Available online at www.worldscientificnews.com

World Scientific News

An International Scientific Journal

WSN 155 (2021) 98-112 EISSN 2392-2192

Pentagonal Graceful Labeling of Some Graphs

S. Mahendran[1] & **K. Murugan**[2]

P.G. & Research Department of Mathematics,
The Madurai Diraviyam Thayumanavar Hindu College, Tirunelveli, India

[1,2]E-mail address: mahe1999bsc@gmail.com , murugan@mdthinducollege.org

ABSTRACT

Numbers of the form $\frac{n(3n-1)}{2}$ for all $n \geq 1$ are called pentagonal numbers. Let G be a graph with p vertices and q edges. Let $f: V(G)\rightarrow\{0,1,2,\ldots,P_q\}$ where P_q is the q^{th} pentagonal number be an injective function. Define the function f^*: E(G) $\rightarrow$ $\{1,5,\ldots,P_q\}$ such that $f^*(uv)=|f(u)-f(v)|$ for all edges $uv\in E(G)$. If $f^*(E(G))$ is a sequence of distinct consecutive pentagonal numbers $\{P_1,P_2,\ldots,P_q\}$, then the function f is said to be pentagonal graceful labeling and the graph which admits such a labeling is called a pentagonal graceful graph. In this paper, pentagonal graceful labeling of some graphs is studied.

***Keywords*:** Pentagonal graceful number, pentagonal graceful labeling, pentagonal graceful graphs

1. INTRODUCTION

Graphs considered in this paper are finite, undirected and simple. Let $G = (V, E)$ be a graph with p vertices and q edges. A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. If the domain of the mapping is the set of vertex (edge/both) then the labeling is called a vertex (edge/total) labeling.

Rosa [17] introduced β-valuation of a graph. Golomb [6] called it as graceful labeling. Let G be a (p,q) graph. A one to one function f: $V(G)\rightarrow\{0,1,2,\ldots,q\}$ is called a graceful labeling

(Received 21 February 2021; Accepted 10 March 2021; Date of Publication 11 March 2021)

of G if the induced edge labeling f': E(G)→{1,2,…,q} defined by f'(e) = $|f(u)-f(v)|$ for each e = uv of G is also one to one. The graph G possessing graceful labeling is called graceful graph.

In [1], certain families of graceful graphs were constructed.

There are several types of graceful labeling and a detailed survey is found in [7]. The concept of pentagonal graceful labeling was introduced by D.S.T. Ramesh and M.P. Syed Ali Nisaya in [16] and further studied in [19]. In this paper, pentagonal graceful labeling of some other graphs is studied.

Labeled graphs are becoming an increasing useful family of mathematical models for a broad range of application like designing X-Ray crystallography, formulating a communication network addressing system, determining an optimal circuit layouts, problems in additive number theory etc. A systematic presentation of diverse applications of graph labeling is given in [2-5, 8, 11, 18, 20-27].

Following definitions are necessary for the present study.

Definition 1.1. [10]: Shrub St(n_1,n_2,…,n_m) is a graph obtained by connecting a vertex $v_0$to the central vertex of each of m numbers of stars.

Definition 1.2. [10]: Banana tree denoted by Bt(n_1,n_2,…,n_m) (m times n) is a graph obtained by connecting a vertex v_0 to one leaf of each of m number of stars.

Definition 1.3. [10]: Coconut tree graph CT(n,m) is obtained by identifying the central vertex of $K1,n$ with a pendant vertex of the path P_m.

Definition 1.4. [12]:F-tree on n+2 vertices, denoted by FP_n, is obtained from a path P_n by attaching exactly two pendant vertices to the vertices n-1 and n of P_n.

Definition 1.5. [12]: Y-tree on n+1 vertices, denoted by Y_n, is obtained from a path P_n by attaching exactly a pendant vertex to the (n-1)th vertex of P_n.

Definition 1.6. [15]: Let $X_i \in N$. Then the caterpillar $S(X_1, X_2, \dots, X_n)$ is obtained from the path $P_n$by joining $X_i$vertices to each of the i^{th} vertex of P_n (1≤ i≤ n).

Definition 1.7. [15]: P_{n-1} (1,2,…,n) is a graph obtained from a path of vertices v_1,v_2,…,v_n having the path length n-1 by joining i pendant vertices at each of its i^{th} vertex.

Definition 1.8. [14]: Twig graph G is obtained from the path P_n by attaching exactly two pendant edges to each internal vertex of the path.

Definition 1.9. [9]: The corona $G_1 \odot G_2$ of two graphs G_1andG_2 where G_1 has m vertices and n edges is defined as the graph G_1obtained by taking one copy of G_1 and m copies of G_2, and the joining by an edge the i^{th}vertex of G_1 to every vertex in the i^{th}copy of G_2.

Definition 1.10. [13]: A subdivision of a graph G is a graph that can be obtained from G by a sequence of edge subdivision.

Definition 1.11. Numbers of the form $\frac{n(3n-1)}{2}$ for all $n \geq 1$ are called pentagonal numbers. The first few pentagonal numbers are 1, 5, 12, 22, 35, 51, 70,…

Definition 1.12. [16]: Let G be a graph with p vertices and q edges. Let $f : V(G) \rightarrow \{0,1,2,\ldots,P_q\}$ where P_q is the q^{th} pentagonal number be an injective function. Define the function f^*: E(G) $\rightarrow \{1,5,\ldots,P_q\}$ such that $f^*(uv) = |f(u)-f(v)|$ for all edges $uv \in E(G)$. If $f^*(E(G))$ is a sequence of distinct consecutive pentagonal numbers $\{P_1,P_2,\ldots,P_q\}$ then the function f is said to be pentagonal graceful labeling and the graph which admits such a labeling is called a pentagonal graceful graph.

2. RESULTS

Previous result 2.1. [16]:

(i) caterpillar $S(X_1, X_2, \ldots, X_n)$ is pentagonal graceful.

Corollary 2.2. When $X_i = m$, $1 \leq i \leq n$, the graph $P_n \odot \overline{K_n}$ is pentagonal graceful for all $n \geq 2$ and $m \geq 1$.

Example 2.3. Pentagonal graceful labeling of $P_3 \odot \overline{K_3}$ is shown in Fig. 1.

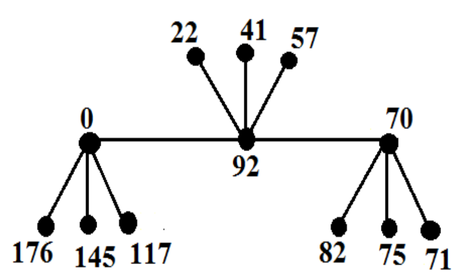

Fig. 1

Corollary 2.4. When m = 1, the graph $P_n \odot K_1$ is called a comb. Comb is pentagonal graceful.

Example 2.5. Pentagonal graceful labeling of $P_4 \odot K_1$ is shown Fig. 2.

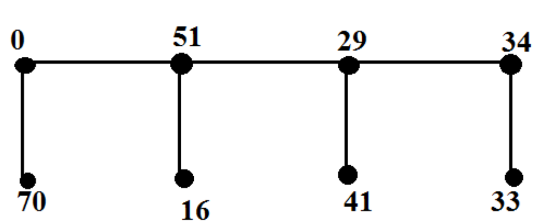

Fig. 2

Corollary 2.6. P_{n-1} $(1,2,\ldots,n)$ is pentagonal graceful.

Example 2.7. Pentagonal graceful labeling of P_4 (1,2,3,4) is shown Fig. 3.

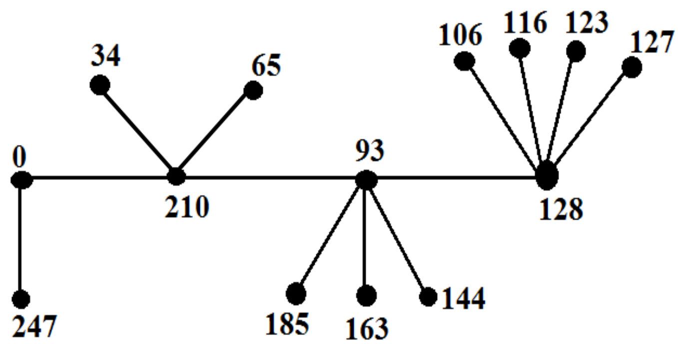

Fig. 3

Corollary 2.8. Twig graph is pentagonal graceful.

Example 2.9. Pentagonal graceful labeling of twig graph is obtained from the path P_5 is shown in Fig. 4.

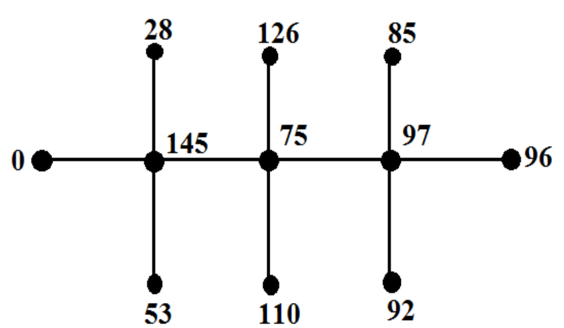

Fig. 4

Theorem 2.10. Shrub St(n_1,n_2,…,n_m) is pentagonal graceful.

Proof: Let G be the graph St(n_1,n_2,…,n_m).

Let V(G) = { v,v_i,v_{ij} : $1\leq i\leq m$, $1\leq j\leq n_i$ } and E(G) = {vv_i ,v_iv_{ij} : $1\leq i\leq m$, $1\leq j\leq n_i$ }.

G has $m + n_1+n_2+ \ldots + n_m+1$ vertices and $m + n_1+n_2+ \ldots + n_m$ edges.

Let t = $m + n_1 + n_2 + \ldots + n_m$.

Let f : V(G)→{0,1,2,…,A_t} be defined as follows.

$f(v) = 0$

$f(v_i) = A_{t-[n_1+n_2+\ldots+n_{i-1}+i-1]}$; $1 \le i \le m$.

$f(v_{ij}) = A_{t-[n_1+n_2+\ldots+n_{i-1}+i-1]} - A_{t-[n_1+n_2+\ldots+n_{i-1}+(i-1)+(j+i-1)]}$; $1 \le i \le m$, $1 \le j \le n_i$.

Let f^* be the induced edge labeling of f.

Then $f^*(vv_i) = A_{t-[n_1+n_2+\ldots+n_{i-1}+i-1]}$; $1 \le i \le m$.

$f^*(v_i v_{ij}) = A_{t-[n_1+n_2+\ldots+n_{i-1}+(i-1)+(j+i-1)]}$; $1 \le i \le m$, $1 \le j \le n_i$.

The induced edge labels $A_1, A_2, \ldots, A_t$ are distinct and consecutive pentagonal numbers.

Hence the Shrub is pentagonal graceful.

Example 2.11. Pentagonl graceful labeling of St(2,3,4,5) is given in Fig. 5.

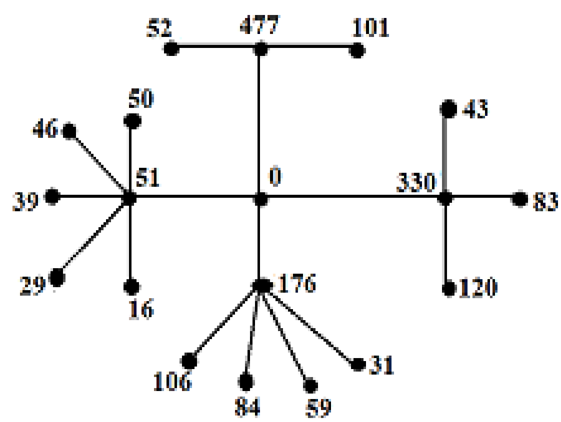

Fig. 5

Theorem 2.12. Banana tree $Bt(n_1, n_2, \ldots, n_m)$ is pentagonal graceful.

Proof: Let G be the graph $Bt(n_1, n_2, \ldots, n_m)$.

Let $V(G) = \{v, v_i, w_i, w_{ij} : 1 \le i \le m, 1 \le j \le n_i - 1\}$ and

$E(G) = \{ vv_i, v_i w_i, w_i w_{ij} : 1 \le i \le m, 1 \le j \le n_i - 1\}$.

G has $m + n_1 + n_2 + \ldots + n_m + 1$ vertices and $m + n_1 + n_2 + \ldots + n_m$ edges.

Let $t = m + n_1 + n_2 + \ldots + n_m$.

Let $f : V(G) \to \{0, 1, 2, \ldots, A_t\}$ be defined as follows

$f(v) = 0$

$f(v_i) = A_{t-i+1}$; $1 \le i \le m$.

$f(w_i) = f(v_i) - A_{t-m-[n_1+n_2+\ldots+n_{i-1}]}$; $1 \le i \le m$.

$f(w_{ij}) = f(w_i) + \mathrm{A}_{t-\mathrm{m}-[n_1+n_2+\ldots+n_{i-1}]-\mathrm{j}}$; $1 \le i \le m$, $1 \le j \le n_i - 1$.

Let f^* be the induced edge labeling of f.

Then $f^*(vv_i) = \mathrm{A}_{t-i-1}$; $1 \le i \le m$.

$f^*(v_i w_i) = \mathrm{A}_{t-\mathrm{m}-[n_1+n_2+\ldots+n_{i-1}]}$; $1 \le i \le m$.

$f^*(w_i w_{ij}) = \mathrm{A}_{t-\mathrm{m}-[n_1+n_2+\ldots+n_{i-1}]-\mathrm{j}}$; $1 \le i \le m$, $1 \le j \le n_i - 1$.

The induced edge labels $\mathrm{A}_1, \mathrm{A}_2, \ldots, \mathrm{A}_t$ are distinct and consecutive pentagonal numbers.

Hence the Banana tree is pentagonal graceful.

Example 2.13. Pentagonal graceful labling of Bt(4,4,4,4,4,4,4,4) is given in Fig. 6.

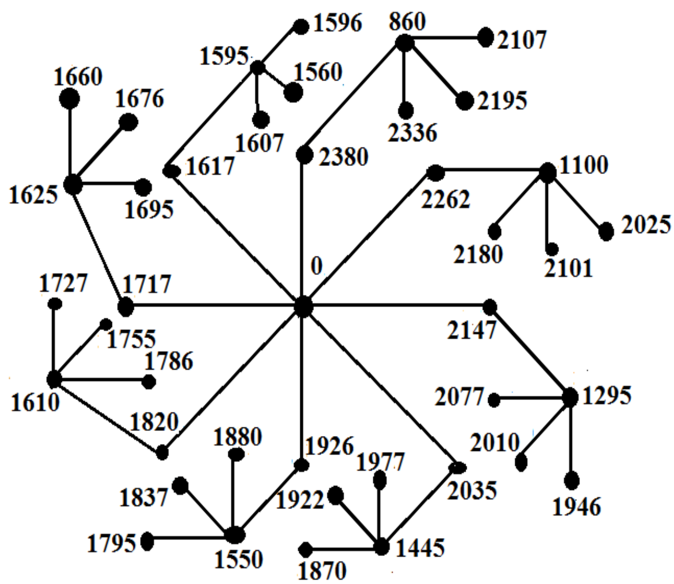

Fig. 6

Theorem 2.14. Coconut tree CT(n,m) is pentagonal graceful for all $n \ge 1$, $m \ge 2$.

Proof: Let G be the graph CT(n,m).

Let V(G) = { v, v_i,u_j : $1 \le i \le n$, $1 \le j \le m$-1} and E(G) = { vv_i,vu_1 ,$u_j u_{j+1}$: $1 \le i \le n$, $1 \le j \le$ m-1}.

G has $n + m$ vertices and $n + m - 1$ edges.

Let $\mathrm{t} = n + m - 1$.

Let f : V(G)$\rightarrow$$\{0,1,2,\ldots,\mathrm{A}_t\}$ be defined as follows

$f(v) = 0$

$f(v_i) = \mathrm{A}_{t-i+1}$; $1 \le i \le n$

$f(u_1) = A_{t-n}$

$f(u_j) = f(u_{j-1}) + A_{t-n-(j-1)}$ if j is odd and $2 \le j \le m-1$.

$\quad = f(u_{j-1}) - A_{t-n-(j-1)}$ if j is even and $2 \le j \le m-1$

Let f^* be the induced edge labeling of f.

Then $f^*(vv_i) = A_{t-i+1}$; $1 \le i \le n$.

$f^*(vu_1) = A_{t-n}$.

$f^*(u_j u_{j+1}) = A_{t-n-j}$; $1 \le j \le m-2$.

The induced edge labels $A_1, A_2, \ldots, A_t$ are distinct and consecutive pentagonal numbers..

Hence Coconut tree is pentagonal graceful.

Example 2.15. Pentagonal graceful labeling of CT(4,5) is given in Fig. 7.

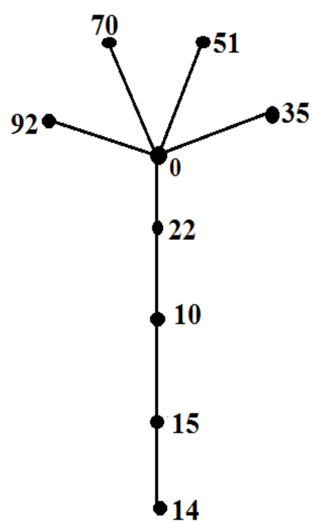

Fig. 7

Theorem 2.16. $K_{1,n} \odot K_1$ is pentagonal graceful.

Proof: Let G be the graph $K_{1,n} \odot K_1$.

Let $V(G) = \{ v, v_i, u_i, w : 1 \le i \le n\}$ and $E(G) = \{ vv_i, v_i u_i, vw : 1 \le i \le n \}$.

G has $2n + 2$ vertices and $2n + 1$ edges.

Let $t = 2n + 1$.

Let $f : V(G) \to \{0,1,2,\ldots,A_t\}$ be defined as follows

$f(v) = 0$

$f(v_i) = A_{t-(i-1)}$; $1 \le i \le n$

$f(w) = A_{t-n}$

$f(u_i) = f(v_i) - A_{n-(i-1)}$; $1 \le i \le n$.

Let f^* be the induced edge labeling of f.

Then $f^*(vv_i) = A_{t-(i-1)}$; $1 \le i \le n$.

$f^*(vw) = A_{t-n}$.

$f^*(v_iu_i) = A_{n-(i-1)}$; $1 \le i \le n$.

The induced edge labels $A_1, A_2, \ldots, A_t$ are distinct and consecutive pentagonal numbers.

Hence $K_{1,n} \odot K_1$ is pentagonal graceful.

Example 2.17. Pentagonal graceful labeling of $K_{1,5} \odot K_1$ is given in Fig. 8.

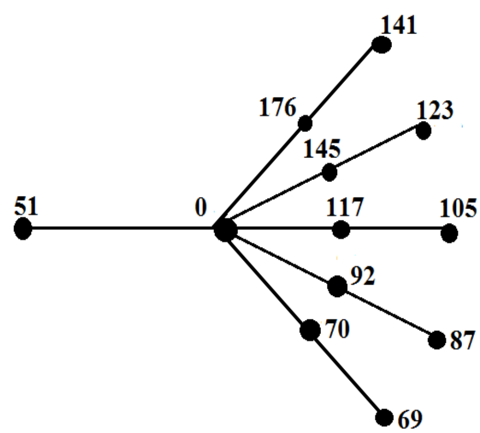

Fig. 8

Theorem 2.18. Let G be the graph obtained by identifying the leaves of $K_{1,n}$ with the central vertex of $K_{1,2}$. Then G is pentagonal graceful for all $n \ge 1$.

Proof: Let G be the graph obtained by identifying the leaves of $K_{1,n}$ with the central vertex of $K_{1,2}$.

Let $V(G) = \{ v, v_i, v_{ij} : 1 \le i \le n, 1 \le j \le 2 \}$ and $E(G) = \{ vv_i, v_iv_{ij} : 1 \le i \le n, 1 \le j \le 2 \}$.

G has $3n + 1$ vertices and $3n$ edges.

Let $t = 3n$.

Let $f : V(G) \rightarrow \{0, 1, 2, \ldots, A_t\}$ be defined as follows.

$f(v) = 0$

$f(v_i) = A_{3(n-(i-1))}$; $1 \le i \le n$.

$f(v_{ij}) = f(v_i) - A_{t-(i-1)n-j}$; $1 \le i \le n$, $1 \le j \le 2$.

Let f^* be the induced edge labeling of f.

Then $f^*(vv_i) = A_{3(n-(i-1))}$; $1 \le i \le n$.

$f^*(v_i v_{ij}) = A_{t-(i-1)n-j}$; $1 \le i \le n$, $1 \le j \le 2$.

The induced edge labels $A_1, A_2, \dots, A_t$ are distinct and consecutive pentagonal numbers.

Hence G is pentagonal graceful for all $n \ge 1$.

Example 2.19. Pentagonal graceful labeling of $K_{1,3} \odot K_{1,2}$ is given in Fig. 9.

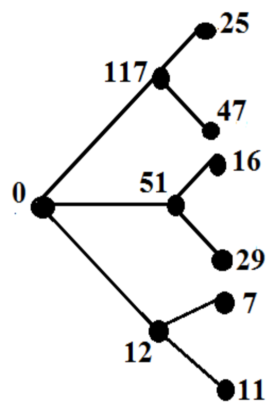

Fig. 9

Theorem 2.20. F-tree FP_n , $n \ge 3$ is pentagonal graceful.

Proof: Let G be FP_n , $n \ge 3$.

Let $V(G) = \{ u, v, v_i : 1 \le i \le n\}$ and $E(G) = \{v_i v_{i+1} : 1 \le i \le n\text{-}1 \} \cup \{ uv_{n-1}, vv_n\}$.

G has $n+2$ vertices and $n+1$ edges.

Let $t = n + 1$.

Let $f : V(G) \to \{0, 1, 2, \dots, A_t\}$ be defined as follows

$f(v_1) = 0$

$f(v_i) = f(v_{i-1}) - A_{t-i+2}$ if i is odd and $2 \le i \le n$.

$\quad = f(v_{i-1}) + A_{t-i+2}$ if i is even and $2 \le i \le n$.

$f(v) = f(v_n) - 1$

$f(u) = f(v_{n-1}) - 5$

Let f^* be the induced edge labeling of f.

Then $f^*(v_i v_{i+1}) = A_{t-i+1}$; $1 \le i \le n\text{-}1$.

$f^*(uv_{n-1}) = A_2$

$f^*(vv_n) = A_1$

The induced edge labels $A_1, A_2, \ldots, A_t$ are distinct and consecutive pentagonal numbers.

Hence F-tree FP_n , $n \geq 3$ is pentagonal graceful.

Example 2.21. Pentagonal graceful labeling of FP_6 is given in Fig. 10.

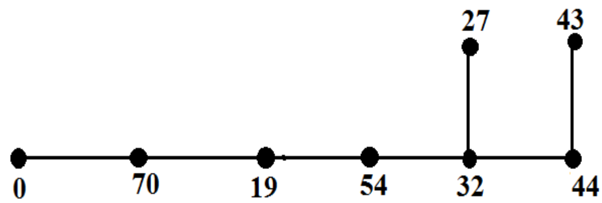

Fig. 10

Theorem 2.22. A Y-tree is pentagonal graceful.

Proof: Let G be the Y-tree.

Let V(G) = $\{v, v_i : 1 \leq i \leq n\}$ and E(G) = $\{v_i v_{i+1}, vv_{n-1} : 1 \leq i \leq n\text{-}1\}$.

G has n +1 vertices and n edges.

Let t = n.

Let f : V(G)→{0,1,2,…,A_t} be defined as follows

$f(v_1) = 0$

$f(v_i) = f(v_{i-1}) - A_{t-i+2}$ if i is odd and $2 \leq i \leq n$.

$= f(v_{i-1}) + A_{t-i+2}$ if i is even and $2 \leq i \leq n$.

$f(v) = f(v_{n-1}) - 1$

Let f^* be the induced edge labeling of f.

Then $f^*(v_i v_{i+1}) = A_{t-i+1}$; $1 \leq i \leq n$-1.

$f^*(vv_{n-1}) = A_1$

The induced edge labels $A_1, A_2, \ldots, A_t$ are distinct and consecutive pentagonal numbers..

Hence the Y-tree is pentagonal graceful.

Example 2.23. Pentagonal graceful labeling of Y_7 is given in Fig. 11.

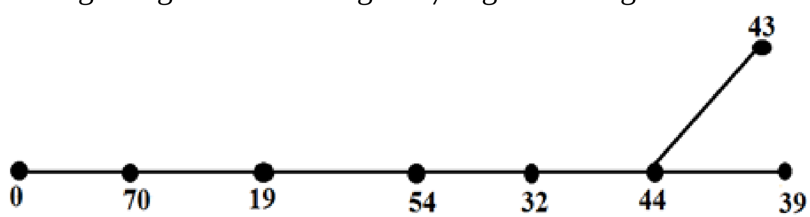

Fig. 11

Theorem 2.24. Let G be the graph obtained by identifying a pendant vertex of P_m with a leaf of $K_{1,n}$. Then G is pentagonal graceful for all m ≥ 2and n ≥ 1.

Proof: Let G be the graph obtained by identifying the pendant vertex v_1 of P_m with a leaf u_n of $K_{1,n}$.

Let V(G) = {u,u_i,v_j : 1 ≤ i ≤ n-1, 1 ≤ j ≤ m } and E(G) = {uu_i,uv_1,v_jv_{j+1}: 1 ≤ i ≤ n-1, 1 ≤ j ≤m-1 }.

G has m + n vertices and m + n – 1 edges.

Let t = m + n - 1.

Let f : V(G)→{0,1,2,…,A_t} be defined as follows

$f(u) = 0$

$f(u_i) = A_{t-(i-1)}$;$1 \leq i \leq n$

$f(v_1) = A_m$

$f(v_j) = f(v_{j-1}) + A_{n-(j-2)}$ if j is odd $2 \leq j \leq m$.

$= f(v_{j-1}) - A_{n-(j-2)}$ if j is even $2 \leq j \leq m$.

Let f* be the induced edge labeling of f.

Then $f^*(uu_i) = A_{t-(i-1)}$; $1 \leq i \leq n$-1.

$f^*(uv_1) = A_m$

$f^*(v_jv_{j+1}) = A_{m-j}$;$1 \leq j \leq m$-1.

The induced edge labelsA_1,A_2,…, A_t are distinct and consecutive pentagonal numbers..

Hence G is pentagonal graceful for all m ≥ 2and n ≥ 1.

Example 2.25. Pentagonal graceful labeling graph obtained by identifying a pendant vertex of P_5 with a leaf of $K_{1,4}$ is given in Fig. 12.

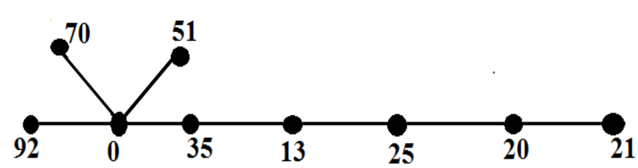

Fig. 12

Theorem 2.26. The graph obtained by subdividing the edges of the star $K_{1,n}$ is pentagonal graceful for all n ≥ 1.

Proof: Let G be the graph obtained by subdividing the edges of the star $K_{1,n}$ for all n ≥ 1.

Let V(G) = {u, v_i,u_i : 1≤ i ≤ n } and E(G) = { uv_i,v_iu_i : 1≤ i ≤ n }.

G has 2n + 1 vertices and 2n edges.

Let t = 2n .

Let f : V(G)→{0,1,2,…,A_t} be defined as follows

$f(u) = 0$

$f(v_i) = A_{t-i+1}$; $1 \leq i \leq n$

$f(u_i) = f(v_i) - A_{n-i+1}$; $1 \leq i \leq n$

Let f^* be the induced edge labeling of f.

Then $f^*(uv_i) = A_{t-i+1}$; 1≤ i ≤ n.

$f^*(v_iu_i) = A_{n-i+1}$; 1≤ i ≤ n.

The induced edge labels A_1,A_2,... , A_t are distinct and consecutive pentagonal numbers.

Hence the graph G is pentagonal graceful for all n ≥ 1.

Example 2.27. Pentagonal graceful labeling of the graph obtained by subdividing the edges of the star $K_{1,4}$ is shown in Fig. 13.

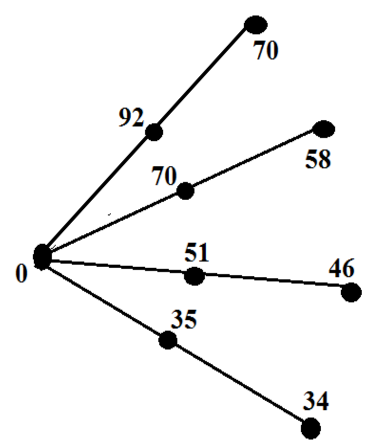

Fig. 13

Theorem 2.28. The graph obtained from $P_n \odot K_1$ by subdividing the edges of the pah P_n is pentagonal graceful for all $n \geq 2$.

Proof: Let G be the graph obtained from $P_n \odot K_1$ by subdividing the edges of the pah P_n.

Let V(G) = { v_i,u_i, w_j : 1≤ i ≤ n , 1≤ j ≤ n-1 } and

E(G) = { v_iw_i , v_j u_j, w_kw_{k+1} : 1≤ i ≤ n-1 , 1≤ j ≤ n , 1≤ k ≤ n-1 }.

G has 3n - 1 vertices and 3n - 2 edges.

Let t = $3n-2$.

Let f : V(G)→{0,1,2,…,A_t} be defined as follows

$f(v_1) = 0$

$f(v_i) = f(w_{i-1}) - A_{t-1-(2(i-2))}$; $2 \le i \le n$

$f(w_j) = f(v_j) + A_{t-2(j-1)}$; $1 \le j \le n$-1

$f(u_i) = f(v_i) + A_{n-i+1}$; $1 \le i \le n$.

Let f^* be the induced edge labeling of f.

Then $f^*(v_i w_i) = A_{t-2(i-1)}$; $1 \le i \le n$-1.

$f^*(v_j u_j) = A_{n-j+1}$; $1 \le j \le n$.

$f^*(w_k w_{k+1}) = A_{t-2k+1}$; $1 \le k \le n$-1.

The induced edge labels $A_1, A_2, \ldots, A_t$ are distinct and consecutive pentagonal numbers.

Hence the graph G is pentagonal graceful for all $n \ge 2$.

Example 2.29. Pentagonal graceful labeling of $P_4 \odot K_1$ by subdividing the edges of the path P_4 is shown in Fig. 14.

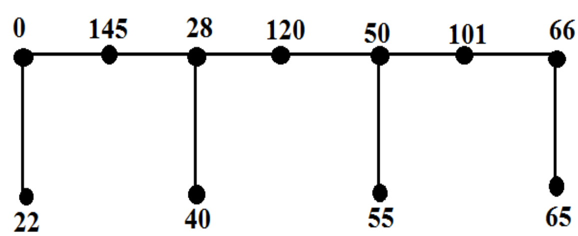

Fig. 14

3. CONCLUSION

In this paper, the authors studied the pentagonal graceful labeling of some graphs. Similar study can be extended for other graphs.

References

[1] B. D. Acharya, Construction of certain infinite families of graceful graphs from a given graceful graph. *Def. Sci. J.* 32(3) (1982) 231-236

[2] J.C. Berbond, Graceful Graphs, Radio Antennae and French Wind Mills. Graph Theory and Combinatories, Pitman, London, (1979), 13-17.

[3] G.S. Bloom and S.W. Golomb, Applications of Numbered Undirected Graphs. *Proceedings of IEEE,* Vol. 65, No. 4 (1977), 562-570

[4] Bloom G.S and Golomb S.W, Numbered Complete Graphs, Unusual Rules and Assoorted Applications, Theory and Application of Graphs, Lecture Notes in Math 642, Springer-Verlag, (1978), 53-65

[5] Frank Werner. Graph Theoretic Problems and their New Applications. *Mathematics,* S 445, (2020) 1-4

[6] S.W. Golomb, How to Number a Graph in Graph theory and Computing, R.C. Read, Ed., Academic Press, New York (1972), 23-37.

[7] Joseph A Gallian, A Dynamic Survey of Graph labeling. *The Electronic Journal of Combinatorics*, 15, 2008, #DS6.

[8] D. Gunasekaran, K. Senbagam, R. Saranya, Labeling of 2-regular graphs by even edge magic. *World Scientific News* 135 (2019) 32-47

[9] V. Lavanya, D.S.T. Ramesh and N. Meena, Bi-Domination in Corona Related Graphs. *Journal of Computer and Mathematical Sciences,* Vol. 10 (11), November (2019), 1650-1653

[10] Monika K, Murugan K, Fuibonacci Sum Labeling of Tree Related Graphs. *Adalya Journal,* Vol. 8, Issue 9, September (2019), 320-323

[11] Muhammed Imran, Adnan Aslam, Sohail Zafar and Waqar Nazeer, Further Results on Edge Irregularity Strength of Graphs, *Indonesian Journal of Combinnatorics* 1(2), (2017) 82-97

[12] G. Muppidathi Sundari & K. Murugan, Extra Skolem Difference Mean Labeling of Some Graphs. *World Scientific News* 145 (2020) 210-221

[13] K. Murugan and A. Subramanian, Labeling of Subdivided Graphs. *American Jr. of Mathematics and Sciences,* Vol. 1, No. 1, January (2012), 143-149

[14] K. Murugan and A. Subramanian, Skolem Difference Mean Graphs, *Mapana J Sci* 11, 4, (2012) 109-120

[15] K. Murugan, Square Graceful Labeling of Some Graphs, *International Journal of Innovative Research in Science, Engineering and Technology*, Vol. 4, Issue 2, February (2015) 511-520.

[16] D.S.T. Ramesh and M.P. Syed Ali Nisaya, Some Important Results on Pentagonal Graceful Graphs, *International Journal of Applied Mathematical Sciences,* Vol. 7, Issue 1, (2014) 71-77.

[17] A. Rosa, On Certain Valuations of the Vertices of a Graph, Theory of Graphs, (International Symposium, Rome, July 1966), Gorden, Breach. N.Y. and Dunad, Paris (1967), 349-355

[18] R.Sivaraman, Graceful Graphs and its Applications, International Journal of Current Research, Vol. 8, Issue 11, (November 2016), 41062-41067.

[19] M.P. Syed Ali Nisaya and D.S.T. Ramesh, Pentagonal Graceful Labeling of Caterpillar Graphs, *International Journal of Engineering Development and Research*, Vol. 6, Issue 4 (2018) 150-154.

[20] G. Muthumanickavel, K. Murugan, Oblong Sum Labeling of Union of Some Graphs. *World Scientific News*, 145 (2020) 85-94

[21] N. Meena, M. Madhan Vignesh, Strong Efficient Co-Bondage Number of Some Graphs. *World Scientific News* 145 (2020) 234-244

[22] M. Vanu Esakki, M. P. Syed Ali Nisaya, Two Modulo Three Sum Graphs. *World Scientific News* 145 (2020) 274-285

[23] Xiaojing Yang, Junfeng Du, Liming Xiong. Forbidden subgraphs for supereulerian and Hamiltonian graphs. *Discrete Applied Mathematics* Volume 288, 15 January 2021, Pages 192-200. https://doi.org/10.1016/j.dam.2020.08.034

[24] Chiba, Shuya, Yamashita, Tomoki 2018. Degree Conditions for the Existence of Vertex-Disjoint Cycles and Paths: A Survey. *Graphs and Combinatorics,* Vol. 34, Issue. 1, p. 1. https://doi.org/10.1007/s00373-017-1873-5

[25] Molla, Theodore, Santana, Michael, Yeager, Elyse 2020. Disjoint cycles and chorded cycles in a graph with given minimum degree. *Discrete Mathematics,* Vol. 343, Issue. 6, p. 111837. https://doi.org/10.1016/j.disc.2020.111837

[26] Kostochka, Alexandr, Yager, Derrek, Yu, Gexin 2020. *Discrete Mathematics and Applications.* Vol. 165, p. 259. https://doi.org/10.1007/978-3-030-55857-4_11

[27] Costalonga, J. P., Kingan, Robert J., Kingan, Sandra R. 2021. Constructing Minimally 3-Connected Graphs. *Algorithms,* Vol. 14, Issue. 1, p. 9. https://doi.org/10.3390/a14010009