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# Analytical Cartesian coordinate solutions of Laplace equations by separation of variable method in mathematical physics

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**ABSTRACT**

This research aimed at solving the Cartesian coordinates of two and three dimensional Laplace equations by separation of variables method. It was painstakingly solved with appropriate boundary conditions of steady states. However, the solution of potential (V) of a partial differential equation (PDE) in three real variables $x, y\ and\ z$ are functionally obtained using separation of variable approach by stating the boundary conditions of the Cartesian coordinates.

**Keywords:** Laplace equations, Cartesian coordinates, Separation of variables, Potential, Dimensions of space, Partial derivatives

## 1. INTRODUCTION

The analytical solution of Cartesian coordinate system of two and three dimensional Laplace equations using the approach of separation of variable is presented in this work.

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The physical state generally known is three dimensional; this is observed by the fact that in reality, the drawing at most three perpendicular straight lines through a point is not obtainable [1]. Nevertheless, the analysis stated above does not limit scientist from studying hyper- space (i.e. Space having dimensions greater than three). Actually, in true geometry of hyper-space have already been confirmed [2-4]. However, hyper-space dimensions are said to be "Compactified" into discrete circles which are smaller than atoms [5].

To solve analytically, the changes in most physical systems, there is a need to apply partial differential equations (PDEs) which is the basic foundation in solving the Laplace equations of Cartesian coordinates using the separation of variable method [6]. The partial differential equation has different categories: hyperbolic, parabolic and elliptic. This classification of PDE's are very important because the analytical and numerical methods for solving field problems are different from the three classes of the equation [7]. The most expected of the elliptic partial differential equation, and indeed of all PDEs in applied physical sciences and physics in particular is Laplace equations [8]. Despite these strong interest in studying Laplace equations, very few analytical solutions have been obtained from it especially in Cartesian and spherical coordinates [9]. The first boundary value problem is popular, known as Poisson's integration for the sphere and secondly the exact solution of Neumann boundary conditions published sixty years ago [10, 11].

The Laplace equations can be solved by separation of variables in eleven coordinate systems. In addition to these eleven coordinate systems, it can be separated into two additional coordinate systems which "bispherical and toroidal coordinates", making the total number of the separations equal thirteen [12, 13]. However, it can generalized that Laplace equations are one of the fundamental equations in mathematical physics and evidently applied in many branches of physics [14, 15]. There are two types of Laplace equations; The homogenous equation that have constant coefficients with many classical solutions which can be solved using separation of variables, the method of characteristics and Fourier transform [16-19], and the non-homogenous equations with constant coefficients solved using operation calculus [20-22]. These methods stated above are applied in comprehensive steps to obtain the solution of Laplace equations of different dimensions.

## 2. TWO - DIMENSIONAL CARTESIAN COORDINATE SOLUTIONS OF LAPLACE'S EQUATION

Two dimensional potential depends on two variables x and y, it is obviously has no simple analytical solution as the case of one dimensional potential, therefore it will be more convenient to use separation of variable method to solve the two dimensional Laplace equation. Thus, the Laplace equation for two dimensional system is stated mathematically as;

$$\nabla^2 V(x,y) = 0 \tag{1}$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{2}$$

It is pertinent to solve the Laplace equation with the following subjected boundary conditions. That is;

$$V(x = 0, 0 \leq y \leq a) = 0 \tag{3a}$$

$$V(x = b, 0 \leq y \leq a) = 0 \tag{3b}$$

$$V(0 \leq x \leq b, \quad y = 0) = 0 \tag{3c}$$

$$V(0 \leq x \leq b, \quad y = a) = 0 \tag{3d}$$

Hence, eq. (2) will be solved using the method of separation of variables which is getting the product of the solution V(x, y);

$$V(x, y) = V(x)\,V(y) \tag{4}$$

Eq. (4) shows that one of the potential is a function of x and other a function of y. Therefore, Eq. (4) will be substituted into eq. (2) thus having that;

$$\frac{\partial^2 V}{\partial x^2}\mathrm{V(y)} + \frac{\partial^2 V}{\partial y^2}\mathrm{V(x)} = 0 \tag{5}$$

Dividing eq. (5) all through by V(x) V(y) and separating V(x) from V(y), therefore having that;

$$-\frac{\partial^2 V}{V(x)\partial x^2} = \frac{\partial^2 V}{V(y)\partial y^2} \tag{6}$$

Since the left hand side of eq. (6) contains only a function x and the right hand side contains only a function y. Hence, for both side of the equation to be equal, it must be equated with a constant $\xi$ thus having that;

$$-\frac{\partial^2 V}{V(x)\partial x^2} = \frac{\partial^2 V}{V(y)\partial y^2} = \xi \tag{7}$$

where ξ in eq. (7) is known as separation constant, hence eq. (7) will be separated thus as;

$$\frac{\partial^2 V}{\partial x^2} + \xi\mathrm{V(x)} = 0 \tag{8}$$

$$\frac{\partial^2 V}{\partial x^2} - \xi\mathrm{V(y)} = 0 \tag{9}$$

The variables having been separated hence, eq. (8) and (9) are referred as separated equations therefore V(x) and V(y) can now be solved separately and then the obtained solution can be substituted into eq. (4) but to achieve that, the boundary conditions in eq. (3) will be required in a separated form. Therefore it can be separated thus as;

$$V\,(0,y) = V\,(0)\,V(y) = 0 \longrightarrow V\,(0) = 0 \qquad (10a)$$

$$V\,(a,b) = V\,(b)\,V(y) = 0 \longrightarrow V\,(b) = 0 \qquad (10b)$$

$$V\,(x,0) = V\,(x)\,V(0) = 0 \longrightarrow V\,(0) = 0 \qquad (10c)$$

$$V\,(x,a) = V\,(0)\,V(a) = V \longrightarrow V\,(0) = V_o\,(inseparable) \qquad (10d)$$

To solve for V (x) and V (y) in eq. (8) and (9) the boundary conditions in eq. (10) will applied and also consider the possible values of ξ that will satisfy eq. (8) and (9) and the boundary condition of eq. (10).

**Case 1:** Considering if $\xi = 0$ eq. (8) now becomes;

$$\frac{d^2V}{dx^2} = 0 \qquad (11)$$

Integrating eq. (11) hence having that;

$$V(x) = Kx + C \qquad (12)$$

The boundary conditions in eq. (10a) and (10b) implies that;

$\mathrm{V}\,(\mathrm{x} = 0) = 0 \longrightarrow 0 = 0 + \mathrm{C} \rightarrow \mathrm{C} = 0$ & $\mathrm{V}\,(\mathrm{x} = 0) = 0 \longrightarrow 0 = \mathrm{K.b} + 0 \rightarrow \mathrm{k} = 0$ because $b \neq 0$. hence, $V(x) = 0$ will be regarded as a trivial solution and can be concluded that $\xi \neq 0$.

**Case 2:** Considering if $\xi < 0$, say $\xi = -\alpha^2$, then eq. (8) becomes;

$$\frac{d^2V}{dx^2} - \alpha^2\,\mathrm{V(x)} = 0 \qquad (13)$$

That is;

$$\frac{dV}{dx} = \pm\,\alpha\,\mathrm{V(x)} \qquad (14)$$

Eq. (14) shows that there are possible solutions corresponding to a plus and minus signs. Hence, for the one of plus sign it is expressed as;

$$\frac{dV}{dx} = \alpha\,\mathrm{V(x)} \qquad (15)$$

Arranging eq. (15), by separating the corresponding variable, therefore having that;

$$\frac{dV}{\mathrm{V(x)}} = \alpha\, \mathrm{dx} \qquad (16)$$

Integrating eq. (17) to have that;

$$\int \frac{dV}{V(x)} = \int \alpha\, dx \qquad (17)$$

Therefore having that;

$$InV(x) = \alpha x + Inz_1 \qquad (18)$$

where $\ln z_1$ becomes constant of integration, thus becomes;

$$V(x) = Z^{1e^{\alpha x}} \qquad (19)$$

Similarly, for the minus sign, it can be obtained by solving eq. (20) hence having that;

$$V(x) = Z^{2e^{-\alpha x}} \qquad (20)$$

Thus, the total solution consists of eq. (19) and (20) that is;

$$V(x) = Z^{1e^{\alpha x}} + z^{2e^{-\alpha x}} \qquad (21)$$

Applying Euler series for hyperbolic trigonometric function, that is;

$$Cosh\alpha x = \frac{e^{\alpha x} + e^{-\alpha x}}{2} \quad and \quad Sinh\alpha x = \frac{e^{\alpha x} - e^{-\alpha x}}{2} \qquad (22)$$

Eq. (22) can equally be expressed thus as;

$$e^{\alpha x} = Cosh\alpha x + Sinh\alpha x \quad and\ e^{-\alpha x} = Cosh\alpha x - Sinh\alpha x \qquad (23)$$

Therefore substituting eq. (23) into eq. (22), having that;

$$V(x) = y_1 Cosh\alpha x + y_2 Sinh\alpha x \qquad (24)$$

where $y_1 = z_1 + z_2\ and\ y_2 = z_1 - z_2$. In view of the given boundary condition, eq. (24) is more preferable to eq. (21) as the solution. Again eq. (10a) and (10b) requires that; $\mathrm{V}(\mathrm{x} = 0) = 0 \longrightarrow 0 = y_1(1) + y_2(0) \rightarrow y_1 = 0$ & $\mathrm{V}(\mathrm{x} = \mathrm{b}) = 0 \longrightarrow 0 = 0 + y_2 Sinh\alpha b$. Knowing that $\alpha \neq 0\ and\ b \neq 0$, therefore $Sinh\alpha b$ cannot be zero.

This is because of the fact that $Sinhx = 0$ if and only if $x = 0$. Hence, $y_2 = 0\ and\ V(x) = 0$. This is also a trivial solution and it can be concluded that $\xi$ cannot be less than zero.

**Case 3:** Considering if $\xi > 0$, say $\xi = \beta^2$, then eq. (8) becomes;

$$\frac{\partial^2 V}{\partial x^2} + \beta^2 \mathrm{V(x)} = 0 \tag{25}$$

Thus, eq. (25) can be rewritten as;

$$\left(\frac{\partial^2}{\partial x^2} + \beta^2\right) V(x) = 0 \quad or \quad \frac{\partial^2 V}{\partial x^2} = \pm \mathrm{j\beta\, V(x)} \tag{26}$$

where j is an imaginary value i.e. $j = \sqrt{-1}$.

Thus, from eq. (14) and (26), there is difference between case 2 and case 3; where in case 2 α is replaced by $j\beta$ in case 3. Hence, taking the same process as in case 2, the solution is now obtained as;

$$V(x) = \mathrm{M_o} e^{j\beta x} + \mathrm{M_1}^{e^{-j\beta x}} \tag{27}$$

Knowing that, $e^{j\beta x} = Cos\beta x + jSin\beta x \quad and \; e^{-j\beta x} = Cos\beta x - jSin\beta x$. Hence, eq. (27) can now be written as;

$$V(x) = P_o Cos\beta x + P^1 Sin\beta x \tag{28}$$

where $P_o = M_o + M_1 \, and \; P_1 = M_o - jM_1$. In view of the given boundary conditions, it is more preferable to use eq. (28). Imposing the conditions in equations (10a) and (10b) therefore having that;

$\mathrm{V\,(x = 0) = 0} \longrightarrow 0 = \mathrm{P_o}(1) + y_2(0) \rightarrow \mathrm{P_o} = 0 \;\&\; \mathrm{V\,(x = b) = 0} \longrightarrow 0 = 0 + P_1 Sinh\beta b$. supposed that $P_1 \neq 0$ otherwise a trivial solution will be obtained.

Thus; $Sin\beta b = 0 = Sinn\pi$ hence having that

$$\beta = \frac{n\pi}{b}, \; n = 1, 2, 3, 4, 5, \dots \dots \tag{29}$$

Note, unlike $Sinhx$, which is zero only when $x = 0$. $Sinx$ is zero at an infinite number of points. It should equally be noted that $n \neq 0$ because $\beta \neq 0$; it has already been considered that the possibility $\beta = 0$ in case 1 where the trivial solutions. Also there is no need considering for $n = -1, -2, -3, -4, -5, ..$ reason is because $\xi = \beta^2$ would remain the same for positive and negative values of n.

Thus, for a given n eq. (28) becomes;

$$V_n(x) = P_n Sin \frac{n\pi x}{b} \tag{30}$$

Having found $V(x)$, hence substituting eq. (29) into the condition stated in case 3, thus having that;

$$\xi = \beta^2 = \frac{n^2\pi^2}{b^2} \tag{31}$$

Therefore eq. (9) which is $\frac{\partial^2 V}{\partial x^2} - \xi V(y) = 0$ can now be solved, and the solution to it is similarly to that of eq. (24) obtained in case 2 that is;

$$V(y) = h_o Cosh\beta y + h_1 Sinh\beta y \tag{32}$$

The boundary condition in eq. (10c) implies that;

$V(y = 0) = 0 \longrightarrow 0 = h_o(1) + 0 \rightarrow h_o = 0$. Hence, the solution for $V(y)$ becomes

$$V_n(y) = h_n Sinh\frac{n\pi y}{b} \tag{33}$$

Substituting eq. (30) and (33), which are solutions to the separated equations in eq. (8) and (9), into the product solution in eq. (4) gives that;

$$V_n(x,y) = P_n h_n Sin\frac{n\pi x}{b} Sinh\frac{n\pi y}{b} \tag{34}$$

This shows that there are many possible solutions $V_1, V_2, V_3, V_4$ and so on, for $n = 1, 2, 3, 4$ and so on. By superposition theorem, if $V_1, V_2, V_3, V_4 \ldots\ldots\ldots V_n$ are solutions of Laplace equation, the linear combination which is;

$$V = q_1V_1 + q_2V_2 + q_3V_3 + \cdots \ldots\ldots\ldots\ldots q_nV_n \tag{35}$$

where $q^1, q^2, q^3, \ldots\ldots\ldots q_n$ are constants and eq. (35) is also a solution of Laplace equation.

Thus the solution to eq. (2) is then given as;

$$V(x,y) = \sum_{n=1}^{\infty} q_n \operatorname{Sin}\frac{n\pi x}{b} \operatorname{Sinh}\frac{n\pi b}{b} \tag{36}$$

where $q_n = P_n h_n$ are the coefficients to be determined from the boundary condition in eq. (10d). Hence, imposing this condition gives that;

$$V(x, y = b) = V_o = \sum_{n=1}^{\infty} q_n \operatorname{Sin}\frac{n\pi x}{b} \operatorname{Sinh}\frac{n\pi b}{b} \tag{37}$$

Which is a Fourier series expansion of $V_o$. Multiplying both sides of eq. (37) by Sin$\frac{m\pi x}{b}$ and integrating over the region $0 < x < b$ gives that;

$$\int_0^b V_o \operatorname{Sin}\frac{m\pi x}{b}dx = \sum_{n=1}^{\infty} q_n \operatorname{Sinh}\frac{n\pi a}{b}\int_0^b \operatorname{Sin}\frac{n\pi x}{b}\operatorname{Sin}\frac{m\pi x}{b}dx \qquad (38)$$

By orthogonality property of Sine function which is;

$$\int_0^{\pi} Sinmx\ Sinnx\ dx = \begin{pmatrix} 0. & m \neq n \\ \pi/2 & m = n \end{pmatrix} \qquad (39)$$

Incorporating this properties in eq. (38) means that all terms on the right-hand side of eq. (38) will vanish except one term in which $m = n$. hence, eq. (38) reduces to;

$$\int_0^b V_o \operatorname{Sin}\frac{n\pi x}{b}dx = q_n \operatorname{Sinh}\frac{n\pi a}{b}\int_0^b Sin^2\frac{n\pi x}{b}\, dx \qquad (40)$$

Integrating only the left-hand side of eq. (40), thus becomes;

$$-V_o \frac{b}{n\pi} Cos \frac{n\pi x}{b} \Big|_0^b = q_n \operatorname{Sinh}\frac{n\pi a}{b}\frac{1}{2}\int_0^b \left(1 - Cos\frac{2\pi x}{b}\right)dx$$

Hence, integrating the equation totally and applying the boundary condition, thus having that;

$$\frac{V_o b}{n\pi}(1 - Cos\, n\pi) = q_n \operatorname{Sinh}\frac{n\pi a}{b}.\frac{b}{2} \qquad (41)$$

Eq. (41) can still be expressed as;

$$q_n \operatorname{Sinh}\frac{n\pi a}{b} = \frac{2V_o}{n\pi}(1 - Cos\, n\pi) \qquad (42)$$

Applying the series that; $\frac{2V_o}{n\pi}(1 - Cos\, n\pi) = \begin{cases} \frac{4V_o}{n\pi}, & n = 1, 3, 5, \dots \dots . \\ 0, & n = 2, 4, 6, \dots . \dots . \end{cases}$

That is;

$$q_n = \begin{cases} \dfrac{4V_o}{n\pi Sinh\, \frac{n\pi a}{b}}, & n = odd \\ 0, & n = even \end{cases} \qquad (43)$$

Substituting eq. (43) into eq. (37), the complete solution of the potential becomes;

$$V(x, y) = \frac{4V_o}{\pi}\sum_{n=1,3,5}^{\infty}\frac{\operatorname{Sin}\frac{n\pi x}{b}\operatorname{Sin}\frac{n\pi y}{b}}{n\, Sinh\frac{n\pi a}{b}} \qquad (44)$$

Eq. (44) is the complete solution of the Laplace equation in two dimensional Cartesian coordinate system. The potential obtained shows that there is no dependence on the z coordinate. Thus, the solution is expressed in Fourier series and also in terms of its orthogonal functions.

## 3. THREE - DIMENSIONAL CARTESIAN COORDINATE SOLUTIONS OF LAPLACE'S EQUATION

In three dimensional system, the potential depends on three variables $x,\ y\ and\ z$. Thus, the Laplace equation in Cartesian coordinate system can be stated mathematically as;

$$\nabla^2 V(x,y,z) = 0 \qquad (45)$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad (46)$$

Therefore eq. (46) can be solve using the separation of variable method, thus expressing as;

$$V\,(x,y,z) = V(x)V(y)V(z) \qquad (47)$$

Differentiating eq. (47) and substituting it into eq. (46). Thus having that;

$$V(y)V(z)\frac{\partial^2 V}{\partial x^2} + V(x)V(z)\frac{\partial^2 V}{\partial y^2} + V(x)V(y)\frac{\partial^2 V}{\partial z^2} = 0 \qquad (48)$$

Dividing eq. (48) all through by $V = V(x)\,V(y)\,V(z)$ thus having that;

$$\frac{1}{V(x)}\frac{\partial^2 V}{\partial x^2} + \frac{1}{V(y)}\frac{\partial^2 V}{\partial y^2} + \frac{1}{V(z)}\frac{\partial^2 V}{\partial z^2} = 0 \qquad (49)$$

Eq. (49) express each terms as a function of only one variable. Hence, taking the partial derivative of the whole expression with respect to x in eq. (49) gives that;

$$\frac{\partial}{\partial x}\left(\frac{1}{V(x)}\frac{\partial^2 V}{\partial x^2}\right) + \frac{\partial}{\partial x}\left(\frac{1}{V(y)}\frac{\partial^2 V}{\partial y^2}\right) + \frac{\partial}{\partial x}\left(\frac{1}{V(z)}\frac{\partial^2 V}{\partial z^2}\right) = 0 \qquad (50)$$

Therefore eq. (50) becomes;

$$\frac{\partial}{\partial x}\left(\frac{1}{V(x)}\frac{\partial^2 V}{\partial x^2}\right) = 0 \qquad (51)$$

From eq. (51) it shows that $\frac{1}{V(x)}\frac{\partial^2 V}{\partial x^2}$ must be independent of $x$, the same holds if $y\ and\ z$ derivatives are taken. That is;

$$\frac{\partial}{\partial y}\left(\frac{1}{V(y)}\frac{\partial^2 V}{\partial y^2}\right) = 0 \quad (52)$$

$$\frac{\partial}{\partial z}\left(\frac{1}{V(z)}\frac{\partial^2 V}{\partial z^2}\right) = 0 \quad (53)$$

This means that each of three terms must be constant. Thus having that;

$$\frac{1}{V(x)}\frac{\partial^2 V}{\partial x^2} = a \quad (54)$$

$$\frac{1}{V(y)}\frac{\partial^2 V}{\partial y^2} = b \quad (55)$$

$$\frac{1}{V(z)}\frac{\partial^2 V}{\partial z^2} = c \quad (56)$$

This implies that the constants must add to zero that is;

$$a + b + c = 0 \quad (57)$$

To solve those equations, it will be more convenient to choose the constants to be in the form $\pm\, \alpha^2$ since the solutions are either exponential or sinusoidal.

**Case 1:** Considering if $a = -\,\alpha^2$ then;

$$\frac{1}{V(x)}\frac{\partial^2 V}{\partial x^2} = -\,\alpha^2 \quad (58)$$

Separating the variable thus having that;

$$\frac{\partial^2 V}{\partial x^2} + \alpha^2 V(x) = 0 \quad (59)$$

Hence, the solution of eq. (59) becomes;

$$V(x) = ASin\alpha x + BCos\alpha x \quad (60)$$

**Case 2:** Considering if $a = +\alpha^2$ then;

$$\frac{1}{V(x)}\frac{\partial^2 V}{\partial x^2} = \alpha^2 \quad (61)$$

Separating the variable thus having that;

$$\frac{\partial^2 V}{\partial x^2} - \alpha^2 V(x) = 0 \quad (62)$$

Hence, the solution of eq. (60) becomes;

$$V(x) = Ae^{\alpha x} - Be^{-\alpha x} \quad (63)$$

where $A$ $and$ $B$ are constants, because $a + b + c = 0$ both signs must occur. If two of the constants are negative (Giving an oscillating solutions), then the third must be positive and will have exponential solutions. If two of the signs are positive then there are two directions with exponential solution. Then the third constant is negative and has oscillating solutions. The sign can be chosen so that the solutions can match the boundary conditions.

That is;

$$a = -\alpha^2 \quad (64)$$

$$b = -\beta^2 \quad (65)$$

$$c = \alpha^2 + \beta^2 \quad (66)$$

Therefore eq. (60) still holds, while y satisfies that;

$$V(x) = CSin\beta y + DCos\beta y \quad (67)$$

While z satisfies that;

$$\frac{\partial^2 V}{\partial z^2} - (\alpha^2 + \beta^2)\, V(z) = 0 \quad (68)$$

For more clarification and simplicity, the parameters in eq. (67) can still be defined as;

$$\alpha^2 + \beta^2 = \gamma^2 \quad (69)$$

Substituting eq. (69) into eq. (68). Thus having that;

$$\frac{\partial^2 V}{\partial z^2} - \gamma^2\, V(z) = 0 \quad (70)$$

Hence, the general solution of eq. (70) is the arbitrary linear combination. That is;

$$V(z) = Ee^{\gamma z} - Fe^{-\gamma z} \quad (71)$$

It is often more useful to use the symmetrical and unsymmetrical combinations. That is;

$$Cosh\gamma z = \frac{e^{\gamma z} + e^{-\gamma z}}{2} \quad (72)$$

$$Sinh\gamma z = \frac{e^{\gamma z} - e^{-\gamma z}}{2} \quad (73)$$

Eq. (72) and (73) can still be expressed in this form;

$$e^{\gamma z} = Cos\gamma z + Sin\gamma z \quad (74)$$

$$e^{-\gamma z} = Cos\gamma z - Sin\gamma z \quad (75)$$

Hence, the solution of eq. (70) becomes;

$$V(z) = E(Cos\gamma z + Sin\gamma z) + F(Cos\gamma z - Sin\gamma z) \quad (76)$$

Solving eq. (76) further therefore having that;

$$V(z) = (E + F)Cos\gamma z + (E - F)\, Sin\gamma z \quad (77)$$

For simplicity sake, the constants in eq. (77) can be represented thus as $E + F = G$ and $E - F = H$. Hence, eq. (77) having;

$$V(z) = GCos\gamma z + H\, Sin\gamma z \quad (78)$$

Substituting eq. (60), (67) and (78) into eq. (47) to obtain the full solution as a linear combination. For any particular value of α and β the solution becomes;

$$V_{\alpha,\beta}(x, y, z) = (ASin\alpha x + BCos\alpha x)(CSin\beta y + DCos\beta y)(GCos\gamma z + H\, Sin\gamma z) \quad (79)$$

Since the Laplace equation is a linear function. Hence, it is more appropriate to express the general solution for potential (V) as a sum of eq. (79) for different α and β. The leading constants may be different for each choice of α and β, so as to have that;

$$V(x, y, z) = \sum_{\alpha,\beta}\left(A_{\alpha,\beta}Sin\alpha x + B_{\alpha,\beta}Cos\alpha x\right)\left(C_{\alpha,\beta}Sin\beta y + DCos\beta y\right)\left(G_{\alpha,\beta}Cos\gamma z + H_{\alpha,\beta}\, Sin\gamma z\right) \quad (80)$$

However, fitting boundary conditions in Cartesian coordinates, hence there is need to consider the potential everywhere inside a conducting cube of side L. Thus, the boundary will be held at zero potential except for the top of the box, which is held at $V_o$. This implies that in x-direction, the potential ($V$) starts and end at zero. That is;

$$V(0, y, z) = 0 \quad (81a)$$

$$V(L, y, z) = 0 \quad (81b)$$

The simplest way to satisfy this boundary condition is to choose the sinusoidal solution in the x-direction, i.e. eq. (60) which is $V(x) = ASin\alpha x + BCos\alpha x$, hence the boundary conditions required will be;

$$V(0, y, z) = V(0)\, V(y)\, V(z) \qquad (82a)$$

$$V(L, y, z) = V(L)\, V(y)\, V(z) \qquad (82b)$$

For all y and z, therefore having that;

$$V(0) = 0$$

$$0 = ASin0 + BCos0$$

Therefore having that; $B = 0$
With $B = 0$ the second boundary condition i.e. eq. (89b) can now be obtain as;

$$V(L) = 0$$

$$0 = ASin\alpha L + BCos\alpha L$$

$$0 = Asin\alpha L$$

Therefore having that; $A = 0$
Since $A = 0$, it implies that the entire potential will vanish. Hence, the possible values of $\alpha$ will be restricted instead. Thus the boundary condition is satisfies if and only if;

$$\alpha L = n\pi \qquad (83)$$

$$\alpha = \frac{n\pi}{L} \qquad (84)$$

For any integer $n$, it is concluded that;

$$V_n(x) = A_n Sin\, \frac{n\pi x}{L} \qquad (85)$$

where the constant $A_n$ depends on the values of $n$.
For y-direction, it is completely analogous. Hence, the vanishing potential is required on both sides. Thus;

$$V(x, 0, z) = 0 \qquad (86a)$$

$$V(x, L, z) = 0 \qquad (86b)$$

For all $x\ and\ z$, therefore it is obtained as;

$$V(0) = 0 \tag{87a}$$

$$V(L) = 0 \tag{87b}$$

Hence, the oscillating solution now becomes;

$$V_m(x) = A_m Sin\,\frac{n\pi y}{L} \tag{88}$$

where the integer m is an independent function of the integer n.

Finally, the boundary condition for $V(z)$ will be considered. Therefore considering that the solution in this direction must be exponential. Hence, eq. (78) which is $V(z) = GCos\gamma z + H\,Sin\gamma z$. Where:

$$\gamma = \sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{n\pi}{L}\right)^2} = \frac{\pi}{L}\sqrt{n^2 + m^2} \tag{89}$$

With the potential on the top of the box equal to $V_o$, therefore the boundary conditions for z are

$$V(x, y, 0) = 0 \tag{90a}$$

$$V(x, y, L) = V_o \tag{90b}$$

Since these relations hold for all $x\ and\ y$, hence condition must be satisfied by the choice obtained that $V(0) and\ V(L)$; the zero condition is completely satisfied by;

$$V(0) = 0$$

$$0 = GCos0 + HSin0$$

Therefore having that; $G = 0$

Hence, the complete solution can be expressed in the form;

$$V(x, y, z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_n Sin\frac{n\pi x}{L}\, C_m \mathrm{Sin}\frac{\mathrm{m}\pi y}{L}\, G_{m,n} Sinh\gamma L \tag{91}$$

$$V(x, y, z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} (A_n\, C_m G_{m,n})\, Sin\frac{n\pi x}{L}\, \mathrm{Sin}\frac{\mathrm{m}\pi y}{L}\, Sinh\gamma L$$

$$V_0 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} (A_n\, C_m G_{m,n})\, Sin\frac{n\pi x}{L}\, \mathrm{Sin}\frac{\mathrm{m}\pi y}{L}\, Sinh\gamma L \tag{92}$$

where it is sufficient to set $A_{nm} = A_n C_m G_{m,n}$ which is a single overall constant for each pair of $m, n$. Hence, considering the final boundary condition, $V(x, y, z) = V_o$, and there is no freedom to satisfy it by choosing γ. Instead, the constant $A_{nm}$ will be chosen.

Thus having that;

$$V_0 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(A_{nm} Sinh\gamma L)\, Sin\frac{n\pi x}{L}\, \text{Sin}\frac{m\pi y}{L} \quad (93)$$

For all $x$ *and* $y$. Where eq. (93) is a double Fourier series. Hence, to solve it, the orthogonality property of sine function will be adopted. That is the explicit integration becomes;

$$\int_0^L \text{Sin}\frac{n_1\pi x}{L}\text{Sin}\frac{n_2\pi x}{L}\, dx = \frac{L}{2}\delta_{n_1 n_2} \quad (94)$$

Considering the x-direction in eq. (93) first by multiplying both sides of the equation by $\text{Sin}\frac{k\pi x}{L}$ and integrating with respect to x using the boundary condi tion starting from $0$ *to* $L$, that is;

$$\int_0^L V_0\, Sin\,\frac{k\pi x}{L}\, dx = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(A_{nm} Sinh\gamma L)\, Sin\frac{m\pi x}{L}\int_0^L \text{Sin}\frac{n\pi x}{L}\text{Sin}\frac{k\pi x}{L}\, dx \quad (95)$$

The left hand side of the eq. (95) is easy to integrate, while the right hand side is integrated using the orthogonality stated in eq. (94). That is;

$$-\frac{LV_0}{k\pi} Cos\,\frac{k\pi x}{L}\,\begin{matrix} L \\ 0\end{matrix} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(A_{nm} Sinh\gamma L)\, Sin\frac{m\pi x}{L}\int_0^L \text{Sin}\frac{n\pi x}{L}\text{Sin}\frac{k\pi x}{L}\, dx \quad (96)$$

Performing the sum over all n, only one term survives and also substituting eq. (89). Hence, having that;

$$-\frac{LV_0}{k\pi} Cos\, k\pi + \frac{LV_0}{k\pi} Cos0 = \frac{L}{2}\sum_{m=1}^{\infty}\left(A_{km}\, Sinh\sqrt{k^2+m^2}\, L\right)\, \text{Sin}\frac{m\pi y}{L} \quad (97)$$

Solving eq. (97) further, hence that;

$$\frac{LV_0}{k\pi}\left(1 - (-1)^k\right) = \frac{L}{2}\sum_{m=1}^{\infty}\left(A_{km}\, Sinh\sqrt{k^2+m^2}\, L\right)\, \text{Sin}\frac{m\pi y}{L} \quad (98)$$

This procedure used in solving for x-direction will be repeated for the y-direction by multiplying eq. (98) by $\text{Sin}\frac{j\pi y}{L}$ and integrating it with respect to y by y taking the boundary condition from $0$ *to* $L$. This becomes;

$$\frac{LV_0}{k\pi}\left(1-\ (-1)^k\right)\int_0^L \mathrm{Sin}\frac{\mathrm{j}\pi y}{L}\,dy = \frac{L}{2}\sum_{m=1}^{\infty}\left(A_{km}\,Sinh\sqrt{k^2+m^2}\,L\right)\int_0^L \mathrm{Sin}\frac{\mathrm{m}\pi y}{L}\mathrm{Sin}\frac{\mathrm{j}\pi y}{L}\,dy \tag{99}$$

$$\frac{LV_0}{k\pi}\left(1-\ (-1)^k\right)\left[-\frac{L}{j\pi}Cos\,\frac{j\pi y}{L}\right]_0^L = \left(\frac{L}{2}\right)^2\sum_{m=1}^{\infty}\left(A_{km}\,Sinh\sqrt{k^2+m^2}\,L\right)\delta_{jm}$$

$$\frac{LV_0}{k\pi}\left(1-\ (-1)^k\right)\left(\frac{L}{j\pi}-\frac{L}{j\pi}Cos\,j\pi\right) = \left(\frac{L}{2}\right)^2 A_{kj}\,Sinh\sqrt{k^2+j^2}\,L$$

$$\frac{4V_0}{jk\pi^2}\left(1-\ (-1)^k\right)\left(1-\ (-1)^j\right) = A_{kj}\,Sinh\sqrt{k^2+j^2}\,L \tag{100}$$

Since these steps carried out for any values of j and k, all of the coefficients $A_{kj}$ has been obtained. Thus eq. (100) can be expressed as;

$$A_{kj} = \frac{4V_0}{jk\pi^2\,Sinh\sqrt{k^2+j^2}\,L}\left(1-\ (-1)^k\right)\left(1-\ (-1)^j\right) \tag{101}$$

Substituting eq. (101)into eq. (92) since $A_{km}$ has been expressed in terms of $A_{kj}$. Thus having that;

$$V(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{4V_0}{mn\pi^2\,Sinh\sqrt{k^2+j^2}\,L}\left(1-\ (-1)^n\right)\left(1-\ (-1)^m\right)Sin\frac{n\pi x}{L}\mathrm{Sin}\frac{\mathrm{m}\pi y}{L}Sinh\gamma z \tag{102}$$

Hence, this simplifies to sums over the odd terms since;

$$\left(1-\ (-1)^n\right) = \begin{cases} 0 & n\ even \\ 2 & n\ odd \end{cases}$$

Thus, from this series eq. (102) can be generalised as;

$$V(x,y,z) = \frac{16V_0}{\pi^2}\sum_{n=odd}^{\infty}\sum_{m=odd}^{\infty}\frac{1}{mn}\frac{Sinh\sqrt{n^2+m^2}\,z}{Sinh\sqrt{n^2+m^2}\,L}\,Sin\frac{n\pi x}{L}\,\mathrm{Sin}\frac{\mathrm{m}\pi y}{L} \tag{103}$$

Eq. (103) is the final potential and also the complete solution of three dimensional Laplace's equation in Cartesian coordinate system. However, the potential obtained is dependence on x, y and z coordinates. And the solution is expressed in Fourier series and also in terms of its orthogonal functions.

## 4. CONCLUSION

Separation of variable method is applied in the paper to solve a Cartesian coordinate Laplace equation of two and three dimensional system with appropriate boundary conditions; because they involve more than one variable. However, the potential function obtained from the partial differential equation equated to zero are expressed in terms of Fourier series and also in terms of their orthogonal functions.

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