and

(iii) $W_{n+1}^k = (A_* \otimes W_n^k) \oplus T(W_n^{k-1})$ for $0 \leq k \leq n$.

In [6] we decompose $bu_{p*}(P_n)$ as a direct sum of W_n^r, for $0 \leq r \leq n-1$ as the following:

Definition 2.10. Given $j_i \in \mathbb{N}_0$ for $i \geq 1$, we define $\beta_{i,n} = \Sigma_{k=i}^n j_k$. (of course, $\beta_{i,n}$ depends on $j_i, \ldots j_n$, but the sequence will be clear from the context.)

Theorem 2.11. [6]

Let $n \geq 1$. Then

$$bu_{p*}(B\mathbb{Z}/p)^{\wedge n} = \bigoplus_{i_1, i_2, \ldots, i_{n+1} = 0}^{p-2} \; \bigoplus_{r=0}^{n-1} W_n^r$$

where: $W_n^r = \oplus_{\beta_{1,n-r}=r} T_{*-2\Sigma_{k=1}^{n+1} i_k}^{j_{n-r}, j_{n-r-1}, \ldots j_1}$

Consequently to explain the action of Ψ^k on $bu_{p*}(P_N)$ it is enough to explain how Ψ^k acts on its summands W_n^r.

3. ADAMS OPERATIONS Ψ ON $bu_{2^*}(P_n)$

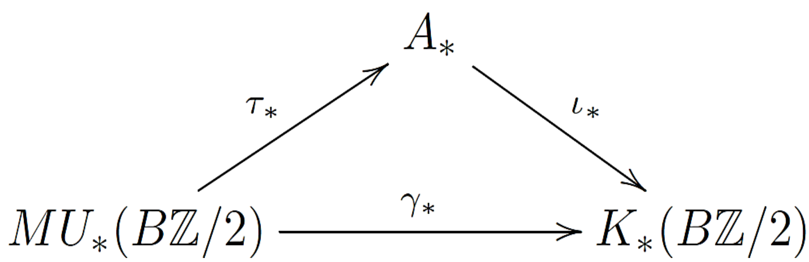
In this section we will explain the action of the operation Ψ^k, for an odd integer k, on A_* when p = 2, mainly by using the infectivity of A_* into the periodic case $K_*(P_1)$, see 2.3, where the analogous results for the periodic case are given in [2, p.123]. After that we will extend this description to $bu_{2^*}(P_N)$ for $n \in \mathbb{N}_0$ using the decomposition of $bu_{2^*}(P_N)$, see 2.11.

Proposition 3.1. The Adams operation

$$\Psi^k: A_* \rightarrow A_*,$$

satisfies $\Psi^k\,(v_{2i-1}) = k^i v_{2i-1}$.

Proof. Let us start from the commutative diagram



which is induced from the diagram in 2.4. We also have a similar diagram when we replace $B\mathbb{Z}/2$ by $\mathbb{R}P^{2n}$. Here, by [2, p.122], γ_* is given by $\gamma_*(\beta_{2i-1}) = u^i\hat{\beta}_{2i-1}$ where β_{2i-1} is in $MU_{2i-1}(\mathbb{R}P^{2n})$ for all $n > i$. The element $\hat{\beta}_{2i-1}$ lies in $K_{-1}(\mathbb{R}P^{2n})$, see [2, section 6.1] for further details. We can choose$\tau_*(\beta_{2i-1}) = v_{2i-1}$, for $v_{2i-1} \in bu_*(\mathbb{R}P^{2n})$. Therefore, by the infectivity of ι_*, (see 2.3), and by the commutatively of this diagram, ι_* maps v_{2i-1} to $u^i\hat{\beta}_{2i-1}$. By 2.5, Ψ^k acts on $MU_{2n+1-2j}\,(\mathbb{R}P^{2n})$ as multiplication by k^{n+1-j}, and by 2.1, we have $\Psi^k\,\gamma_*\,(\beta_{2i-1}) = \gamma_*\left(\Psi^k\,(\beta_{2i-1})\right) = \gamma_*\left(k^i\beta_{2i-1}\right) = k^iu^i\hat{\beta}_{2i-1}$and $\Psi^k\,(\gamma_*\,(\beta_{2i-1})) = \Psi^k\left(u^i\hat{\beta}_{2i-1}\right) = k^iu^i\Psi^k\left(\hat{\beta}_{2i-1}\right)$. So $\Psi^k\left(\hat{\beta}_{2i-1}\right) = \hat{\beta}_{2i-1}$, (this result also can be found in [2, Corollary 6.1.9]).

Thus

$$\Psi^k\left(\iota_*(\upsilon_{2i-1})\right) = \Psi^k\left(u^i\beta_{2i-1}\right) = k^iu^i\hat{\beta}_{2i-1} = \iota_*\left(k^i\upsilon_{2i-1}\right).$$

Since $\Psi^k\left(\iota_*(\upsilon_{2i-1})\right) = \iota_*\left(\Psi^k(\upsilon_{2i-1})\right)$, and ι_* is injective, we conclude that $\Psi^k\,(\upsilon_{2i-1}) = k^i\upsilon_{2i-1}$ on $bu_*\,(\mathbb{R}P^{2n})$ for n > i and so on $A_* = \lim_{\to n} bu_*\,(\mathbb{R}P^{2n})$.

For p=2, 2.11 and 2.9 shows that $bu_{p*}\,(B\mathbb{Z}/p)^{\wedge n}$ is constructed from the summands W_n^r, for $0 \leq r \leq n-1$, where each of these is constructed from the summands $T_*^{j_r,j_r-1,\ldots,j_1}$, see 2.9.

Consequently to explain the action of Ψ^k on $bu_{p*}\,(P_n)$ it is enough to explain how Ψ^k acts on its summands W_n^r. We will start with $W_n^0 = A_*^r$

Proposition 3.2. Let Let $0 < m \leq t$, and consider

$$\Psi^k: A_*^r \rightarrow A_*^r.$$

Then, for $x \in A_s^r$,

$$\Psi^k(x) = \begin{cases} k^{t+m}x & \text{if } r = 2m, \ \ s = 2t, \text{and} \\ k^{t+m+1}x & \text{if } r = 2m+1, \ \ s = 2t+1. \end{cases}$$

Proof. Let us consider the case r = 2m. The analogous calculations for r = 2m+1 are similar. By [6], A_{2t}^{2m} if an $\mathbb{F}_2$- vector space with basis

$$\{v_{2i_1-1} \otimes v_{2i_2-1} \otimes \ldots \otimes v_{2i_{2m}-1} : t = \sum_{j=1}^{2m} i_j - m\}.$$

Let x be any element of A_*^{2m} of degree 2t, then x is a linear combination of $v_{2i_1-1} \otimes v_{2i_2-1} \otimes \ldots \otimes v_{2i_{2m}-1}$ for $t = \sum_{j=1}^{2m} j_j - m$. By 2.1 and 3.1, we have

$$\Psi^k\left(v_{2i_1-1} \otimes v_{2i_2-1} \otimes \ldots \otimes v_{2i_{2m}-1}\right) = k^{\sum_{j=1}^{2m} i_j} v_{2i_1-1} \otimes v_{2i_2-1} \otimes \ldots \otimes v_{2i_{2m}-1}$$

and this shows that $\Psi^k(x) = k^{t+m}x$

Let B_* be a $\mathbb{Z}_2[u]$-module, which is concentrated in odd degrees or in even degrees, with an action of Ψ^k. What is the Ψ^k action on $A_* \otimes B_*$? The answer will be explained in the following Proposition for the relevant actions of Ψ^k on B_*.

Proposition 3.3. Let B_* be as above, such that Ψ^k acts on $x_j \in B_j$ as multiplication by k^t, for j = 2t or j = 2t + 1. Then Ψ^k acts on $(A_* \otimes B_*)_{2t}$ as multiplication by k^t, whereas it acts on $(A_* \otimes B_*)_{2t+1}$ as multiplication by k^{t+1}.

Proof. Let $y \in (A_* \otimes B_*)_{2t}$. Then y is a linear combination of $v_{2i+1} \otimes x_{2j+1}$ for $t = i + j + 1$. By 3.1, we have

$$\Psi^k\left(v_{2i+1} \otimes x_{2j+1}\right) = k^{i+1+j}\left(v_{2i+1} \otimes x_{2j+1}\right)$$

and this shows that $\Psi^k(y) = k^t y$. Similarly, any $y \in (A_* \otimes B_*)_{2t+1}$ is a linear combination of $v_{2i+1} \otimes x_{2j}$ for $t = i + j$. Therefore

$$\Psi^k(y) = k^{\,t=i+j} y = k^{t+1} y$$

By this we see that, if B_* is concentrated in odd degrees with the above action of Ψ^k, then Ψ^k preserves the same action on $A_* \otimes B_*$ as on B_*, and otherwise this is not true.

We have a free resolution of A_* by $\mathbb{Z}_2[u]$-modules, as described in [8] and [6],

$$0 \rightarrow \bigoplus_{j>0} \mathbb{Z}_2[u]\langle a_{2j-1} \rangle \xrightarrow{d} \bigoplus_{j>0} \mathbb{Z}_2[u]\langle b_{2j-1} \rangle \xrightarrow{\epsilon} A_* \rightarrow 0$$

where: $\epsilon\left(b_{2j-1}\right) = v_{2j-1}$ and $d\left(a_{2j-1}\right) = 2b_{2j-1} - ub_{2j-3}$ for $j > 0$. Let us define an action of Ψ^k on the other terms of this resolution, $\bigoplus_{j>0} \mathbb{Z}_2[u]\langle b_{2j-1} \rangle$ and $\bigoplus_{j>0} \mathbb{Z}_2[u]\langle a_{2j-1} \rangle$, as follows:

$$\Psi^k(b_{2j-1}) = k^j b_{2j-1} \text{ and } \Psi^k(a_{2j-1}) = k^j a_{2j-1}.$$

Then

$\Psi^k \epsilon(b_{2j-1}) = \Psi^k(v_{2j-1}) = k^i v_{2j-1} = \epsilon(k^i b_{2j-1}) = \epsilon\Psi^k(b_{2j-1})$ and $\Psi^k d(a_{2j-1}) = \Psi^k(2b_{2j-1} - ub_{2j-3}) = k^j(2b_{2j-1} - ub_{2j-3}) = d(k^j a_{2j-1}) = d\Psi^k(a_{2j-1})$.

This shows that ϵ and d respect the action of Ψ^k, and therefore this resolution is compatible with the Ψ^k-actions.

Proposition 3.4. Consider

$$\Psi^k : T^1_* \to T^1_*.$$

Then Ψ^k acts on $x \in T^1_{2t+1}$ as multiplication by k^{t+1}.

Proof. As described in [6], x can be written as $\sum_{t=i+j-1} v_{2i-1} \otimes a_{2j-1}$, for $i, j > 0$. Therefore

$$\Psi^k(x) = \Psi^k\left(\sum_{t=i+j-1} v_{2i-1} \otimes a_{2j-1}\right) = \sum_{t=i+j-1} k^{i+j} v_{2i-1} \otimes a_{2j-1} = k^{t+1}x.$$

If we replace A_* by a $\mathbb{Z}_2[u]$-module B_* in 3.4, where we already know the action of Ψ^k on B_* let us think about the action of Ψ^k on $T(B_*)_*$ and how it depends on the action of Ψ^k on B_*. To explain that we have the following proposition.

Proposition 3.5. Let B_* be as above, such that Ψ^k acts on $x \in B_s$ as multiplication by k^t, for $s = 2t$ or $s = 2t + 1$. Then Ψ^k has the same action on as on $T(B_*)_*$ as on B_*.

Proof. Let us start again from the free resolution of A_*, which was introduced in [6]. By applying $(B_* \oplus -)$ and shifting by (-1), we get $T(B_*)_* \subset B_* \otimes \left(\otimes_{j>0} \mathbb{Z}_2[u]\langle a_{2i-1}\rangle\right)$.

Let $y \in T(B_*)_{2t}$, then y can be written as a linear combination of elements of the form $x_{2i} \oplus u^k a_{2j-1}$, for $t = i + k + j$.

Since

$$\Psi^k(x_{2i} \oplus u^k a_{2j-1}) = k^{i+k+j} x_{2i} \oplus u^k a_{2j-1} = k^t x_{2i} \otimes u^k a_{2j-1},$$

we have $\Psi^k(y) = k^t y$. Similarly, for $T(B_*)_{2t+1,}$ where y here can be written as a linear combination of elements of the form $x_{2i+1} \oplus u^k a_{2j-1}$ for $t = i + k + j$, and $\Psi^k(x_{2i+1} \oplus u^k a_{2j-1}) = k^{i+k+j} x_{2i+1} \otimes u^k a_{2j-1}$.

Remark 3.6.

- From 3.5, we see that Ψ^k preserves the same action on $T(A^r_*)$ as on A^r_*.

- Since $T_*^{j_1} = T(T_*^{j_1-1})$, then, inductively on j_1, and by 3.5, we can deduce that Ψ^k preserves the same action on $T_*^{j_1}$ as on A_*.

Proposition 3.7. Consider

$$\Psi^k\colon T_*^{j_r,j_{r-1},\dots,j_1} \to T_*^{j_r,j_{r-1},\dots,j_1}$$

and $x_s \in T_s^{j_r,j_{r-1},\dots,j_1}$. Then

$$\Psi^k(x_s) = \begin{cases} k^{t+r_1}x_s, & \text{if } r = 2r_1,\ \ s = 2t, \text{and} \\ k^{t+r_1+1}x_s, & \text{if } r = 2r_1 + 1,\ \ s = 2t + 1 \end{cases}$$

Proof. We prove this by induction on r, where the case r = 1 is considered in 3.6. By 3.3, replacing B_* by T^{j_1}, we deduce that Ψ^k acts on $x_{2t} \in A_* \otimes T^{j_1}$ as multiplication by k^{t+1}, and 3.5 shows that Ψ^k preserves the same action on T^{j_2,j_1} as on $A_* \otimes T^{j_1}$. That is, Ψ^k acts on

$$x_{2t} \in T_*^{j_2,j_1}$$

as multiplication by k^{t+1}. Let us assume the statement is true for $r = 2r_1$, That is, Ψ^k acts on

$$x_{2t} \in T_*^{j_{2r_1},j_{2r_1-1},\dots,j_1}$$

as multiplication by k^{t+r_1}. Then, by 3.3, replacing B_* by $T_*^{j_{2r_1},j_{2r_1-1},\dots,j_1}$, Ψ^k acts on

$$x_{2t+1} \in A_* \otimes T_*^{j_{2r_1},j_{2r_1-1},\dots,j_1}$$

as multiplication by k^{t+r_1+1}, where 3.5 shows that Ψ^k preserves the same action on $T_*^{j_{2r_1},j_{2r_1-1},\dots,j_1}$ as on $A_* \otimes T_*^{j_{2r_1},j_{2r_1-1},\dots,j_1}$.

Now again by 3.3, replacing B_* by $T_*^{j_{2r_1+1},j_{2r_1},\dots,j_1}$, we deduce that Ψ^k preserves the same action on $A_* \otimes T_*^{j_{2r_1+1},j_{2r_1},\dots,j_1}$ as on $T_*^{j_{2r_1+1},j_{2r_1},\dots,j_1}$. That is, Ψ^k acts on

$$x_{2t} \in A_* \otimes T_*^{j_{2r_1+1},j_{2r_1},\dots,j_1}$$

as multiplication by k^{t+r_1+1}, and finally, 3.5 yields that Ψ^k acts on

$$x_{2t} \in \otimes T_*^{j_{2r_1+1},j_{2r_1},\dots,j_1}$$

as multiplication by k^{t+r_1+1}. This completes the proof.

Now $bu_{2^*}(P_n)$ is constructed from the summands W_n^r, for $0 \le r \le n-1$, where each of these is constructed from the summands $T_*^{j_r,j_{r-1},\dots,j_1}$, see 2.9 and 2.11. Consequently to explain the action of Ψ^k on $bu_{2^*}(P_n)$ it is enough to explain how Ψ^k acts on its summands W_n^r.

We have $T_*^{j_r,j_{r-1},\dots,j_1}$ is non-zero just in degrees $s \geq 2\beta_{1,r} + r$ where s and r both are odd or both are even, and $W_n^r = \otimes_{\beta_{1,n-r=r}} T_*^{j_{n-r},j_{n-r-1},\dots,j_1}$ Next we will consider the action of Ψ^k on W_n^r in degrees $s \geq 2\beta_{1,n-r} + n - r$ when s and $n - r$ both are even or both are odd. Otherwise the graded group W_n^r is zero.

Theorem 3.8. Let $0 \leq r \leq n - 1$ and consider

$$\Psi^k : W_n^r \to W_n^r.$$

Then, for $x_s \in W_n^r$,

$$\Psi^k(x_s) = \begin{cases} k^{t+r_1} x_{s,} & \text{if } n - r = 2r_1,\ \ s = 2t, \text{and} \\ k^{t+r_1+1} x_{s,} & \text{if } n - r = 2r_1 + 1,\ \ s = 2t + 1 \end{cases}$$

Proof. The proof follows from 3.7, where $W_n^r = \otimes_{\beta_{1,n-r=r}} T_*^{j_{n-r},j_{n-r-1},\dots,j_1}$

To complete this section, let us consider some summands W_n^r as examples and explain how Ψ^k acts on each of them.

Remark 3.9. We consider the special cases when $r = 0$ and $r = n - 1$ for W_n^r

- When $r = 0$, Ψ^k preserves the same action on W_n^r as on A_*^n, where the action of Ψ^k on A_*^n is described in 3.2.
- When $r = n - 1$, Ψ^k preserves the same action on W_n^r as on A_*, see 2.9(ii) and 3.6.

Example 3.10. For $n = 5$ we get $bu_{2^*}(P_5) = \otimes_{r=0}^{4} W_5^r$ is non-zero in degrees $s \geq 2\beta_{1,5-r} + 5 - r$ when s and $5 - r$ both are even or both are odd. Then, respectively, in degree $s = 2t + 1$ and $s = 2t$, Ψ^k acts on W_5^4 and on W_5^3 as multiplication by k^{t+1}, and on W_5^2 and W_5^1 as multiplication by k^{t+2}, whereas Ψ^k acts on W_5^0 in degree $s = 2t + 1$ as multiplication by k^{t+3}.

4. THE ADAMS OPERATION Ψ^k ON $bu_{p^*}(B\mathbb{Z}/p)^{\wedge n}$

The Atiyah-Hirzebruch spectral sequences for bu_{p^*} and KU_* of $\sum^\infty B\mathbb{Z}/p$ both collapse for dimensional reasons and the map between them is injective so that $bu_{p^*}(\sum^\infty B\mathbb{Z}/p)$ injects into $KU_*(\sum^\infty B\mathbb{Z}/p)$ which, by the universal coefficient theorem for KU-theory [14] and the calculations of [15], is given by $KU_{2j+1}(\sum^\infty B\mathbb{Z}/p) \cong \oplus_{j=1}^{p-1} \mathbb{Z}/p^\infty$ ([17] §2; see also [16] Chapter I, §2) and is zero in even dimensions.

When p is odd it will be convenient to replace $bu\mathbb{Z}_p$ by bu_p, connective unitary K-theory with p-adic integers coefficients and similarly for $KU\mathbb{Z}_p$. These p-adic spectra possess Adams decompositions [13] (see also [1] and [16]).

$$bu_p \simeq \vee_{i=1}^{p-1} \Sigma^{2i-2} lu \text{ and } KU_p \simeq \vee_{i=1}^{p-1} \Sigma^{2i-2} LU$$

where: $lu_*(\Sigma^\infty S^0) \cong \mathbb{Z}_p[\upsilon]$ such that $\deg(\upsilon) = 2p-2$ corresponds to u^{p-1} and multiplication by u translates the summand Σ^{2i-2} lu to Σ^{2i} lu for $0 \le i \le p-2$ and Σ^{2p-4}lu to lu. LU-theory is obtained from lu by localising to invert v.

The injection mentioned above maps $lu_{2i-1}(\Sigma^\infty B\mathbb{Z}/p)$ into $LU_{2i-1}(\Sigma^\infty B\mathbb{Z}/p) \cong \mathbb{Z}/p^\infty$. Therefore this group must be cyclic and an order-count in the collapsed Atiyah-Hirzebruch spectral sequence shows that the non-zero groups $lu_{2k(p-1)+2i-1}(\Sigma^\infty B\mathbb{Z}/p) \cong \mathbb{Z}/p^{k+1}$ for $i = 1, \dots, p-1$ generated by $v_{2k(p-1)+2i-1}$. In $KU_{2r+1}(\Sigma^\infty B\mathbb{Z}/p)$ the element $\upsilon v_{2k(p-1)+2i-1}$ has order p^{k+1}, by Bott periodicity, so we may choose $v_{2(k+1)(p-1)+2i-1}$ so that $\upsilon v_{2k(p-1)+2i-1} = p v_{2(k+1)(p-1)+2i-1}$. this leads to

$$lu_*(\Sigma^\infty B\mathbb{Z}/p) \cong \frac{\mathbb{Z}_p[\upsilon]\langle v_1, v_3, v_5 \dots \rangle}{(pv_1, pv_3, \dots pv_{2p-3}, \upsilon v_{2i-1} - pv_{2(p-1)+2i-1})}$$

where $\deg(v_{2i-1}) = 2i-1$

In 1972, Holzsager [4] split the space $\Sigma B\mathbb{Z}/p$ with p-adic conceits into the wedge of p – 1 spaces B_i, where B_i has homology only in dimensions $2k(p-1)+2i$, for all natural numbers k. So the spectrum $\Sigma^\infty B\mathbb{Z}/p$ splits as $\Sigma^\infty B\mathbb{Z}/p \simeq \vee_{i=1}^{p-1} \Sigma^\infty B_i$, see also [6]. Here the spectrum B_i has stable cells in dimension $2k(p-1)+2i-\epsilon$, for $\epsilon = 0,1$ such that $2k(p-1)+2i-\epsilon \ge 0$. The splitting of $B\mathbb{Z}/p$ as a spectrum is also written as $B\mathbb{Z}/p \simeq \vee_{i=1}^{p-1} B_i$.

By [5], for the case E = lu the Adams summand and $X = B\mathbb{Z}/p$, we have the Thom isomorphism $lu_{q+2}(T(\xi)) \cong lu_q(B\mathbb{Z}/p)$, that is, $lu_*(T(\xi)) \cong lu_*(\Sigma^2 B\mathbb{Z}/p)$. This isomorphism is induced by a homotopy equivalence $lu \wedge T(\xi) \simeq lu \wedge \Sigma^2 B\mathbb{Z}/p$. By applying the splitting of $B\mathbb{Z}/p$ and substituting $T(\xi) = \frac{B\mathbb{Z}/p}{B^1}$ in this homotopy equivalence we get

$$lu \wedge (B_1 \vee B_2 \vee \dots \vee B_{p-1})/(B^1) \simeq lu \vee \Sigma^2(B_1 \vee B_2 \vee \dots \vee B_{p-1})$$

Both sides of the last equivalence are wedges of $p-1$ pieces, and by comparing the dimensions of bottom cells we deduce the following homotopy equivalence $lu \wedge \Sigma^2 B_i \simeq lu \wedge B_{i+1}$ for $1 \le i < p-1$. Inductively on i, we get $lu \wedge \Sigma^{2(i-1)} B_1 \simeq lu \wedge B_i$.

In this section it would be more interesting if we use the splitting $bu_p \simeq \vee_{i=1}^{p-1} \Sigma^{2i-2}$lu, the Holzsager splitting $B\mathbb{Z}/p \simeq \vee_{i=1}^{p-1} B_i$, the homotopy equivalence $lu \wedge \Sigma^{2(i-1)} B_1 \simeq lu \wedge B_i$ and the injectivity of $lu_*(B\mathbb{Z}/p)$ into the periodic case $LU_*(B\mathbb{Z}/p)$ to consider the action of the Adams operation Ψ^k on $lu_*(B_1)$ where the analogous results for the periodic case also are given in [2]. After that we will extend this description to $bu_{p*}(P_n)$ for $n \in \mathbb{N}_0$ using the decomposition of $bu_{p*}(P_n)$ see 2.11

Notation 4.1. [6]

In this section we write T for the graded group $T(lu_*(B_1))$ which is calculated from the following free $\mathbb{Z}_p[\upsilon]$-resolution of $lu_{2(p-1)*+1}(B_1)$

$$0 \to \oplus_{j\geq 0}\mathbb{Z}_p[\upsilon]\langle a_{2j(p-1)+1}\rangle \xrightarrow{d} \oplus_{j\geq 0}\mathbb{Z}_p[\upsilon]\langle b_{2j(p-1)+1}\rangle \xrightarrow{\epsilon} lu_{2(p-1)*+1}(B_1) \to 0$$

where:

$\epsilon(b_{2j(p-1)+1}) = v_{2j(p-1)+1}$ and $d(a_{2j(p-1)+1}) = pb_{2j(p-1)+1} - \upsilon b_{2(j-1)(p-1)+1}$ for all $j \geq 0$.

Similar to the previous section, ϵ and d respect the action of Ψ^k, and therefore this resolution is compatible with the Ψ^k-actions.

By applying $T(lu_*(B_1))_* \otimes_{\mathbb{Z}_p[\upsilon]}$ - instead of $lu_*(B_1)\oplus_{\mathbb{Z}_p[\upsilon]}$- to the previous free resolution of $lu_*(B_1)$ with shifting by (-1) and by using induction on n, we can calculate the graded group T_*^n. This is non-zero just in degrees $2t(p-1)+2n+1$.

For a prim number p, $bu_{p*}(B\mathbb{Z}/p)^n$ is constructed from the summands W_n^r, for $0 \leq r \leq n-1$, that is,

$$bu_{p*}(B\mathbb{Z}/p)^{\wedge n} = \oplus_{i_1,i_2,\dots,i_{n+1}=0}^{p-2}\oplus_{r=0}^{n-1}W_n^r$$

where $W_n^r = \oplus_{\beta_{1,n-r}=r}T_{*-2\sum_{k=1}^{n+1}i_k}^{j_{n-r},j_{n-r-1},\dots,j_1}$ and T is referred to the graded group $T\big(lu_*(B_1)\big)_*$.

Of course W_n^r depends on $i_1, i_2, \dots, i_{n+1}$. Consequently to explain the action of Ψ^k on $bu_{p*}(B\mathbb{Z}/p)^{\wedge n}$ it is enough to explain how Ψ^k acts on $T_*^{j_n,j_{n-1},\dots,j_1}$.

Proposition 4.2. The Adams operation

$$\Psi^k: lu_*(B_j) \to lu_*(B_j),$$

satisfies $\Psi^k\left(v_{2i(p-1)+2_{j-1}}\right) = k^{i(p-1)+j}v_{2i(p-1)+2_{j-1}}$, for $j = 1, \dots, p-1$ and $i \geq 0$.

Proof. the proof is similar to 3.1 by using the injectivity of $lu_*(B\mathbb{Z}/p)$ into the periodic case $LU_*(B\mathbb{Z}/p)$.

Proposition 4.3. Let $0 < r_1 \leq t$, and consider

$$\Psi^k: lu_*(B_1)^r \to lu_*(B_1)^r.$$

Then, for $x \in lu_s(B_1)^r$,

$$\Psi^k(x) = \begin{cases} k^{t+r_1}x & \text{if } r = 2r_1,\ \ s = 2t, \text{and} \\ k^{t+r_1+1}x, & \text{if } r = 2r_1+1,\ \ s = 2t+1. \end{cases}$$

where: $t = \sum_{i=1}^{r} j_i(p-1) + r_1$ and $v_{2j_i(p-1)+1}$ is the generator of $lu_{2j_i(p-1)+1}(B_1)$ for $j_i \geq 0$.

Proof. The proof is similar to 3.2, where $lu_{2t}(B_1)^r$ is an $\mathbb{F}_p$-vector space with basis

$$\left\{v_{2j_1(p-1)+1}\otimes v_{2j_2(p-1)+1}\otimes \dots \otimes v_{2j_r(p-1)+1} : t = \sum_{i=1}^{r} j_i(p-1) + r_1\right\}$$

Similar to 3.7, we can deduce the following result where T here is referred to the graded group $T(lu_*(B_1))$.

Proposition 4.4. Consider

$$\Psi^k : T_*^{j_r,j_{r-1},\dots,j_1} \to T_*^{j_r,j_{r-1},\dots,j_1}$$

and $x_s \in T_s^{j_r,j_{r-1},\dots,j_1}$. Then

$$\Psi^k(x_s) = \begin{cases} k^{t+r_1}x_s, & \text{if } r = 2r_1,\ s = 2t, \text{and} \\ k^{t+r_1+1}x_s, & \text{if } r = 2r_1+1,\ s = 2t+1. \end{cases}$$

where $t = \sum_{i=1}^{r+\beta_{1,r}} L_i(p-1) + \beta_{1,r} + r_1, \beta_{i,n}$ as in 2.10 and $j_i, L_i \geq 0$.

Proof. The proof is by induction on r where $T_*^{j_r,j_{r-1},\dots,j_1}$ is a graded group, which is non-zero just in degrees $2\sum_{i=1}^{r+\beta_{1,r}} L_i(p-1) + 2\beta_{1,r} + r$ for $L_i \geq 0$, see [6] and [7].

we will conclude this paper by an example to explain the action of the Adams Operation Ψ^k on $bu_{p*}(B\mathbb{Z}/p)^{\wedge n}$ for some cases.

Example 4.5. For $n = p = 3$, $bu_9(B\mathbb{Z}/3)^{\wedge 3} = \oplus_{i_1,i_2,i_3,i_4=0}^{1} \oplus_{r=0}^{2} W_3^r$ where $W_3^r = \oplus_{\sum j_i = r} T_{9-2\sum_{k=1}^{4} i_k}^{j_{3-r},j_{2-1},\dots,j_1}$. Since $T_*^{j_n,j_{n-1},\dots,j_1}$ is non-zero just in degrees $2t(p-1) + 2\beta_{1,n} + n$ for $t \geq 0$. Therefore, we have $bu_9(B\mathbb{Z}/3)^{\wedge 3} = T_9^2 \oplus (T_7^{0,0,0}) \oplus (T_5^2)^6 \oplus (T_3^{0,0,0})^4$.

By 4.4, Ψ^k acts on T_9^2 and on $T_7^{0,0,0}$ as multiplication by k^5, whereas Ψ^k acts on T_5^2 and on $T_3^{0,0,0}$ as multiplication by k^3.

5. CONCLUSIONS

For a prim number p, $bu_{p*}(B\mathbb{Z}/p)^n$ is constructed from the summands W_n^r, for $0 \leq r \leq n-1$, that is, $bu_{p*}(B\mathbb{Z}/p)^{\wedge n} = \oplus_{i_1,i_2,\dots,i_{n+1}=0}^{p-2} \oplus_{r=0}^{n-1} W_n^r$, where each of these is constructed from the summands $T_*^{j_r,j_{r-1},\dots,j_1}$. That is, $W_n^r = \oplus_{\beta_{1,n-r}=r} T_{*-2\sum_{k=1}^{n+1} i_k}^{j_{n-r},j_{n-r-1},\dots,j_1}$, see 2.9 and 2.11.

Consequently, to explain the action of Ψ^k on $bu_{p*}(B\mathbb{Z}/p)^n$, when p=2, it is enough to explain how Ψ^k acts on its summands W_n^r, which is explained in Theorem 3.8. But in case of any prime p, of course W_n^r depends on $i_1, i_2, \dots, i_{n+1}$. Consequently to explain the action of Ψ^k on $bu_{p*}(B\mathbb{Z}/p)^{\wedge n}$ we were content with studding of the action of Ψ^k on $T_*^{j_n,j_{n-1},\dots,j_1}$.

Finally, we concluded this paper by giving an example to explain the action of the Adams operation Ψ^k on $bu_{p*}(B\mathbb{Z}/p)^{\wedge n}$ when $n = p = 3$.

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