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Some results on centered triangular sum graphs

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ABSTRACT

A centered triangular sum labeling of a graph G is a one-to-one function f : V (G) → N ∪ {0} that induces a bijection f^{*}: E(G) $\rightarrow \{B_1, B_2, \dots B_q\}$ of the edges of G defined by $f^{*}(uv) = f(u) + f(v)$, for all e = uv ∈ E(G). The graph which admits such labeling is called a centered triangular sum graph.

***Keywords*:** Centered triangular numbers, centered triangular sum labeling, centered triangular sum graphs

1. INTRODUCTION AND DEFINITIONS

The graph considered in this paper are finite, undirected and without loops or multiple edges. Let G = (V, E) be a graph with p vertices and q edges. Undefined terms are used in the sense of Harary [8], Parthasarathy [17] and Bondy and U.S.R. Murthy [3]. For number theoretic terminology, we refer to [1] and [16].

Graph labeling is one of the fascinating areas of graph theory with wide ranging applications. Graph labeling was first introduced in 1960's. A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. If the domain of the mapping is the set of vertices (edges / both) then the labeling is called the vertex (edge / total) labeling. Most popular graph labeling trace their origin to one introduced by Rosa [19].

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Rosa called a function (labeling) f a β-valuation of a graph in the year 1966 and Golomb [7] called it as graceful labeling. There are several types of graph labeling and a detailed survey is found in [6].

The concept of a sum graph was introduced by Harary [9] in 1990 and was defined as a graph whose vertices can be labeled with distinct positive integers so that the sum of the labels on each pair of adjacent vertices is the label of some other vertex. In 1991, Harary et al. [11] defined a real sum graph. One of the earliest interesting results was due to Ellingham [5] who proved the conjecture of Harary [9].

In [14], the concept of centered triangular sum labeling was introduced. Jeyanthi et al. [13] introduced centered triangular mean labeling. For more information related to sum graphs, see [2, 12, 15, 18, 22-33]. The following definitions are necessary for present study.

Definition 1.1: A graph G is a finite non-empty set of objects called vertices together with a set of unordered pairs of distinct vertices of G called edges. The vertex set and the edge set of G are denoted by V(G) and E(G) repectively. The number of elements of V(G) = p is called the order of G and the number of elements of E(G) = q is called the size of G. A graph of order p and size q is called a (p,q) - graph. If e = uv is an edges of G, we say that u and v are adjacent and that u and v are incident with e.

Definition 1.2: The degree of a vertex v in a graph G is defined to be the number of edges incident on v and is denoted by deg(v). A graph is called r-regular if deg(v) = r for each $v \in$ V(G). The minimum of {deg v: $v \in$ V(G)} is denoted by δ and maximum of {deg v: $v \in$ V(G)} is denoted by △. A vertex of degree 0 is called an isolated vertex, a vertex of degree is called a pendant vertex or an end vertex.

Definition 1.3: A graph in which any two distinct points are adjacent is called a complete graph. The complete graph with n points is denoted by K_n.

Definition 1.4: A Path P_n is obtained by joining u_i to the consecutive vertices u_{i+1} for $1 \leq i \leq$ n-1.

Definition 1.5: A closed trail whose origin and internal vertices are distinct is called a Cycle. A cycle of length n is called n-cycle. It is denoted by C_n.

Definition 1.6: A connected acyclic graph is called a tree

Definition 1.7: The Y- Tree is a graph obtained from path by appending an edge to a vertex of a path adjacent to an end point and it is denoted by Y_n where n is the number of vertices in the tree.

Definition 1.8: Let P_n be the path on n vertices. Then the Twig graph obtained from the path P_n by attaching exactly two pendant edges to each internal vertex of the path and it is denoted by TW(P_n).

Definition 1.9: A (n, m) Balloon tree is a graph obtained by connecting one leaf of each of n-copies of a $K_{1,m}$ star graph. Let us denote it by $BL_{n,m}$.

Definition 1.10: F-Tree on $n+2$ vertices denoted by FP_n, is obtained from a path P_n by attaching exactly two pendant vertices to the n-1 and n^{th} vertex of P_n.

Definition 1.11: The complete bipartite graph $K_{1,n}$ is called a Star graph.

Definition 1.12: A lobster graph is a tree having the property that the removal of leaf nodes leaves a caterpillar graph.

Definition 1.13: A caterpillar is a tree with a path P_m: $v_1, v_2, \ldots, v_m$, called spine with leaves (pendant vertices) known as feet attached to the vertices of the spine by edges known as legs. If every spine vertex v_i is attached with n_i number of leaves then the caterpillar is denoted by $S(n_1, n_2, \ldots, n_m)$.

Definition 1.14: Shrub $St(n_1, n_2, \ldots, n_m)$ is a graph obtained by connecting a vertex v_0 to the central vertex of each of m number of stars.

Definition 1.15: The graph $P_m@P_n$ is obtained from P_m and m copies of P_n by identifying one pendant vertex of the i^{th} copy of P_n with i^{th} vertex of P_m where P_m is a path of length of m−1.

Definition 1.16: Let G be a graph with fixed vertex v and let (P_m: G) be the graph obtained from m copies of and the path P_m: $u_1, u_2, \ldots, u_m$ by joining u_i with the vertex v of the i^{th} copy of G by means of an edge for $1 \leq i \leq n$

Definition 1.17: Banana tree $Bt(n_1, n_2, \ldots, n_m)$ is a graph obtained by connecting a vertex v_0 to one leaf of each of m number of stars.

Definition 1.18: A centered triangular number is a centered figurate number that represents a triangle with a dot in the center and all other dots surrounding the center in successive triangular layers. If the nth centered triangular number is denoted by B_n, then $B_n = \frac{1}{2}(3n^2 - 3n + 2)$.
The first few centered triangular numbers are: 1, 4, 10, 19, 31, 46, 64, 85, 109, 136, 166, 199, 235, 274, ...

Definition 1.19: A Sum labeling is an injective function $f : V(G) \to N \cup \{0\}$ that induces a bijection $f^+ : E(G) \to \{1, 2, \ldots q\}$ of edges G defined by $f^+(uv) = f(u) + f(v)$, for all $e = uv \in E(G)$. The graph which admits such labeling is called a sum graph.

Definition 1.20: A centered triangular sum labeling of a graph G is a one-to-one function $f : V(G) \to N \cup \{0\}$ that induces a bijection $f^* : E(G) \to \{B_1, B_2, \ldots B_q\}$ of the edges of G defined by $f^*(uv) = f(u) + f(v)$, for all $e = uv \in E(G)$. The graph which admits such labeling is called a centered triangular sum graph.

2. SOME KNOWN RESULTS [14]

Theorem 2.1: The path P_n admits centered triangular sum labeling.

Theorem 2.2: The comb $P_n \odot K_1$ admits centered triangular sum labeling.

Theorem 2.3: The star $K_{1,n}$ graph admits centered triangular sum labeling.

Theorem 2.4: S($K_{1,n}$), the subdivision of the star $K_{1,n}$ admits centered triangular sum labeling.

Theorem 2.5: The bistar $B_{m,n}$ admits centered triangular sum labeling.

Theorem 2.6: Coconut tree admits centered triangular sum labeling.

3. MAIN RESULTS

Theorem 3.1: Any Y_n tree is a centered triangular sum graph.

Proof: Let G be a Y_n tree .
Let V(G) = $\{v_i : 1 \leq i \leq n\}$ and
E(G) = $\{v_i v_{i+1} : 1 \leq i \leq n-2 \; and \; v_{n-2} v_n\}$.
Here G has n vertices and n-1 edges.
Define f: V(G) → $\{0,1,\ldots B_{n-1}\}$ as follows
$f(v_1) = 0$
For 2≤ $i \leq n-1$, $f(v_i) = B_{i-1} - f(v_{i-1})$ and
$f(v_n) = B_{n-1} - f(v_{n-2})$.
Clearly f is injective and f induces a bijective function f^* : E(G) →$\{1,4,10,\ldots B_{n-1}\}$ as
$f^*(v_i v_{i+1}) = B_i$, 1≤ $i \leq n-2$ and
$f^*(v_{n-2} v_n) = B_{n-1}$.
Hence the edge labels are 1,4,… B_{n-1}.
Thus f is a centered triangular sum labeling of G.
Therefore, G = Y_n tree is a centered triangular sum graph.

Example 3.2: The centered triangular sum labeling of Y_7 is shown in Fig. 1.

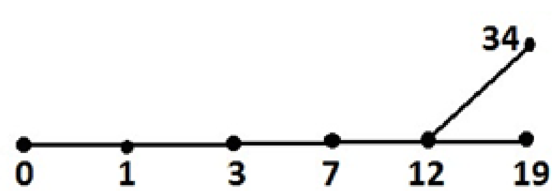

Fig. 1

Theorem 3.3: Any Twig graph TW(P_n) is a centered triangular sum graph.

Proof: Let G be a Twig graph TW(P_n).

Let V(G) = { v_i, u_j, w_j : $1 \leq i \leq n$ and $2 \leq j \leq n$-1 } and

E(G) = $\{v_i v_{i+1} : 1 \leq i \leq n-1\}$ ∪ $\{ v_j u_j, v_j w_j : 2 \leq j \leq n\text{-}1\}$.

Here G has 3n-4 vertices and 3n-5 edges.

Define f: V(G) ⟶ $\{0,1,\dots B_{3n-5}\}$ as follows

$f(v_1) = 0$

For $2 \leq i \leq n$, $f(v_i) = B_{i-1} - f(v_{i-1})$

For $2 \leq j \leq n$-1, $f(u_j) = B_{n+(2j-4)} - f(v_j)$ and

$f(w_j) = B_{(n+1)+(2j-4)} - f(v_j)$.

Clearly f is injective and f induces a bijective function f^* : E(G) ⟶$\{1,4,\dots B_{3n-5}\}$ as

$f^*(v_i v_{i+1}) = B_i$, $1 \leq i \leq n-1$

$f^*(v_j u_j) = B_{n+2(j-2)}$ and $f^*(v_j w_j) = B_{(n+1)+2(j-2)}$, $2 \leq j \leq n$-1.

Hence the edge labels are 1,4,… B_{3n-5}.

Thus f is a centered triangular sum labeling of G.

Therefore, G = TW(P_n) is a centered triangular sum graph.

Example 3.4: The centered triangular sum labeling of TW(P_4) is shown in Fig. 2.

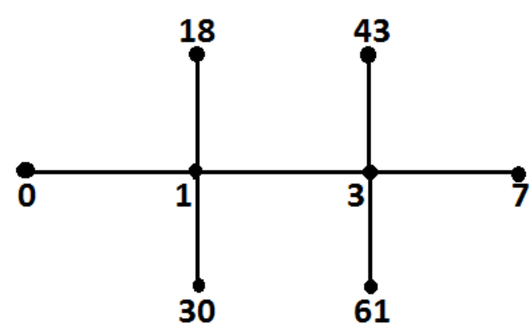

Fig. 2

Theorem 3.5: The balloon tree $BL_{2,m}$ where m≥ 1 is a centered triangular sum graph.

Proof: Let G be a balloon tree $BL_{2,m}$ where m≥ 1.

Let V(G) = $\{v_{00}, v_{10}, v_{1j}, v_{20}, v_{2j} : 1 \leq j \leq m\}$ and

E(G) = $\{v_{00}v_{10}, v_{10}v_{1j}, v_{00}v_{20}, v_{20}v_{2j} : 1 \leq j \leq m\}$.

Here G has 2m+3 vertices and 2m+2 edges.

Define f: V(G) → $\{0,1,\ldots B_{2m+2}\}$ as follows

$f(v_{10})$ = 0

$f(v_{1j}) = B_j$, $1 \leq j \leq m$

$f(v_{00}) = B_{m+1}$

$f(v_{20}) = B_{m+2} - B_{m+1}$

$f(v_{2j}) = B_{m+3+(j-1)}$- $f(v_{20})$, $1 \leq j \leq m$.

Clearly f is injective and f induces a bijective function f^*: E(G) →$\{1,4,\ldots B_{2m+2}\}$ as

$f^*(v_{10}v_{1j}) = B_j$, $1\leq j \leq m$

$f^*(v_{10}v_{00}) = B_{m+1}$

$f^*(v_{00}v_{20}) = B_{m+2}$ and

$f^*(v_{20}v_{2j}) = B_{m+3+(j-1)}$, $1\leq j \leq m$.

Hence the edge labels are $1,4,\ldots B_{2m+2}$.

Thus f is a centered triangular sum labeling of G.

Therefore, G = $BL_{2,m}$ is a centered triangular sum graph.

Example 3.6: The centered triangular sum labeling of $BL_{2,2}$ is shown in Fig. 3.

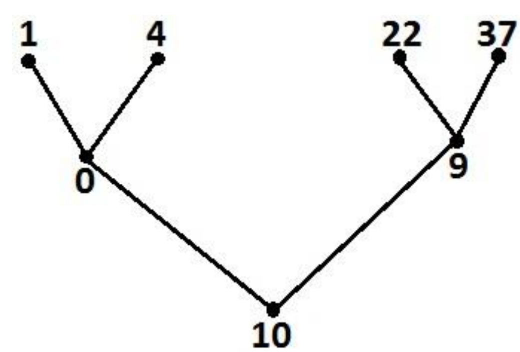

Fig. 3

Theorem 3.7: A F- tree FP_n , n≥ 3 is a centered triangular sum graph.

Proof: Let G be a F- tree FP_n , n≥ 3.

Let V(G) = $\{u, v, v_i : 1 \leq i \leq n\}$ and

E(G) = $\{v_iv_{i+1}: 1 \leq i \leq n-1\} \cup \{uv_{n-1}, vv_n\}$.

Here G has n+2 vertices and n+1 edges.

Define f: V(G) ⟶ $\{0,1,\ldots B_{n+1}\}$as follows

$f(v_1) = 0$

For $2 \leq i \leq n,\ f(v_i) = B_{i-1}\text{-}\, f(v_{i-1})$

f(u) = $B_n\text{-}\, f(v_{n-1})$ and

$f(v) = B_{n+1}\text{-}\, f(v_n)$.

Clearly f is injective and f induces a bijective function f^*: E(G) ⟶$\{1,4,\ldots B_{n+1}\}$ as

$f^*(v_i v_{i+1}) = B_i$, $1 \leq i \leq n-1$

$f^*(u) = B_n$,

$f^*(v) = B_{n+1}$.

Hence the edge labels are $1,4,\ldots B_{n+1}$.

Thus f is a centered triangular sum labeling of G.

Therefore, G = FP_n is a centered triangular sum graph.

Example 3.8: The centered triangular sum labeling of FP_6 is shown in Fig. 4.

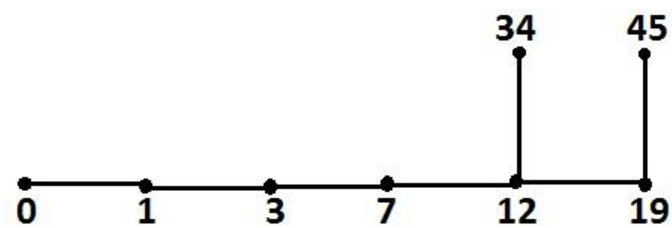

Fig. 4

Theorem 3.9: Let G be a graph obtained by identifying a pendant vertex of P_m with a leaf of $K_{1,n}$. Then G is a centered triangular sum graph for all values of m and n.

Proof: Let V(G) = $\{v,\ v_i,\ u_j : 1 \leq i \leq n,\ 1 \leq j \leq m\}$ and

E(G) = $\{vv_i, vu_1, u_j u_{j+1} : 1 \leq i \leq n,\ 2 \leq j \leq m-1\}$.

Here G has m + n vertices and m + n - 1 edges.

Define f: V(G) ⟶ $\{0,1,\ldots,B_{m+n-1}\}$as follows

$f(v) = 0$

$f(v_i) = B_i$, $1 \leq i \leq n$

$f(u_1) = B_n\text{-}\, f(v)$

For $2 \leq j \leq m,\ f(u_j) = B_{n+(j-1)}\text{-}\, f(u_{j-1})$.

Clearly f is injective and f induces a bijective function f^*: E(G) ⟶$\{1,4,\ldots B_{m+n-1}\}$ as

$f^*(vv_i) = B_i$, $1 \leq i \leq n$

$f^*(vu_1) = B_n$ and

$f^*(u_j u_{j+1}) = B_{n+(j-1)}$, $2 \leq j \leq m$-1.

Hence the edge labels are 1,4,… B_{m+n-1}.

Thus f is a centered triangular sum labeling of G.

Therefore, G is a centered triangular sum graph.

Example 3.10: A pendent vertex of P_5 with a leaf of $K_{1,6}$ is shown in Fig. 5.

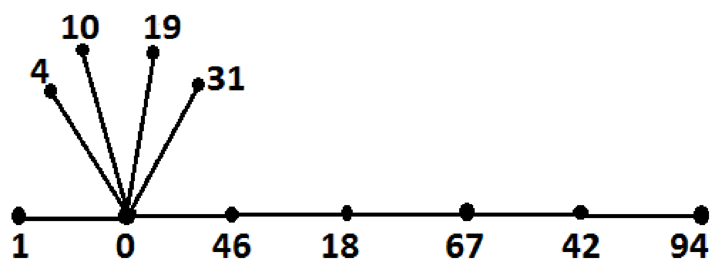

Fig. 5

Theorem 3.11: The lobster G obtained by joining the centres of k copies of a star to a new vertex w is a centered triangular sum graph.

Proof: Let G be a lobster obtained by joining the centres of k stars $K_{1,n}$ with a vertex w.

Denote the root vertex of i^{th} star $K_{i,n}$ as w_i, i = 1,2,…k and the pendent vertices of i^{th} star as w_{ij}, i = 1,2,…k, j = 1,2,…n.

That is, V(G) = $\{w,\ w_i,\ w_{ij}\ : 1 \leq i \leq k, 1 \leq j \leq n\}$ and

E(G) = $\{ww_i, w_i w_{ij} : 1 \leq i \leq k, 1 \leq j \leq n\}$.

Here G has nk+k+1 vertices and nk+k edges.

Define f : V(G) ⟶ $\{0,1,\ldots B_{nk+k}\}$as follows

$f(w) = 1$

$f(w_i) = B_i$ -1, $1 \leq i \leq k$ and

$f(w_{ij}) = B_{k+j+m}$- $f(w_i)$, $1 \leq i \leq k, 1 \leq j \leq n,\ m = (i-1)n.$

Since $f(w_{ij}) < f(w_{ij+1})$ for some i,j, we have $f(w_{ij}) + f(w_i) < f(w_{ij+1}) + f(w_i)$ and one can see that

$f^*(G) = \{B_1, B_2, \ldots B_{nk+k}\}$.

Thus G is a centered triangular sum graph.

Example 3.12:

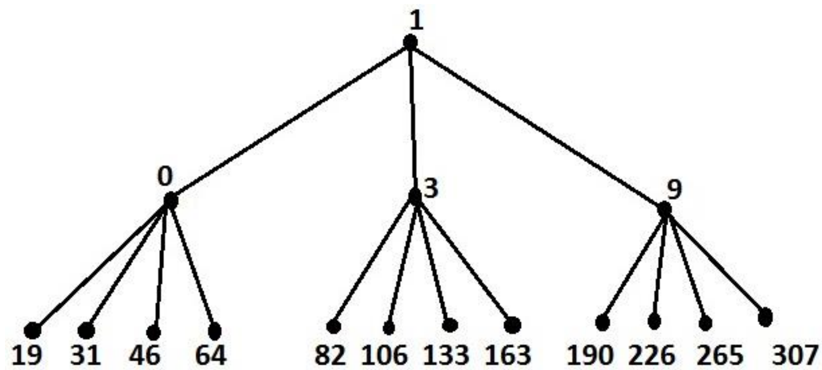

Fig. 6

Theorem 3.13: The caterpillar S($n_{1,}n_2, \dots n_m$) where m ≥ 3 is a centered triangular sum graph.

Proof: Let G be a caterpillar S($n_{1,}n_2, \dots n_m$) graph, m ≥ 3.

Let V(G) = $\{v_i,\ v_{ij}\ : 1 \leq i \leq m,\ 1 \leq j \leq n_i\}$ and

E(G) = $\{v_t v_{t+1}, v_i v_{ij} : 1 \leq t \leq m-1, 1 \ \leq \ i \ \leq \ m, 1 \ \leq \ j \ \leq \ n_i\}$.

Here G has $n_1 + n_2 + \cdots + n_m + m$ vertices and $n_1 + n_2 + \cdots + n_m + m - 1$ edges.

Let k = $n_1 + n_2 + \cdots + n_m + m$-1.

Define f: V(G) ⟶ $\{0,1,\dots B_k\}$as follows

$f(v_1) = 0$

$f(v_i) = B_{i-1} - B_{i-2} + B_{i-3} - \cdots + (-1)^i B_1$ for $2 \leq i \leq m$

$f(v_{1j}) = B_{m-1+j}$ for $1 \ \leq \ j \ \leq \ n_i = n_1$

$f(v_{ij}) = B_{m-1+n_1+n_2+\cdots+n_{i-1}+j} + (-1)^{i-1} (B_1 - B_2 + B_3 - \cdots + (-1)^i B_{i-1})$;

$2 \leq i \leq m, 1 \ \leq \ j \ \leq \ n_i$.

Clearly f is injective and f induces a bijective function f^*: E(G) ⟶$\{1,4,\dots B_k\}$ as

$f^*(v_t v_{t+1}) = B_t$, $1 \leq t \leq m-1$,

$f^*(v_1 v_{1j}) = B_{m-1+j}$ for $1 \ \leq \ j \ \leq \ n_i = n_1$ and

$f^*(v_i v_{ij}) = B_{m-1+n_1+n_2+\cdots+n_{i-1}+j}$, $2 \leq i \leq m, 1 \ \leq \ j \ \leq \ n_i$.

Hence the edge labels are 1,4,… B_k.

Thus f is a centered triangular sum labeling of G.

Therefore, G is a centered triangular sum graph.

Example 3.14: The centered triangular sum labeling of $S(3,4,5,6)$ is shown in Fig. 7.

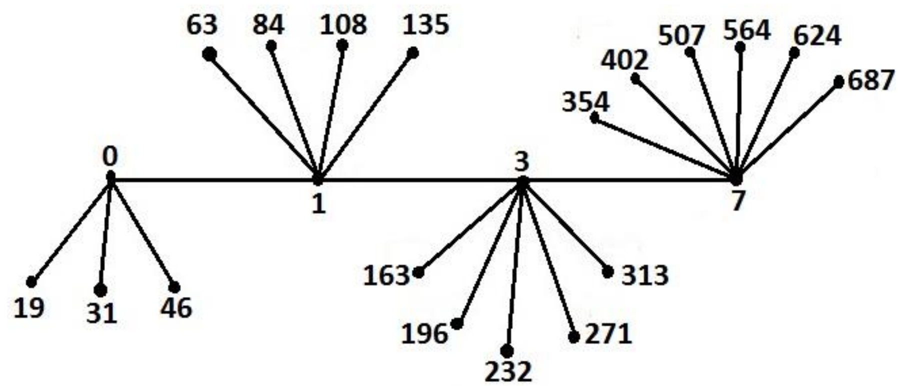

Fig. 7

Theorem 3.15: The Shrub $St(n_{1,}n_2, \dots n_m)$ is a centered triangular sum graph.

Proof: Let G be a Shrub $St(n_{1,}n_2, \dots n_m)$ graph.

Let V(G) = $\{v_o,\ v_i,\ v_{ij}\ : 1 \le i \le m, 1 \le j \le n_i\}$ and

E(G) = $\{v_0 v_i, v_i v_{ij} : 1 \le i \le m, 1 \le j \le n_i\}$.

Here G has $n_1 + n_2 + \cdots + n_m + m + 1$ vertices and $n_1 + n_2 + \cdots + n_m + m$ edges.

Let k = $n_1 + n_2 + \cdots + n_m + m$.

Define f : V(G) → $\{0,1,\dots B_k\}$as follows

$f(v_0) = 0$

$f(v_i) = B_i$, $1 \le i \le m$

$f(v_{ij}) = B_{m+n_1+n_2+\cdots+n_{i-1}+j} - B_i,\ 1 \le i \le m, 1 \le j \le n_i$.

Clearly f is injective and f induces a bijective function f^* : E(G) →$\{1,4,\dots B_k\}$ as

$f^*(v_0 v_i) = B_i$, $1 \le i \le m$ and

$f^*(v_i v_{ij}) = B_{m+n_1+n_2+\cdots+n_{i-1}+j}$, $1 \le i \le m, 1 \le j \le n_i$.

Hence the edge labels are $1,4,\dots B_k$.

Thus f is a centered triangular sum labeling of G.

Therefore, G is a centered triangular sum graph.

Example 3.16: The centered triangular sum labeling of St(4,5,6,7) is shown in Fig. 8.

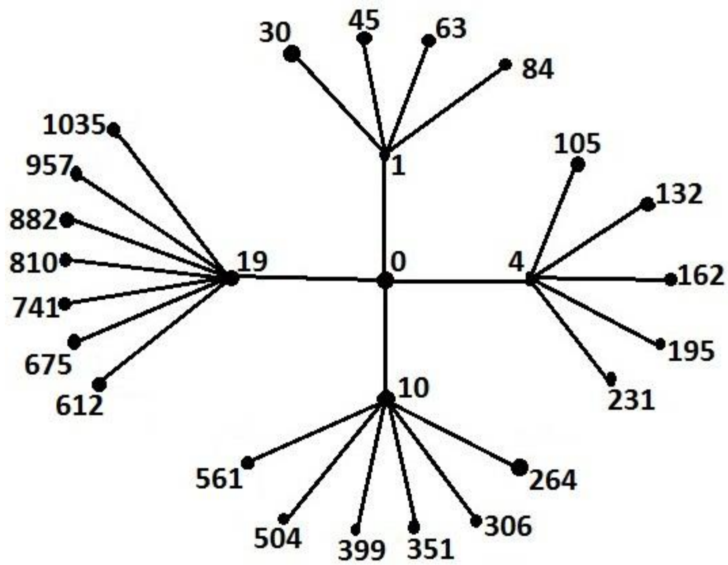

Fig. 8

Theorem 3.17: The graph $P_m@P_n$ is a centered triangular sum graph.

Proof: Let G be a $P_m@P_n$ graph.

Let V(G) = $\{v_i,\ v_{ij}\ : 1 \le i \le m,\ 1 \le j \le n\}$ with $v_i = v_{i1}$ $(1 \le i \le m)$ and

E(G) = $\{v_i v_{i+1}, v_{ij} v_{i\,j+1} : 1 \le i \le m-1, 1 \le j \le n-1\}$.

Here G has mn vertices and $mn - 1$ edges.

Define f: V(G) $\rightarrow$ $\{0,1,\ldots B_{mn-1}\}$as follows

$f(v_1) = f(v_{11}) = 0$

$f(v_i) = f(v_{i1}) = B_{i-1} - f(v_{i-1}),\ 2 \le i \le m$

$f(v_{12}) = B_m,$

$f(v_{i2}) = B_{m+i-1} - f(v_i),\ 1 \le i \le m$

$f(v_{ij}) = (B_{(j-1)m+i-1} - B_{(j-2)m+i-1} + B_{(j-3)m+i-1} - \cdots + (-1)^{j-1} B_{m+i-1}) +$
$(-1)^{j-1} (B_{i-1} - B_{i-2} + B_{i-3} - \cdots + (-1)^i B_1), 1 \le i \le m, 3 \le j \le n.$

Clearly f is injective and f induces a bijective function f^*: E(G) $\rightarrow \{1,4,\ldots B_{mn-1}\}$ as

$f^*(v_i v_{i+1}) = B_i$, $1 \le i \le m-1$

$f^*(v_{i1} v_{i2}) = B_{m+i-1}$,$1 \le i \le m$ and

$f^*(v_{ij} v_{ij+1}) = B_{mj+i-1}$, $1 \le i \le m, 2 \le j \le n-1$.

Hence the edge labels are $1,4,\ldots B_{mn-1}$.

Thus f is a centered triangular sum labeling of G.

Therefore, G = $P_m@P_n$ is a centered triangular sum graph.

Example 3.18: The centered triangular sum labeling of $P_4@P_4$ is shown in Fig. 9.

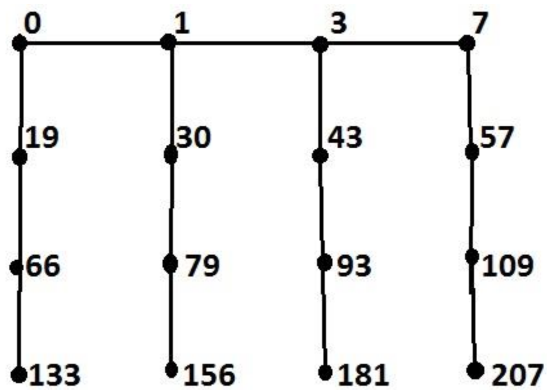

Fig. 9

Theorem 3.19: The graph $(P_n: K_{1,m})$ is a centered triangular sum graph for all n>1 and m≥ 1.

Proof: Let G be a $(P_n: K_{1,m})$ graph.

Let V(G) = $\{ v_i,\ w_i, w_{ij}\ : 1 \leq i \leq n, 1 \leq j \leq m\}$ and

E(G) = $\{v_i v_{i+1}, v_j w_j, w_j w_{jk} : 1 \leq i \leq n-1, 1 \leq j \leq n, 1 \leq k \leq m\}$.

Here G has $2n + mn$ vertices and $2n + mn - 1$ edges.

Let t = $2n + mn - 1$.

Define f : V(G) → $\{0,1,\ldots B_t\}$ as follows

$f(v_1) = 0$

For $2 \leq i \leq n,\ f(v_i) = B_{i-1} - f(v_{i-1})$

$f(w_i) = B_{n-1+i} - f(v_i)$, $1 \leq i \leq n$ and

$f(w_{ij}) = B_{2n-4+3i+j} - f(w_i)$; $1 \leq i \leq n,\ 1 \leq j \leq m$.

Clearly f is injective and f induces a bijective function f^* : E(G) →$\{1,4,\ldots B_t\}$ as

$f^*(v_i v_{i+1}) = B_i$, $1 \leq i \leq n-1$,

$f^*(v_j w_j) = B_{n-1+j}$, $1 \leq j \leq n$ and

$f^*(w_i w_{ij}) = B_{2n-4+3i+j}$; $1 \leq i \leq n, 1 \leq j \leq m$.

Hence the edge labels are 1,4,… B_t.

Thus f is a centered triangular sum labeling of G.

Therefore, G = $(P_n: K_{1,m})$ is a centered triangular sum graph.

Example 3.20: The centered triangular sum labeling of $(P_5: K_{1,3})$ is shown in Fig. 10.

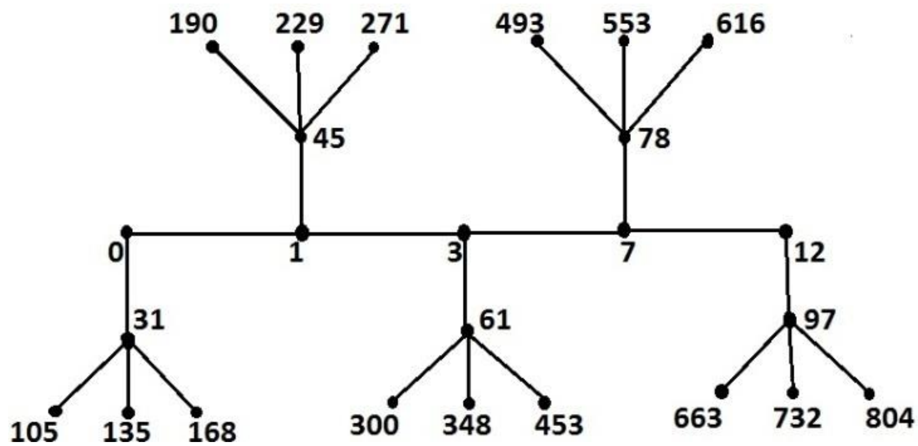

Fig. 10

Theorem 3.21: Banana tree *Bt*(n,n,…n)(m times) is a centered triangular sum graph for all n>1.

Proof: Let G be a Banana tree $Bt(\mathrm{n,n,...n})(\mathrm{m\ times})$ graph.

Let V(G) = $\{ v, v_i, w_i, w_{ij} : 1 \le i \le m, 2 \le j \le n\}$ and

E(G) = $\{vv_i, v_iw_i, w_iw_{ij} : 1 \le i \le m, 2 \le j \le n\}$.

Define f : V(G) → {0,1,… } be defined as follows

$f(v) = 0$

$f(v_i) = B_{i,}\ 1 \le i \le m$

$f(w_i) = B_{m+i} - B_i$, $1 \le i \le m$ and

$f(w_{ij}) = B_{2m-3+2i+j} - f(u_i), 1 \le i \le m, 2 \le j \le n$.

Clearly f is injective and f induces a bijective function f^* : E(G) →{1,4,… } as

$f^*(vv_i) = B_i$, $1 \le i \le m$,

$f^*(v_iw_i) = B_{m+i}$,$1 \le i \le m$ and

$f^*(w_iw_{ij}) = B_{2m-3+2i+j}$, $1 \le i \le m, 2 \le j \le n$.

Hence the edge labels are B_1, B_2,… are distinct and consecutive centered triangular numbers.

Thus f is a centered triangular sum labeling of G.

Therefore G = *Bt*(n,n,…n)(m times) is a centered triangular sum graph.

Example 3.22: The centered triangular sum labeling of $Bt(4,4,4,4)$ is shown in Fig. 11.

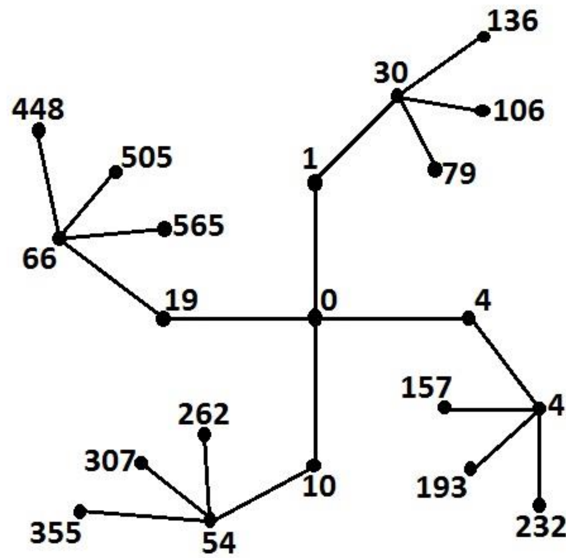

Fig. 11

4. CONCLUSIONS

In this paper, we have studied the centered triangular sum labeling of some tree related graphs. This work contributes several new results to the theory of graph labeling. The centered triangular sum can be verified for many other graphs. Also some more centered triangular sum labeling can be investigated.

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