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Second order triangular graceful graphs

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ABSTRACT

Let G $=$ (V, E) be a graph with p vertices and q edges. A second order triangular graceful labeling of a graph G is an one to one function $\varphi: V(G) \rightarrow \{0,1,2,\dots,B_q\}$ where B_q is the q^{th} second order triangular number, ie., $B_q = \frac{1}{6}q(q+1)(2q+1)$, that induces a bijection φ^*: E(G) $\rightarrow \{B_1, B_2, \dots, B_q\}$ of the edges of G defined by $\varphi^*(uv) = |\varphi(u) - \varphi(v)|$ $\forall$ e $=$ uv $\in$ E(G). A graph which admits such labeling is called a second order triangular graceful graph. In this paper, we introduce second order triangular graceful labeling and we prove that star, subdivision of star, n$K_{1,3}$, nK_2, bistar, path, comb, coconut tree, shrub and Y-tree are second order triangular graceful graphs.

***Keywords*:** Second order triangular number, Second order triangular graceful labeling, Second order triangular graceful graph

1. INTRODUCTION AND DEFINITIONS

The graph considered in this paper are finite, undirected and without loops or multiple edges. Let G = (V, E) be a graph with p vertices and q edges. Terms not defined here are used in the sense of Harary [8] and K. R. Parthasarathy [14]. For number theoretic terminology, we refer to [2, 5] and [13].

Graph labeling is one of the fascinating areas of graph theory with wide ranging applications. Graph labeling was first introduced in 1960's. A graph labeling is an assignment

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of integers to the vertices (edges / both) subject to certain conditions. If the domain of the mapping is the set of vertices (edges / both) then the labeling is called the vertex (edge / total) labeling.

Most popular graph labeling trace their origin to one introduced by Rosa [17]. Rosa called a function (labeling) f a β-valuation of a graph G with q edges if f is an injection from the vertices of G to the set $\{0,1,2,\ldots,q\}$ such that each edge xy is assigned the label $|f(x)-f(y)|$, the resulting edge labels are distinct and Golomb [7] called it as graceful labeling. Acharya [1] constructed certain infinite families of graceful graphs.

There are several types of graph labeling and a detailed survey is found in [6].

T. Tharmaraj and P.B. Sarasija [20, 21] introduced square graceful labeling and further studied in [10]. The concept of polygonal graceful labeling was introduced by D.S.T. Ramesh and M. P. Syed Ali Nisaya [15, 16, 19]. For more information related to graph labeling and its applications, see [3, 4, 9, 11, 12, 18, 22-32]. The following definitions are necessary for present study.

Definition 1.1: A graph G is a finite non-empty set of objects called vertices together with a set of unordered pairs of distinct vertices of G called edges. The vertex set and the edge set of G are denoted by V(G) and E(G) respectively. The number of elements of V(G) = p is called the order of G and the number of elements of E(G) = q is called the size of G. A graph of order p and size q is called a (p, q) - graph. If e = *uv* is an edges of G, we say that *u* and *v* are adjacent and that *u* and *v* are incident with e.

Definition 1.2: The degree of a vertex *v* in a graph G is defined to be the number of edges incident on v and is denoted by deg(v). A graph is called r-regular if deg(*v*) = r for each *v*∈ V(G). The minimum of {deg *v* : *v*∈ V(G) } is denoted by δ and maximum of {deg *v* : *v* ∈ V(G)} is denoted by △. A vertex of degree 0 is called an isolated vertex, a vertex of degree is called a pendant vertex or an end vertex.

Definition 1.3: A graph in which any two distinct points are adjacent is called a complete graph. The complete graph with n points is denoted by K_n.

Definition 1.4: A path P_n is obtained by joining u_i to the consecutive vertices u_{i+1} for $1 \leq i \leq n-1$.

Definition 1.5: The complete bipartite graph $K_{1,n}$ is called a Star graph

Definition 1.6: A graph, which can be formed from a given graph G by breaking up each edge into exactly two segments by inserting intermediate vertices between its two ends is called a sub division graph. It is denoted by S(G).

Definition 1.7: nG is a graph which contains n copies of the graph G. That is, $nG = \bigcup_{i=1}^{n} G_i$ where each $G_i = G$.

Definition 1.8: The bistar B(m, n) is the graph obtained from K_2 by joining m pendant edges to one end of K_2 and n pendant edges to the other end of K_2. The edge of K_2 is called the central edge of B(m, n) and the vertices of K_2 are called the central vertices of B(m, n).

Definition 1.9: A closed trail whose origin and internal vertices are distinct is called a Cycle. A cycle of length n is called n-cycle. It is denoted by C_n.

Definition 1.10: A connected acyclic graph is called a tree

Definition 1.11: The graph obtained by joining a single pendant edge to each vertex of a path P_n is called a Comb graph. It is denoted by $P_n \odot K_1$.

Definition 1.12: A coconut tree CT(m, n) is the graph obtained from the path P_m by appending n new pendant edges at an end vertex of P_m.

Definition 1.13: The Y- Tree is a graph obtained from path by appending an edge to a vertex of a path adjacent to an end point and it is denoted by Y_n where n is the number of vertices in the tree.

Definition 1.14: Shrub St($n_1, n_2, \ldots, n_m$) is a graph obtained by connecting a vertex v_0 to the central vertex of each of m number of stars.

Definition 1.15: A Second order triangular number is a number obtained by adding all the squares of positive integers less than or equal to a given positive integer n. If the n^{th} second order triangular number is denoted by B_n, then B_n=1^2+2^2+…+n^2. That is $B_n = \frac{1}{6}$n (n + 1) (2 n + 1). The second order triangular numbers are 1, 5, 14, 30, 55, 91, 140, 204, 285, 385, 506, 650,…

2. MAIN RESULTS

Definition 2.1: A second order triangular graceful labeling of a graph G is an one to one function $\varphi: V(G) \to \{0,1,2, \ldots, B_q\}$ where B_q is the q^{th} second order triangular number, ie., $B_q = \frac{1}{6} q(q+1)(2q+1)$, that induces a bijection φ^*: E(G) → $\{B_1, B_2, \ldots, B_q\}$ of the edges of G defined by $\varphi^*(uv) = |\varphi(u) - \varphi(v)|$ ∀ e = uv ∈ E(G). A graph which admits such labeling is called a second order triangular graceful graph.

Example 2.2: Second Order Triangular Graceful graph is shown in Figure 1.

Theorem 2.3:

The star $K_{1,n}$ is a second order triangular graceful graph for all n ≥ 1.

Proof:

Let G be a star graph $K_{1,n}$∀ n ≥ 1. Let v be the unique vertex in one partition of G and $v_1, v_2, \ldots, v_n$ be the n vertices in the other. Hence G has (n+1) vertices and n edges. Define $\varphi: V(G) \to \{0,1,2, \ldots, B_n\}$ by φ(v)=0 and $\varphi(v_i)$=B_i where 1≤ $i \le n$. Clearly φ is one to one. The induced edge function φ^*:E(G)→ $\{B_1, B_2, \ldots, B_n\}$ is defined as $\varphi^*(e_i)$=B_i where 1≤ $i \le n$.

Clearly φ^* is a bijection and φ^*(E(G))→ {B_1,B_2,…,B_n}.Thus G admits second order triangular graceful labeling. Hence the star $K_{1,n}$ is a second order triangular graceful graph for all n ≥ 1.

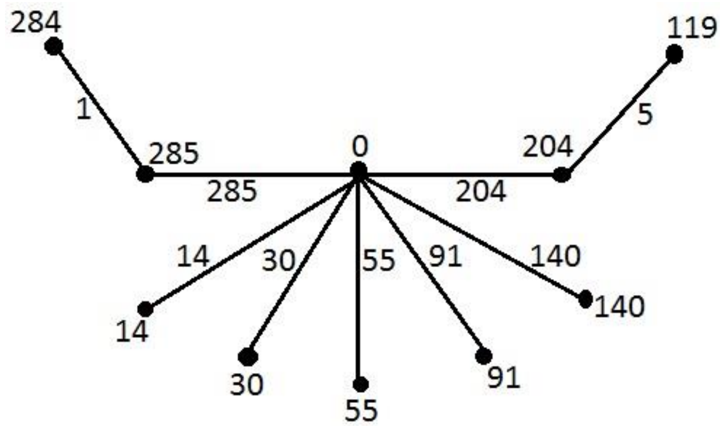

Figure 1.

Example 2.4: The second order triangular graceful labeling of $K_{1,7}$ is shown in Figure 2.

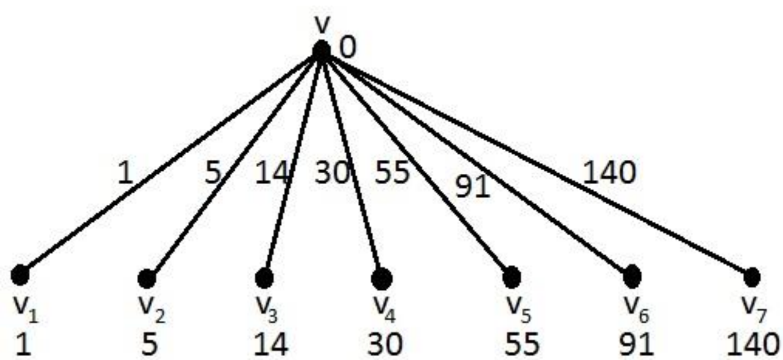

Figure 2.

Theorem 2.5:

S($K_{1,n}$), the subdivision of the star $K_{1,n}$ is a second order triangular graceful graph for all n≥ 1.

Proof: Let G be a subdivision graph of the star $K_{1,n}$ for all n≥ 1.

Let V(G)={ v,v_i,u_i : 1≤ $i \leq n$} and E(G)={ vv_i, v_iu_i : 1≤ $i \leq n$}.

Then G has 2n+1 vertices 2n edges. Define $\varphi: V(G) \rightarrow \{0,1,2,\dots,B_{2n}\}$ as follows.

$\varphi(v)$ = 0

$\varphi(v_i) = B_{2n-(i-1)}$ where 1≤ $i \leq n$

$\varphi(u_i)$ = $B_{2n-(i-1)}$-B_i where 1≤ $i \le n$

Clearly φ is one to one. The induced edge function φ^* : E(G)→ {B_1,B_2,…,B_{2n}} is defined as follows.

$\varphi^*(vv_i)$=$B_{2n-(i-1)}$ where 1≤ $i \le n$

$\varphi^*(v_iu_i)$=B_i where 1≤ $i \le n$

Clearly φ^* is a bijection and φ^*(E(G))→ {B_1,B_2,…,B_{2n}}.Therefore G admits second order triangular graceful labeling. Hence the graph S$(K_{1,n})$, for all n≥ 1 is a second order triangular graceful graph.

Example 2.6: The second order triangular graceful labeling of S($K_{1,7}$) is shown in Figure 3.

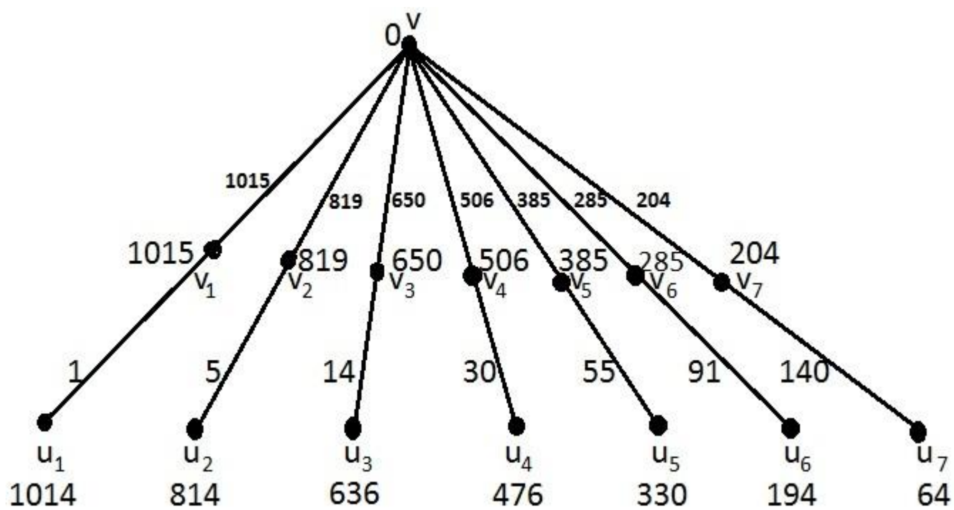

Figure 3.

Theorem 2.7:

n$K_{1,3}$ is a second order triangular graceful graph for all n≥ 1.

Proof: Let G be a graph which contains n copies of $K_{1,3}$.

Let V(G) = {x_i,u_i, v_i, w_i: where 1≤ $i \le n$} and E(G) = { x_iu_i, x_iv_i, x_iw_i: *where* 1≤ $i \le n$}.

Hence G has 4n vertices and 3n edges. Define $\varphi: V(G) \to \{0,1,2, , \dots , B_{3n}\}$ as follows.

$$\varphi\,(x_i) = \begin{cases} B_{3n} - 2(n-i) & if\ 1 \le i < n \\ 0 & if\ i = n \end{cases}$$

$$\varphi\,(u_i) = \begin{cases} \varphi(x_i) - B_{3i-2} & if\ 1 \le \mathrm{i} < \mathrm{n} \\ B_{3i-2} & if\ \mathrm{i} = \mathrm{n} \end{cases}$$

$$\varphi\,(v_i) = \begin{cases} \varphi(x_i) - B_{3i-1} & if\ 1 \le \mathrm{i} < \mathrm{n} \\ B_{3i-1} & if\ \mathrm{i} = \mathrm{n} \end{cases}$$

$$\varphi\,(w_i)=\begin{cases}\varphi(x_i)-B_{3i} & if\ \ 1 \le \mathrm{i} < \mathrm{n} \\ B_{3i} & if\ \ \mathrm{i}=\mathrm{n}\end{cases}$$

Clearly φ is one to one. The induced edge function φ^*: E(G)→ $\{B_1, B_2, \ldots, B_{3n}\}$ is defined as follows.

$$\varphi^*(x_i u_i)=\begin{cases} B_1 & if\ i=1 \\ B_4 & if\ i=2 \\ \quad . \\ \quad . \\ \quad . \\ B_{3n-2} & if\ i=n \end{cases}$$

ie, $\varphi^*(x_i u_i)=B_{3i-2}$ $where$ 1≤ $i \le n.$

$$\varphi^*(x_i v_i)=\begin{cases} B_2 & if\ i=1 \\ B_5 & if\ i=2 \\ \quad . \\ \quad . \\ \quad . \\ B_{3n-1} & if\ i=n \end{cases}$$

ie,$\varphi^*(x_i v_i)$= B_{3i-1} $where$ 1≤ $i \le n.$

And $\varphi^*(x_i w_i)=\begin{cases} B_{3,} & if\ i=1 \\ B_6, & if\ i=2 \\ \quad . \\ \quad . \\ \quad . \\ B_{3n,} & if\ i=n \end{cases}$

ie., $\varphi^*(x_i w_i)=B_{3i} where$ 1≤ $i \le n.$ Clearly φ^* is a bijection and φ^*(E(G))→ $\{B_1, B_2, \ldots, B_{3n}\}$. Therefore G admits second order triangular graceful labeling. Hence the graph n$K_{1,3}$ for all n≥ 1 is a second order triangular graceful graph.

Example 2.8:

The second order triangular graceful labeling of $4K_{1,3}$ is shown in Figure 4.

Theorem 2.9:

nK_2 is a second order triangular graceful graph for all n≥ 1.

Proof:

Let G be a graph which contains n copies of K_2.

Let V(G)=$\{v_{i1}, v_{i2}$:1≤ $i \le n\}$ and E(G)=$\{v_{i1}v_{i2}$:1≤ $i \le n\}$.

Hence G has 2n vertices and n edges. Define $\varphi: V(G) \rightarrow \{0,1,2,\ldots,B_n\}$ as follows.

$\varphi(v_{11}) = 0$

$\varphi(v_{12}) = B_n$

$\varphi(v_{i1}) = \sum_{j=1}^{i-1}(n-j)$ where $2 \leq i \leq n$

$\varphi(v_{i2}) = B_{n-(i-1)} + \varphi(v_{i1})$ where $2 \leq i \leq n$

Clearly φ is one to one. The induced edge function φ^*:E(G)→ $\{B_1,B_2,\ldots,B_n\}$ is defined as follows.

$\varphi^*(v_{i1}v_{i2}) = B_{n-(i-1)}, where\ 1 \leq i \leq n$

Clearly φ^* is a bijection and φ^*(E(G))= $\{B_1,B_2,\ldots,B_n\}$. Therefore G admits second order triangular graceful labeling. Hence the graph nK_2 for all n ≥ 1 is a second order triangular graceful graph.

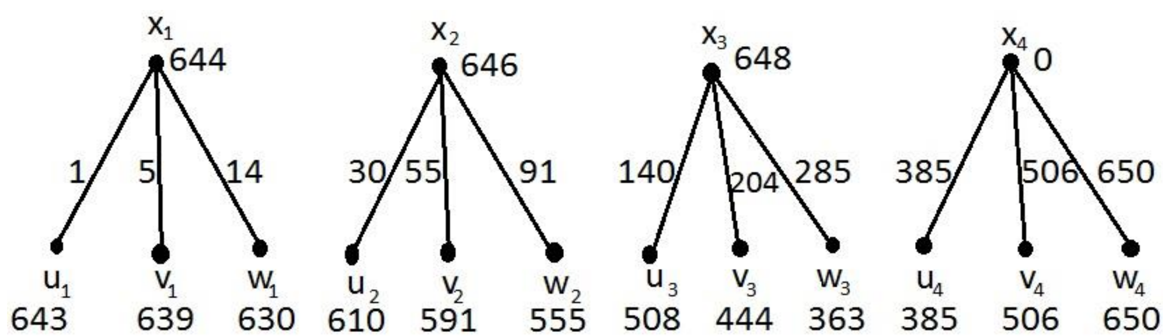

Figure 4.

Example 2.10: The second order triangular graceful labeling of $9K_2$ is shown in Figure 5

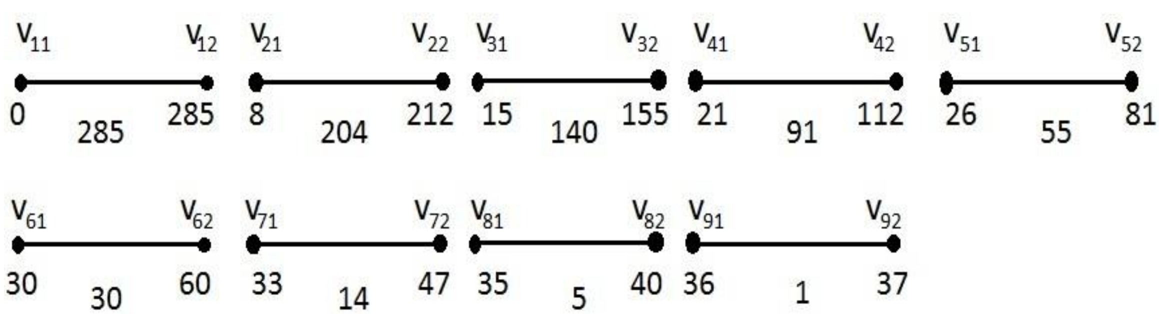

Figure 5.

Theorem 2.11:

The bistar B(m, n) is a second order triangular graceful graph for all m, n ≥ 1.

Proof: Let G be a bistar B(m ,n). Let V(G)={ u, v, u_i, v_j: 1≤ $i \leq m$;1≤ $i \leq n$} and E(G)={uv, uu_i, vv_j: 1≤ $i \leq m$;1≤ $i \leq n$}. Hence G has m+n+2 vertices and m+n+1 edges.

Define $\varphi: V(G) \rightarrow \{0,1,2,\ldots,B_{m+n+1}\}$ as follows.

φ(u) = 0

φ(v) = B_{m+n+1}

$\varphi(u_i)$ = $B_{m+n+1-i}$ where 1≤ $i \leq m$

$\varphi(v_j)$ = $B_{m+n+1} - B_j$ where 1≤ $j \leq n$

Clearly φ is one to one. The induced edge function φ^* : E(G)→ $\{B_1, B_2, \ldots, B_{m+n+1}\}$ is defined as follows.

φ^* (uv) =B_{m+n+1},

φ^* (uu_i) =$B_{m+n+1-i}$ where 1≤ $i \leq m$

$\varphi(vv_j)$ =B_j where 1≤ $j \leq n$

Clearly φ^* is a bijection and φ^*(E(G))= $\{B_1, B_2, \ldots, B_{m+n+1}\}$.

Therefore G admits second order triangular graceful labeling. Hence the graph B(m, n) for all m, n≥ 1 is a second order triangular graceful graph.

Example 2.12: The second order triangular graceful labelling of B (9,5) is shown in Figure 6

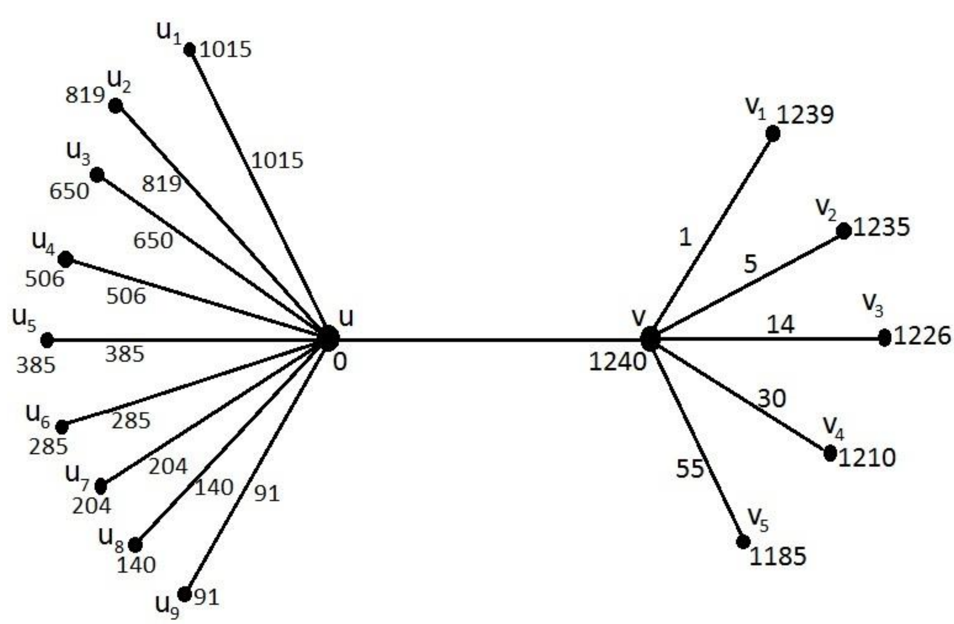

Figure 6.

Theorem 2.13:

The path P_n on n vertices is a second order triangular graceful graph for all n ≥ 2.

Proof: Let G be a path P_n on n vertices where n≥ 2. Let E(G)={v_1,v_2,…,v_n} and V(G)={$v_i v_{i+1}$:1≤ $i \le n-1$}. Then G has n vertices and n-1 edges. Let s = n-1.

Define $\varphi: V(G) \to \{0,1,2,,\ldots,B_s\}$ as follows

$\varphi(v_1)$ = 0

$$\varphi(v_i) = \begin{cases} \varphi(v_{i-1}) - B_{s-(i-2)} \; if\ i\ is\ odd\ 2 \le i \le n \\ \varphi(v_{i-1}) + B_{s-(i-2)} \; if\ i\ is\ even\ 2 \le i \le n \end{cases}$$

Clearly φ is one to one. The induced edge function φ^* : E(G)→ $\{B_1,B_2,\ldots,B_s\}$ is defined as

φ^* $(v_i v_{i+1})$ = B_{n-i} ,$1 \le i \le n-1$.

Clearly φ^* is a bijection and φ^*(E(G)) $= \{B_1,B_2,\ldots,B_{n-1}\}$. Therefore G admits second order triangular graceful labeling. Hence the path P_n on n vertices is a second order triangular graceful graph for all n ≥ 2.

Example 2.14: The second order triangular graceful labeling of P_9 is shown in Figure 7.

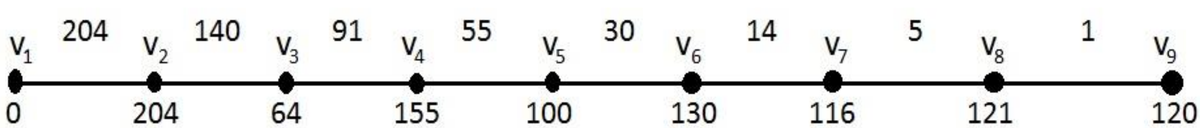

Figure 7.

Theorem 2.15

The comb graph $P_n \odot K_1$ is a second order triangular graceful graph for all n≥ 2.

Proof: Let G be a comb graph $P_n \odot K_1$. Then V(G)={u_i, w_i: where 1≤ $i \le n$} and E(G) = { $u_i u_{i+1}$: *where* 1≤ $i \le n-1\}\cup\{u_i w_i$: where 1≤ $i \le n$}

Hence G has 2n vertices and 2n-1edges.

Let s = 2n-1

Define $\varphi: V(G) \to \{0,1,2,\ldots,B_s\}$ as follows.

$\varphi(u_1)$ = 0

$$\varphi(u_i) = \begin{cases} \varphi(u_{i-1}) - B_{s-(i-2)\ if\ i\ is\ odd\ 2\le i\le n} \\ \varphi(u_{i-1}) + B_{s-(i-2)\ if\ i\ is\ even\ 2\le i\le n} \end{cases}$$

$\varphi(w_1) = B_{2s+1}$

$\varphi(w_i) = \varphi(u_i) + B_{s+(i-1)}$, 2≤ $i \le n$

Clearly φ is one to one. The induced edge function φ^*:E(G)→ $\{B_1,B_2,\ldots,B_{2n-1}\}$ is defined as follows.

φ^* $(u_i u_{i+1})$ = B_{n-i} , 1≤ $i \le n-1$

$\varphi(u_1w_1) = B_{2s+1}$

$\varphi(u_iw_i) = B_{s+(i-1)}$, $2\leq i \leq n$

Clearly φ^* is a bijection and φ^*(E(G))→ $\{B_1,B_2,\ldots,B_{2n-1}\}$. Therefore G admits second order triangular graceful labeling. Hence the comb $P_n \odot K_1$ is a second order triangular graceful graph for all n≥ 2.

Example 2.16: The second order triangular graceful labeling of $P_7 \odot K_1$ is shown in figure 8.

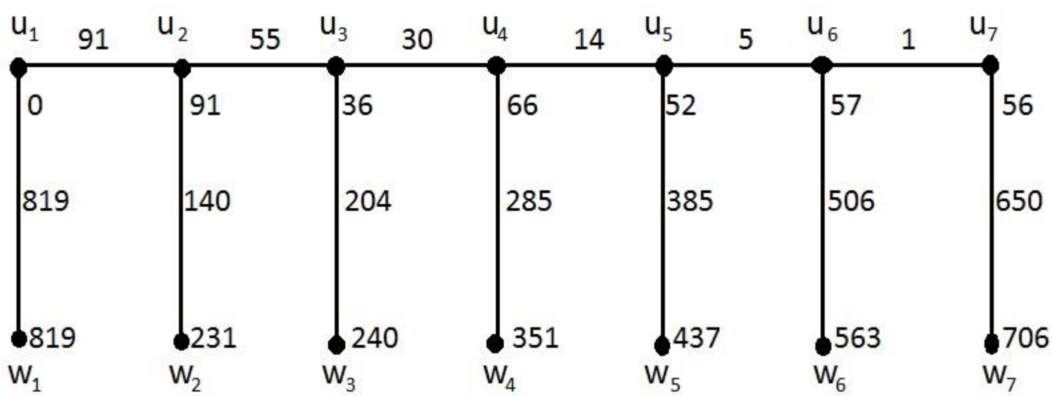

Figure 8.

Theorem 2.17:

Coconut tree CT(m ,n) is a second order triangular graceful graph for all m, n≥ 1

Proof: Let G be a coconut tree CT(m ,n).Then V(G)=$\{w_j, v_i, 1\leq j \leq m;\ 1\leq i \leq n\}$ and E(G) = $\{v_1w_j, v_iv_{i+1};\ 1\leq j \leq m;\ 1\leq i \leq n-1\}$. Hence G has m+n vertices and m+n-1 edges.

Let S = m+n. Define $\varphi: V(G) \rightarrow \{0,1,2,\ldots,B_s\}$ as follows.

$\varphi(v_1) = 0$

$$\varphi(v_i) = \begin{cases} \varphi(v_{i-1}) - B_{n-(i-2)} \text{ if } i \text{ is odd } 2\leq i\leq n \\ \varphi(v_{i-1}) + B_{n-(i-2)} \text{ if } i \text{ is even } 2\leq i\leq n \end{cases}$$

$\varphi(w_i) = B_{s-(j-1)}$; $1\leq j \leq m$

Clearly φ is one to one. The induced edge function φ^*: E(G)→ $\{B_1,B_2,\ldots,B_{m+n-1}\}$ is defined as follows.

$\varphi^*(v_iv_{i+1}) = B_{n-i}$; $1 \leq i \leq n-1$

$\varphi^*(v_1w_j) = B_{s-(j-1)}$; $1\leq j \leq m$ and $s = m+n$

Clearly φ^* is a bijection and φ^*(E(G))→ $\{B_1,B_2,\ldots,B_{m+n-1}\}$.

Therefore G admits second order triangular graceful labeling.

Hence the graph CT (m, n) is a second order triangular graceful graph.

Example 2.18: The second order triangular graceful labelling of CT (5, 4) is shown in Figure 9.

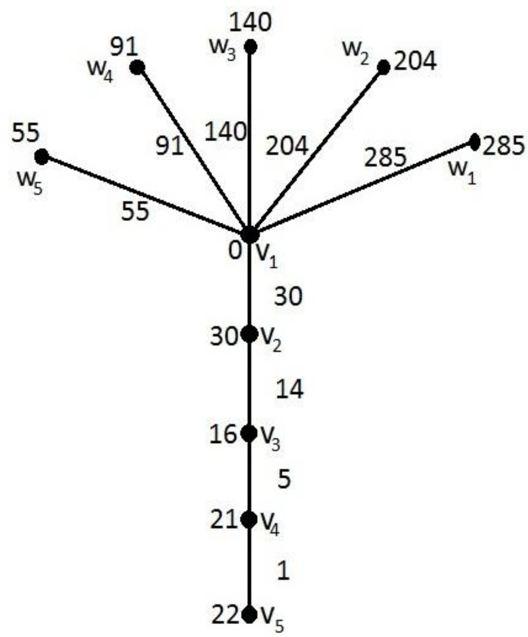

Figure 9.

Theorem 2.19

Shrub St($n_{1,}n_{2,....,}n_s$) is a second order triangular graceful graph.

Proof:

Let $K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_s}$ be a s different stars that are attached by their central vertices to one vertex v. Then G has $n_1+n_2+\ldots+n_s$+s+1 vertices and $n_1+n_2+\ldots+n_s$+s edges. We give v the label 0. Also we give the central vertices of these stars the labels:

$B_{n_1+n_2+\cdots+n_s+s,}B_{n_1+n_2+\cdots+n_s+s-1,\ldots,}B_{n_1+n_2+\cdots+n_s+s-(s-1)}$ respectively and finally we give the end vertices of the first star the labels:

$B_{n_1+n_2+\cdots+n_s+s} - B_1, B_{n_1+n_2+\cdots+n_s+s} - B_2,\ B_{n_1+n_2+\cdots+n_s+s} - B_{n_1}$,the end vertices of the second star the labels:

$B_{n_1+n_2+\cdots+n_s+s-1} - B_{n_1+1},\ B_{n_1+n_2+\cdots+n_s+s-1} - B_{n_1+2,\ldots,}B_{n_1+n_2+\cdots+n_s+s-1} - B_{n_1+n_2}$, the end vertices of the third star the labels:

$B_{n_1+n_2+\cdots+n_s+s-2} - B_{n_1+n_{2+1}}, B_{n_1+n_2+\cdots+n_s+s-2} - B_{n_1+n_{2+2,\ldots,}}B_{n_1+n_2+\cdots+n_s+s-2} - B_{n_1+n_{2+n_3}}$ and so on, until the last star, we give its end vertices the labels

$B_{n_1+n_2+\cdots+n_s+1} - B_{n_1+n_2+\cdots+n_{s-1}+1}, B_{n_1+n_2+\cdots+n_s+1} - B_{n_1+n_2+\cdots+n_{s-1}+2}, \ldots, \ldots, B_{n_1+n_2+\cdots+n_s+1} - B_{n_1+n_2+\cdots+n_{s-1}+n_s}$.

Clearly the edge labels are the second order triangular graceful numbers $B_1, B_2, \ldots, B_{n_1+n_2+\cdots+n_{s-1}+n_{s+s}}$ and also the edge labels are all distinct.

Hence G is a second order triangular graceful graph.

Example 2.20: Second order triangular graceful labeling of the Shrub St(2, 3, 3, 2, 3) is shownin Figure 10.

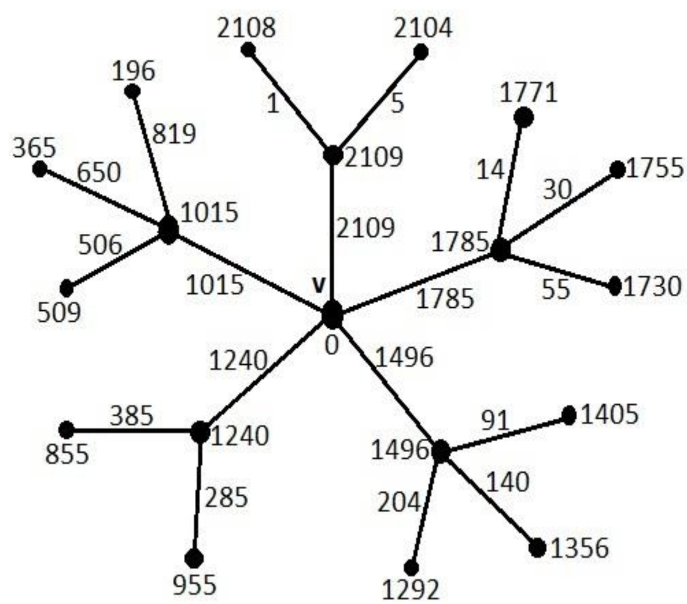

Figure 10.

Theorem 2.21

Any Y-tree Y_n is a second order triangular graceful graph.

Proof: Let G be a Y-tree Y_n. Let V(G)={v_i: $1 \le i \le n$} and

E(G) = $\{v_i v_{i+1}: 1 \le i \le n-2 \text{ and } v_{n-2}v_n\}$. Hence G has n vertices and n-1 edges.

Let s = n. Define $\varphi: V(G) \to \{0,1,2,\ldots,B_{n-1}\}$ as follows.

$\varphi(v_1) = 0$

$$\varphi(v_i) = \begin{cases} \varphi(v_{i-1}) - B_{s-(i-2)} \text{ if } i \text{ is odd } 2 \le i \le n \\ \varphi(v_{i-1}) + B_{s-(i-2)} \text{ if } i \text{ is even } 2 \le i \le n \end{cases}$$

$\varphi(v_n) = \varphi(v_{n-2}) + 1$

Clearly φ is one to one. The induced edge function φ^*:E(G)→ $\{B_1, B_2, \ldots, B_{n-1}\}$ is defined as

$\varphi^*(v_i v_{i+1}) = B_{n-i}$,$1 \le i \le n-2$

$\varphi^*(v_{n-2}v_n) = B_1$

Clearly φ^* is a bijection and φ^*(E(G))→ $\{B_1, B_2, \ldots, B_{n-1}\}$.

Therefore G admits second order triangular graceful labeling.

Hence the graph Y-tree is a second order triangular graceful graph.

Example 2.22: The second order triangular graceful labeling of Y_7 is shown in Figure 11.

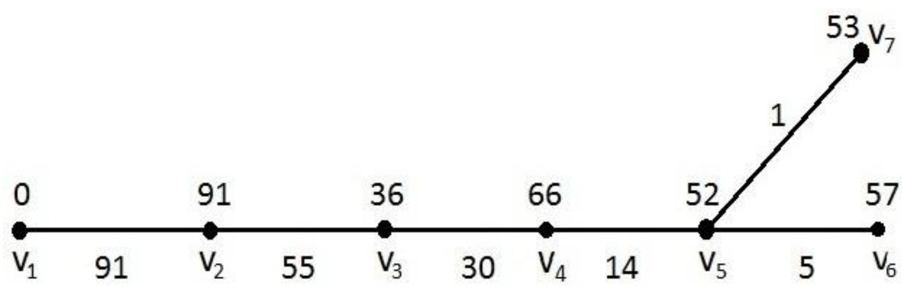

Figure 11.

3. CONCLUSIONS

In this paper, we have introduced and studied the second order triangular graceful labeling of some graphs. This work contributes several new results to the theory of graph labeling.

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