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Some Special Results for Square Pyramidal Graceful Graphs

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ABSTRACT

Numbers of the form $\frac{n(n+1)(2n+1)}{6}$ for all n≥1 are called square pyramidal numbers. Let G be a graph with p vertices and q edges. Let τ : V(G) →{0, 1, 2… M_k} where M_k is the k^{th} square pyramidal number be an injective function. Define the function τ^*:E(G)→{1,5,14,.., M_k} such that $\tau^*(uv) = |\tau(u)-\tau(v)|$ for all edges uv∈E(G). If τ^*(E(G)) is a sequence of distinct consecutive square pyramidal numbers $\{M_1, M_2, \ldots, M_k\}$, then the function τ is said to be square pyramidal graceful labeling and the graph which admits such a labeling is called a square pyramidal graceful graph. In this paper, some special results for square pyramidal graceful graphs is studied.

Keywords: Square pyramidal graceful number, square pyramidal graceful labeling, square pyramidal graceful graphs

1. INTRODUCTION AND DEFINITIONS

Graphs considered in this paper are finite, undirected and (simple) without loops or multiple edges. Let G = (V, E) be a graph with p vertices and q edges. Graph labeling is one of the fascinating areas of graph theory with wide ranging applications. Graph labeling was first introduced in 1960's. A graph labeling is an assignment of integers to the vertices (edges / both)

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subject to certain conditions. If the domain of the mapping is the set of vertices (edges / both) then the labeling is called the vertex (edge / total) labeling. For number theoretic terminology, we refer to [1] and [2]. Terms not defined here are used in the sense of Parthasarathy [3] and Bondy and B. R. Murthy [4].

Most popular graph labeling trace their origin to one introduced by Rosa [5]. Rosa called a function (labeling) f a β-valuation of a graph G with q edges if f is an injection from the vertices of G to the set $\{0,1,2,\ldots,q\}$ such that each edge xy in G is assigned the label $|f(x)-f(y)|$, the resulting edge labels are distinct consecutive numbers and Golomb [6] called it as graceful labeling. Acharya [7] constructed certain infinite families of graceful graphs.

Labeled graphs are becoming an increasing useful family of mathematical models for a broad range of application like designing X-Ray crystallography, formulating a communication network addressing system, determining an optimal circuit layouts, problems in additive number theory etc. For more information related to graph labeling and its applications, see [9-45]. There are several types of graph labeling and a detailed survey is found in [8].

The following definitions are necessary for present study.

Definition 1.1: Let G be a (p, q) graph. A one to one function f: V(G) →{0,1,2,..,q} is called a graceful labeling of G if the induced edge labeling f' : E(G) →{1,2,…,q} defined by $f'(e) = |f(u)-f(v)|$ for each e = uv of G is also one to one. The graph G graceful labeling is called graceful graph.

Definition1.2: Bistar is the graph obtaining by joining the apex vertices of two copies of star $K_{1,n}$.

Definition 1.3: Let $v_1, v_2,\ldots, v_n$, be the n vertices of a path P_n. From each vertex v_i, i =1,2,…,n there are m_i , i = 1,2,…,n pendent vertices say $v_{i1}, v_{i2},\ldots , v_{im_i}$. The result graph is a caterpillar and is denoted by B $(m_1, m_2,\ldots, m_n)$.

Definition 1.4: A coconut tree CT(n, m) is the graph obtained from the path P_m by appending n new pendant edges at an end vertex of P_m.

Definition 1.5: A path P_n is obtained by joining u_i to the consecutive vertices u_{i+1} for $1 \leq i \leq n-1$.

Definition 1.6: Let G be a graph with p vertices and q edges. Let τ : V(G) →{0, 1, 2… M_k} where M_k is the k^{th} square pyramidal number be an injective function. Define the function τ^*: E(G)→{1,5,14,.., M_k} such that $\tau^*(uv) = |\tau(u) - \tau(v)|$ for all edges uv ∈ E(G). If τ^*(E (G)) is a sequence of distinct consecutive square pyramidal numbers $\{M_1, M_2, \ldots, M_k\}$, then the function τ is said to be square pyramidal graceful labeling and the graph which admits such a labeling is called a square pyramidal graceful graph.

Definition 1.7: A graph G is a finite non-empty set of objects called vertices together with a set of unordered pairs of distinct vertices of G called edges. The vertex set and the edge set of G are denoted by V(G) and E(G) repectively. The number of elements of V(G) = p is called the order of G and the number of elements of E(G) = q is called the size of G. A graph of order p and size q is called a (p,q) - graph. If e = *uv* is an edges of G, we say that *u* and *v* are adjacent and that *u* and *v* are incident with e.

Definition 1.8: The degree of a vertex *v* in a graph G is defined to be the number of edges incident on v and is denoted by deg(v). A graph is called r-regular if deg(*v*) = r for each *v*∊ V(G). The minimum of {deg *v* : *v*∊ V(G) } is denoted by δ and maximum of {deg *v* : *v* ∊ V(G)} is denoted by △. A vertex of degree 0 is called an isolated vertex, a vertex of degree is called a pendant vertex or an end vertex.

Definition 1.9: A connected acyclic graph is called a tree.

Definition 1.10: A graph in which any two distinct points are adjacent is called a complete graph. The complete graph with n points is denoted by K_n.

Definition 1.11: A graph *G* is said to be connected if for every pair *u*,*v* of vertices a *u*-*v* path. Otherwise *G* is disconnected.

Definition 1.12: A graph that has neither self-loop nor parallel edges is called a simple graph.

Definition 1.13: The complete bipartite graph $K_{1,n}$ is called a Star and it has *n* + 1 vertices and *n* edges.

2. RESULTS

Theorem 2.1: Let G be a path with *m* vertices. Then G is square pyramidal graceful for all *m* ≥ 3.

Proof: Let G be a path with m vertices.

Let V (G) = {v_i: 1≤ i ≤ m} be the vertex set of G and

E(G) = {$v_i v_{i+1}$:1 ≤ i ≤ m-1} be the edge set of G.

Hence G has m vertices and m-1 edges.

Let k = m-1.

Define a function τ: V(G)→{0,1,2,…,M_k} as follows

τ (v_1) = 0

τ (v_2) = M_k

τ (v_i) = τ (v_{i-1}) - $M_{k-(i-2)}$ if i is odd and 3≤ i ≤ m.

= τ (v_{i-1}) + $M_{k-(i-2)}$ if i is even and 3≤ i≤ m.

Let τ* be the induced edge labeling of τ.

Then τ*($v_1 v_2$) = M_k

$\tau^*(v_i v_{i+1}) = M_{k-(i-1)}$; 2≤ i≤ m-1.

The induced edge lables $M_1, M_2, \ldots, M_k$ are distinct and consecutive square pyramidal numbers.

Hence the graph G is a square pyramidal graceful.

Example 2.2: Square pyramidal graceful labeling of the path P_7 is shown in Fig. 1.

Fig. 1

Theorem 2.3: The bistar B(n_1, n_2) where n_1≥1 and n_2≥1 is square pyramidal graceful.

Proof: Let P_2 be a path on two vertices and let v_1 and v_2 be the vertices of P_2.

From v_1 there are n_1 pendent vertices say $v_{11}, v_{12}, \ldots, v_{1n}$ and from v_2, there are n_2 pendent vertices say $v_{21}, v_{22}, \ldots, v_{2n_2}$.

The resulting graph is a bistar B(n_1, n_2) .

Let G = (V,E) be the bistar B(n_1, n_2).

Let V(G) = $\{v_i$: i=1,2$\}$ ∪ $\{v_{1j}$: 1≤ j ≤ $n_1\}$ ∪ $\{v_{2j}$: 1≤ j≤ $n_2\}$ and

E(G) = $\{v_1 v_2\}$ ∪ $\{v_1 v_{1j}$: 1≤ j ≤ $n_1\}$ ∪$\{v_2 v_{2j}$: 1≤ j≤ $n_2\}$.

Then G has n_1+ n_2+2 vertices and n_1+ n_2+1 edges.

Let n_1+ n_2+1 = k (say)

Now label the vertices v_1, v_2 of P_2 as 0 and 1.

Then label the n, vertices adjacent to v_1 other than v_2 as $M_k, M_{k-1}, M_{k-2}, \ldots, M_{k-n_1+1}$.

Next label the n_2 vertices adjacent to v_2 other than v_1 as M_{k-n_1}+1,…, $M_{k-n_1-n_2+1}$+1.

We shall prove that G admits square pyramidal graceful labeling.

From the definition, it is clear that max τ (v)ϵ$\{0,1,2,\ldots, M_k\}$for all v ∈V(G)

Also from the definition, all the vertices of G have different labeling.

Hence τ is one to one.

It remains to show that the edges values are of the form $\{M_1, M_2, \ldots, M_k\}$.

The induced edges function τ*:E(G)→$\{1,5,\ldots, M_k\}$ is defined as follows

$\tau^*(v_i v_{ij}) = M_{k-(j-1)}$ if i = 1 and 1 ≤ j≤ n_1

$\tau^*(v_i v_{ij}) = M_{k-(n_1+j-1)}$ if i = 2 and 1≤ j ≤ n_2.

And $\tau^*(v_1 v_2) = M_1$.

Clearly τ^* is one to one and τ^*(E(G))={ $M_1, M_2, \ldots, M_k$}.

Therefore G is admits square pyramidal graceful labeling.

Hence the graph B(n_1, n_2) is square pyramidal graceful.

Example 2.4: The square pyramidal graceful labeling of the bistar graph B(5,2) is shown in Fig. 2.

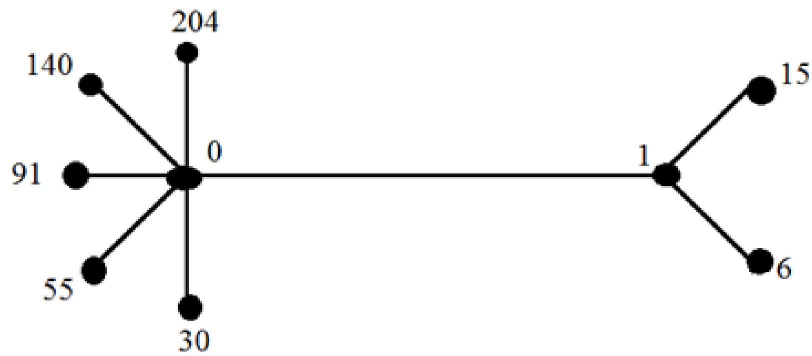

Fig. 2

Theorem 2.5: Caterpillars are square pyramidal graceful.

Proof: Let $v_1, v_2, \ldots, v_m$ be the m vertices of the path P_m.

From each vertex v_i, i=1,2,…,m, there are n_i, i=1,2,…,m, pendent vertices say $v_{i1}, v_{i2}, \ldots, v_{in_i}$.

The resultant graph is a caterpillar and is denoted as B($n_1, n_2, \ldots, n_m$).

Assume m ≥ 3.

Clearly B($n_1, n_2, \ldots, n_m$) has $n_1+n_2+\ldots+n_m$+(m-1)edges

Let k = $n_1+n_2+\ldots+n_m$+(m-1)

Defined τ: V(B($n_1, n_2, \ldots, n_m$)) →{0,1,2,…, M_k} as follows.

$\tau(v_1) = 0$

$\tau(v_{1i}) = M_{k-(i-1)}$, where i=1,2,…,n

$\tau(v_2) = \tau(v_1) + M_{k-n_1}$,

$\tau(v_{2i}) = \tau(v_2) - M_{k-n_1-i}$, where i=1,2,…, n_2.

$\tau(v_3) = \tau(v_2) - M_{k-n_1-n_2-1}$.

$\tau(v_{3i}) = \tau(v_3) + M_{k-n_1-n_2-1-i}$, where i=1,2,…, n_3.

$\tau(v_4) = \tau(v_3) + M_{k-n_1-n_2-n_3-2}$.

$\tau(v_{4i}) = \tau(v_4) - M_{k-n_1-n_2-n_3-2-i}$ where i = 1,2,…, n_4 and so on.

$\tau(v_m) = \tau(v_{m-1}) - M_{k-n_1-n_2-L-n_{m-1}-(m-2)}$ if m is odd

$\tau(v_m) = \tau(v_{m-1}) + M_{k-n_1-n_2-L-n_{m-1}-(m-2)}$ if m is even

$\tau(v_{mi}) = \tau(v_m) - M_{k-n_1-n_2-L-n_{m-1}-(m-2)-i}$ if m is even and $1 \leq i \leq n_m$

$\tau(v_{mi}) = \tau(v_m) + M_{k-n_1-n_2-L-n_{m-1}-(m-2)-i}$ if m is odd and $1 \leq i \leq n_m$

For i = n_m, $\tau(v_{mn_m}) = \tau(v_m) \pm M_{k-n_1-n_2-L-n_{m-1}-(m-2)-n_m}$

$$= \tau(v_m) \pm M_{k-n_1-n_2-L-n_{m-1}-(m-2)} = \tau(v_m) \pm M_1$$

Clearly the vertex labels are distinct and the resulting edge labels are of the form $\{M_1, M_2, \ldots, M_k\}$.

Thus caterpillar are square pyramidal graceful.

Example 2.6: The square pyramidal graceful labeling of a caterpillar graph B(1,2,1,2) is shown in Fig. 3.

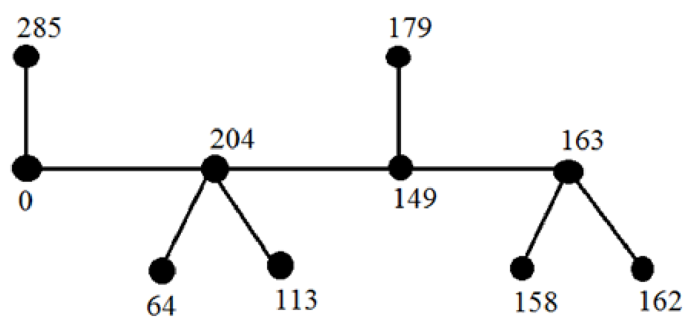

Fig. 3

Theorem 2.7: The caterpillar B $(n_1, 0, n_2)$ is square pyramidal graceful for all $n_1, n_2 \geq 1$.

Proof: Let v_1, v_2, v_3 be the three vertices of P_3.

From v_1 there are n_1 pendent vertices say $u_1, u_2, \ldots, u_{n_1}$ and from v_3, there are n_2 pendent vertices say $w_1, w_2, \ldots, w_{n_2}$.

The resulting graph is denoted as B $(n_1, 0, n_2)$.

Let it be G = (V,E).

Then G has $n_1 + n_2 + 3$ vertices and $n_1 + n_2 + 2$ edges.

Let k = $n_1 + n_2 + 2$.

Define τ : V(G)→$\{0,1,2,\ldots,M_k\}$ as follows.

τ (v_1)= M_k

τ (v_2) = 0

τ (v_3) = M_{k-n_1-1}

τ (u_i) = M_k − M_{k-i} where $1 \leq i \leq n_1$,

τ (w_j) = M_{k-n_1-1} +M_j , where $1 \leq j \leq n_2$.

We shall prove that G admits square pyramidal graceful labeling.

From the definition, it is clear that max τ (v) is M_k for all v∈V(G) and τ (v)∈{0,1,2,…, M_k}.

Also from the definition, all the vertices of G have different labeling.

Hence τ is one to one.

It remains to show that the edge values are of the form {M_1, M_2,…, M_k}.

The induced edge function τ *: E (G) → {1,5,…, M_k} is defined as follows τ*(v_1v_2) = M_k.

τ *(v_2v_3) = M_{k-n_1-1}

τ * (v_1u_i) = M_{k-i} where $1 \leq i \leq n_1$.

τ *(v_3w_j) = M_j , where $1 \leq j \leq n_2$.

Clearly τ * is one to one and τ *(E(G))={ M_1, M_2,…, M_k}.

Therefore G admits square pyramidal graceful labeling.

Hence the graph B $(n_1, 0, n_2)$ is square pyramidal graceful for all $n_1, n_2 \geq 1$.

Example 2.8: The square pyramidal graceful labeling of a caterpillar graph B(4,0,3) is shown in Fig. 4.

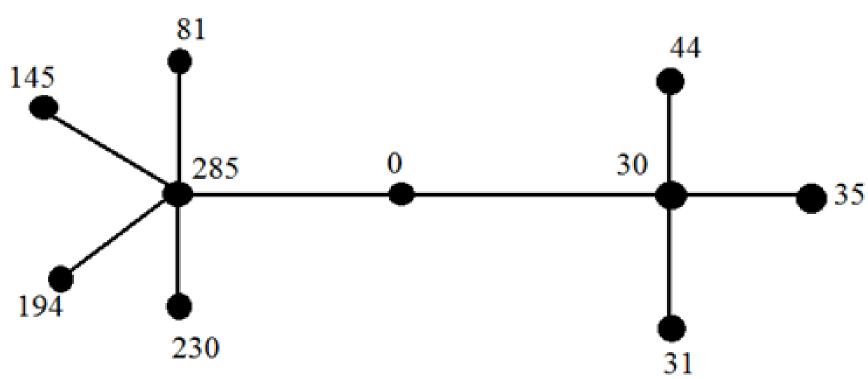

Fig. 4

Corollary 2.9: The caterpillar B $(n_1, 1, n_2)$ is square pyramidal graceful for all $n_1, n_2 \geq 1$.

Example 2.10: The square pyramidal graceful labeling of a caterpillar graph B(4,1,3) is shown in Fig. 5.

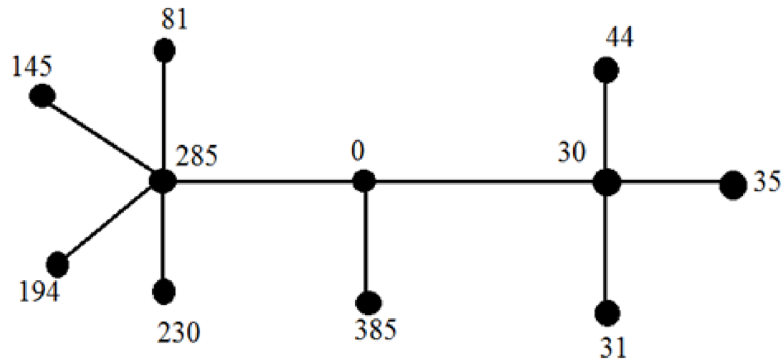

Fig. 5

Theorem 2.11: Coconut tree CT(n,m) is square pyramidal graceful for all $n \geq 1$, $m \geq 2$.

Proof: Let G be the graph CT(n,m).

Let V(G) = $\{v, v_i, u_j : 1 \leq i \leq n, 1 \leq j \leq m\text{-}1\}$ and E(G) = $\{vv_i, vu_1, u_ju_{j+1} : 1 \leq i \leq n, 1 \leq j \leq m\text{-}1\}$.

G has $n + m$ vertices and $n + m - 1$ edges.

Let $k = n + m - 1$.

Let $\tau : V(G) \rightarrow \{0,1,2,\ldots,M_k\}$ be defined as follows

$\tau(v) = 0$

$\tau(v_i) = M_{k-i+1}$; $1 \leq i \leq n$

$\tau(u_1) = M_{k-n}$

$\tau(u_j) = \tau(u_{j-1}) + M_{k-n-(j-1)}$ if j is odd and $2 \leq j \leq m\text{-}1$.

$= \tau(u_{j-1}) - M_{k-n-(j-1)}$ if j is even and $2 \leq j \leq m\text{-}1$

Let τ^* be the induced edge labeling of τ.

Then $\tau^*(vv_i) = M_{k-i+1}$; $1 \leq i \leq n$.

$\tau^*(vu_1) = M_{k-n}$.

$\tau^*(u_ju_{j+1}) = M_{k-n-j}$; $1 \leq j \leq m\text{-}2$.

The induced edge labels $M_1, M_2, \ldots, M_k$ are distinct and consecutive square pyramidal numbers.

Hence Coconut tree is square pyramidal graceful.

Example 2.12: The square pyramidal graceful labeling of a graph coconut tree CT(7,4) is shown in Fig. 6.

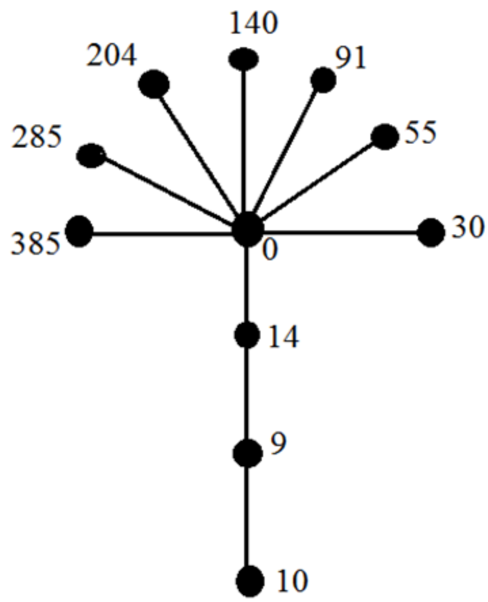

Fig. 6

Theorem 2.13: Olive trees are square pyramidal graceful.

Proof: Let u_0 be the root of the Olive tree$O(T_n)$.

Let $u_{11}, u_{12}, \dots, u_{1n}$ be the vertices in the first level.

Then there are n edges in the first level.

Also, let $u_{22}, u_{23}, \dots, u_{2n}$ be the vertices in the second level.

Hence $u_{1i}u_{2i}, i = 2,3,4, \dots, n$ be the $n-1$ edges in the second level.

Let $u_{33}, u_{34}, \dots, u_{3n}$be the vertices in the third level.

Thus $u_{2j}u_{3j}, j = 3,4, \dots, n$ be the $n-2$ edges in the third level.

Proceeding like this, u_{nn}be the unique vertex in the n^{th} level and the corresponding edge will be $u_{n-1\,n}u_{nn}$.

Now the total number of edges in $O(T_n)$ is k(say).

Consider the vertex function $\tau : V(O(T_n)) \rightarrow \{0,1,2, \dots, M_k\}$. Label the vertices as $\tau\,(u_0) = 0$ and $\tau\,(u_{1i}) = M_{k-(i-1)}$, $1 \le i \le n$ so as the edge values are $M_k, M_{k-1}, \dots, M_{k-(n-1)}$.

$\tau\,(u_{2i})$, $i = 2,3, \dots, n$ are obtained by $\tau\,(u_{1i}) - x$ (x must be distinct and suitably chosen for each i) so as the edge values are $M_{k-n}, M_{k-(n+1)}, \dots, M_{k-(2n-2)}$.

$\tau\,(u_{3i})$, $i = 3,4, \dots, n$ are obtained by $\tau\,(u_{2i}) + y, i = 3,4, \dots, n$ (y's are distinct and suitably chosen for each i) so as the edge values are $M_{k-(2n-1)}, M_{k-(2n)}, \dots, M_{k-(3n-4)}$.

Proceeding like this (that is, alternatively subtracting and adding suitable, distinct positive quantities with the τ values of the previous level), in the last level τ (u_{nn}) will obtain the value from τ $(u_{n-1\,n})$ in such a way that the values of the edge must be $1 = M_1$. Then the edge values of the olive tree are $\{M_1, M_2, \dots, M_k\}$.

For the above process, we shall give the following algorithm also.

Define τ $: V(T) \to \{0,1,2, \dots, M_k\}$ as follows.

$\tau\,(u_0) = 0$

$\tau\,(u_{1i}) = M_{k-(i-1)}, \;\; 1 \le i \le n$

For $2 \le t \le n$ *and* $t \le i \le n$,

$$\tau\,(u_{ti}) = \begin{cases} \tau\,(u_{t-1\,i}) - M_{(K_{t-1}-(i-t+1))} & \textit{if } t \textit{ is even} \\ \tau\,(u_{t-1\,i}) + M_{(K_{t-1}-(i-t+1))} & \textit{if } t \textit{ is odd} \end{cases}$$

where $K_i = m - 2(n-1) - \sum_{j=1}^{i-2}\big(n - (j+1)\big),\; i \ge 3$ and

$K_i = m - i(n-1),\;\; i = 1,2.$

Clearly τ is injective and the set of edge labels which are absolute differences of the labels of the adjacent vertices are $M_1, M_2, \dots, M_k$.

Hence olive trees are square pyramidal graceful labeling.

Example 2.14: The square pyramidal graceful labeling of the graph olive tree $O(T_4)$ is shown in Fig. 7. Here $k = 10$ *and* $n = 4$.

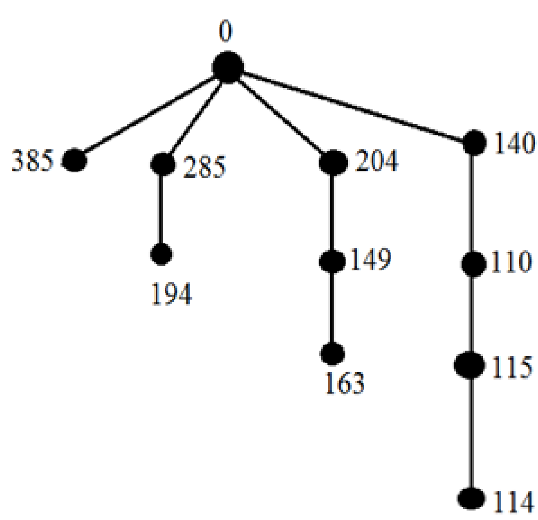

Fig. 7

Theorem 2.15: The star $K_{1,n}$ is square pyramidal graceful for all n.

Proof: Let V($K_{1,n}$) = {u_i : 1≤ i≤ n+1}.

Let E($K_{1,n}$) = { u_{n+1} u_i : 1≤ i≤ n}.

Define an injection τ : V($K_{1,n}$)→{0,1,2,3…, M_k } by τ (u_i) = M_i if 1≤ i≤ n and τ (u_{n+1}) = 0.

Then τ induces a bijection $τ_p$: E($K_{1,,n}$) →{1,5,14,…, M_k }.

Hence the star $K_{1,n}$ is square pyramidal graceful for all n.

Example 2.16: A square pyramidal graceful labeling of star graph $K_{1,6}$ is shown in Fig. 8.

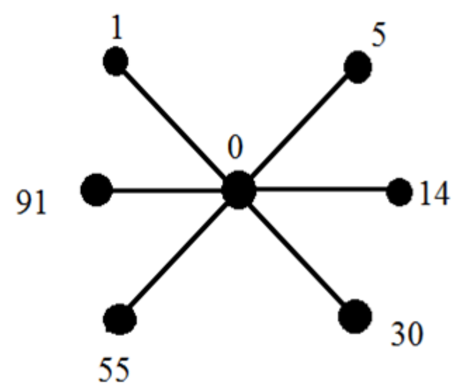

Fig. 8

3. CONCLUSIONS

In this paper, we have introduced some special results for square pyramidal graceful graphs and studied graceful labeling of some graphs. This work contributes several new results to the theory of graph labeling. The square pyramidal graceful can be verified for many other graphs.

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