Analytical models for quark stars with the MIT Bag model equation of state

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ABSTRACT

Two classes of exact solutions to the Einstein-Maxwell system is found in terms of elementary function. This is achieved by choosing a particular form for the measure of anisotropy with the MIT bag model equation of state relating the radial pressure to the energy density consistent with quark stars. These solutions contain the models found previously in the limit of vanishing charge/measure of anisotropy. Isotropic exact solutions regained include models by Komathiraj and Maharaj; Mak and Harko; and Misner and Zапolsky. A physical analysis of the matter and electromagnetic variables indicates that the model is well behaved and regular.

Keywords: Einstein-Maxwell system, exact solutions, MIT bag model EOS

1. INTRODUCTION

The first study of quark stars was performed by Itoh [1] for static matter in equilibrium. The physical processes governing the behaviour of quark matter with ultrahigh densities is still under investigation. Since we do not observe free quarks, in an attempt to describe the quark confinement mechanism, Chodos et al. [2] proposed the phenomenological MIT Bag model where one assumes that the quark confinement is caused by a universal pressure called the ‘bag
pressure’ at the boundary of the region containing quarks. The equation of state (EOS) in the bag model has a simple linear form

\[ p_r = \frac{1}{3} (\rho - 4B), \]  

(1)

where \( \rho \) is the density, \( p_r \) radial pressure and \( B \) is the bag constant. However, theoretical works of realistic stellar models [3-5] it has been suggested that superdense matter may be anisotropic, at least in some density ranges. The review of Weber [6] highlights models of compact astrophysical objects composed of strange quark stars. Some recent investigations for compact objects with a quark equation of state include the treatments of Malavar [7, 8], Takisa et al. [9] and Paul et al. [10].

Incorporation of electromagnetic field and anisotropy makes the system of field equations even more difficult to solve unless one adopts some simplifying techniques to make them tractable. In an earlier work, by identifying a conformal Killing vector, Mak and Harko [11] developed a relativistic model of an isotropic quark star. The work was later extended by Komathiraj and Maharaj [12] who provided a more general class of exact solutions by incorporating an electromagnetic field in the system of field equations.

In a more recent work, Maharaj et al. [13] have made a further generalization of [12] model by incorporating anisotropic stress into the system. In a subsequent paper, Sunzu et al. [14] performed a detailed physical analysis of the solution obtained in [13] and discussed its relevance in the context of compact quark stars candidates. It is interesting to note that the class of solutions generated in [13] for an assumed form of the anisotropic parameter \( \Delta = A_0 + A_1 x + A_2 x^2 + A_3 x^3 \) can be reduced to the charged isotropic stellar solutions of [11] and [12].

In this work, we choose a different form of the measure of anisotropy which, interestingly, provides much simpler analytic solutions.

The objective of this paper is to generate new classes of exact solutions with a linear quark matter equation of state for charged anisotropic stars. We build new models by specifying a particular form for one of the gravitational potentials and the measure of anisotropy that has been used in a recent paper by Maharaj et al. [13] and Malavar [15]. The advantage of this approach is that one can regain the charged isotropic stellar model simply by setting the anisotropy to zero. It is interesting to note that many previously found explicit solutions of the Einstein-Maxwell system with anisotropic stress e.g., solutions obtained by [16-20] do not have their corresponding isotropic analogues.

The paper has been organized as follows: In Section 2, the Einstein-Maxwell system of equations is expressed for static spherically symmetric spacetime according to Durgapal and Bannerji [21] transformation by incorporating the linear quark matter equation of state. In Section 3, particular forms for one of the gravitational potentials and the measure of anisotropy are chosen.

This helps to deduce the master differential equation in the remaining gravitational potential which governs the behaviour of the model. Two new class of solutions, in terms of elementary functions, have been obtained. The charged isotropic solutions found earlier in [11, 12, 22] have been shown to be special cases of general class of solutions obtained. The physical features of the model generated in this paper are illustrated briefly in Section 4. Finally decided in Section 5.
2. FIELD EQUATIONS

The generic form of the line element of a spherically symmetric relativistic fluid sphere in Schwarzschild coordinates \((x^a) = (t, r, \theta, \phi)\) is given by

\[
ds^2 = -e^{2\mu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),
\]

where \(\mu(r)\) and \(\lambda(r)\) are yet to be determined. The Einstein-Maxwell system of field equations corresponding to the line element (1), are obtained as:

\[
\frac{1}{r^2}(1 - e^{-2\lambda}) + \frac{2\lambda'}{r}e^{-2\lambda} = \rho + \frac{1}{2}E^2,
\]

\[
- \frac{1}{r^2}(1 - e^{-2\lambda}) + \frac{2\mu'}{r}e^{-2\lambda} = p_r - \frac{1}{2}E^2,
\]

\[
e^{-2\lambda} \left( \mu'' + \mu' + \frac{\mu'}{r} - \frac{\mu'\lambda'}{r} \right) = p_t + \frac{1}{2}E^2,
\]

\[
\frac{1}{r^2}e^{-\lambda}(r^2E)' = \sigma.
\]

In the above \(\rho\) is the energy density, \(p_r\) is the radial pressure, \(p_t\) is the tangential pressure, \(E\) is the electric field intensity and \(\sigma\) is the proper charge density, and a prime (') denotes derivative with respect to the radial coordinate \(r\).

Now by introducing the Durgapal and Bannerji [21] transformations

\[
A^2y^2(x) = e^{2\mu(r)}, \quad Z(x) = e^{-2\lambda(r)}, \quad x = Cr^2,
\]

the Einstein-Maxwell system of field equations (3)-(6) can be written as

\[
\frac{1 - Z}{x} - 2\dot{Z} = \frac{\rho}{C} + \frac{1}{2}E^2
\]

\[
4Z \frac{\dot{y}}{y} + \frac{Z - 1}{x} = \frac{p_r}{C} - \frac{1}{2C} E^2
\]

\[
4xZ \frac{\dot{y}}{y} + (4Z + 2x\dot{Z}) \frac{\dot{y}}{y} + \dot{Z} = \frac{p_t}{C} + \frac{1}{2C} E^2
\]

\[
\sigma^2 = \frac{4CZ}{x} (x\dot{E} + E)^2
\]

where \(A\) and \(C\) are arbitrary constants, and dots denote derivative with respect to the new coordinate \(x\). In these equations and hereafter we have used units where \(8\pi G = c = 1\).
The energy density $\rho$, radial pressure $p_r$ and the tangential pressure $p_t$ are measured relative to the comoving fluid 4-velocity $u^a = e^{-\mu}\delta^a_0$.

The system of equations (7)-(10), including the MIT bag model EOS (1) becomes

$$\rho = 3p_r + 4B,$$  \hfill (11)

$$\frac{p_r}{C} = Z\frac{\dot{y}}{y} - \frac{\dot{Z}}{2} - \frac{B}{C},$$  \hfill (12)

$$p_t = p_r + \Delta,$$  \hfill (13)

$$\frac{\Delta}{C} = \frac{4xZ\dot{y}}{y} + (6Z + 2x\dot{Z})\frac{\dot{y}}{y} + \left[2\left(\dot{Z} + \frac{B}{C}\right) + \frac{Z - 1}{x}\right],$$  \hfill (14)

$$\frac{E^2}{2C} = \frac{1 - Z}{x} - 3Z\frac{\dot{y}}{y} - \frac{\dot{Z}}{2} - \frac{B}{C},$$  \hfill (15)

where $\Delta = p_t - p_r$ represents the measure of anisotropy. The system of equation (11)-(15) governs the gravitational behavior of a charged quark star with anisotropic matter. We describe one possible integration procedure that leads to an exact solution of the Einstein-Maxwell system (11)-(15). Note that other procedures are possible; the approach in this work has the advantage of producing a first order differential equation that has solutions in terms of elementary functions.

The mass of a self-gravitating object for a given radius is an important measure for comparison with observational data. In this case, the mass contained within a radius $x$ of the sphere is obtained as

$$m(x) = \frac{1}{4C^2}\int_0^x \sqrt{x}\rho(x)dx.$$  \hfill (16)

3. GENERATING NEW SOLUTIONS

Note that the system (11)-(15) comprises five independent equations in seven unknowns $Z$, $y$, $\rho$, $p_r$, $p_t$, $\Delta$ and $E$. Naturally, the equivalent system (11)-(15) can be solved if two of these unknowns are assumed a priori. We seek to solve the system by making explicit choices for the gravitational potential $Z$ and the anisotropic parameter $\Delta$. Accordingly, we make the following assumptions:

$$y(x) = (a + x^n)^{2n},$$  \hfill (17)

$$\Delta = \frac{2\alpha C}{ax^{n-1} + x^{2n-1}},$$  \hfill (18)
where $a, n$ and $\alpha$ are constants. The choice (17) ensures that the metric function is regular at the centre and is well behaved within the stellar interior. A similar choice has been used in [12, 13]. As far as the second choice is concerned, it is a reasonable assumption in the sense that $\Delta$ vanishes at the center (i.e., $p_r = p_t$ at the origin) which is consistent with the physical requirement for a realistic stellar model.

Substitution of (17) and (18) into (14) yields

$$
\dot{Z} + \left[ \frac{1}{2x} + \frac{2(2n - 1)(nx^{2n-1})}{x(ax^{n-1} + x^{2n-1})} + \frac{n(4 - 3n + 8n^2)x^{2n-1}}{2x(ax^{n-1} + (1 + 2n^2)x^{2n-1})} \right] Z
- \left( \frac{1 - \frac{2Bx}{C}}{2x(ax^{n-1} + x^{2n-1})} - \frac{\alpha}{(ax^{n-1} + (1 + 2n^2)x^{2n-1})} \right) = 0,
$$

(19)

Once (19) is integrated we can directly find the remaining quantities $\rho, p_r, p_t$ from the system (11)-(13) and $E$ from (15) as $\Delta$ is known from (18). Equation (19) can be integrated in terms of elementary functions for specific values of the model parameter $n$ as discussed in the following.

3.1. The case $n = \frac{1}{2}$

In this case (17) gives the first metric function

$$
y(x) = a + \sqrt{x}
$$

Equation (19) can be integrated to give the second metric function

$$
Z = \frac{3(2a + \sqrt{x}) - \frac{Bx}{C}(4a + 3\sqrt{x}) + 3\alpha x^{3/2}}{3(2a + 3\sqrt{x})}
$$

Consequently, we generate an exact analytical model for the Einstein Maxwell system as:

$$
e^{2\mu} = A^2(a + \sqrt{x})^2,
$$

(20)

$$
e^{2\lambda} = \frac{3(2a + 3\sqrt{x})}{3(2a + \sqrt{x}) - \frac{Bx}{C}(4a + 3\sqrt{x}) + 3\alpha x^{3/2}},
$$

(21)

$$
\rho = \frac{3C(6a^2 + 10a\sqrt{x} + 3x)}{2\sqrt{x}(a + \sqrt{x})(2a + 3\sqrt{x})^2} + \frac{B\left(16a^3 + 47a^2\sqrt{x} + 48ax + 18x^2\right)}{2(a + \sqrt{x})(2a + 3\sqrt{x})^2}
- \frac{3\alpha \sqrt{x}(3a^2 + 4a\sqrt{x})}{2(a + \sqrt{x})(2a + 3\sqrt{x})^2},
$$

(22)
\[ p_r = \frac{C(6a^2 + 10a\sqrt{x} + 3x)}{2\sqrt{x}(a + \sqrt{x})(2a + 3\sqrt{x})^2} - \frac{B\left(16a^3 + 81a^2\sqrt{x} + 120ax + 54x^2\right)}{6(a + \sqrt{x})(2a + 3\sqrt{x})^2} \]

\[ - \frac{\alpha\sqrt{x}(3a^2 + 4a\sqrt{x})}{2(a + \sqrt{x})(2a + 3\sqrt{x})^2}, \tag{23} \]

\[ p_t = \frac{C(6a^2 + 10a\sqrt{x} + 3x)}{2\sqrt{x}(a + \sqrt{x})(2a + 3\sqrt{x})^2} - \frac{B\left(16a^3 + 81a^2\sqrt{x} + 120ax + 54x^2\right)}{6(a + \sqrt{x})(2a + 3\sqrt{x})^2} \]

\[ + \frac{\alpha\sqrt{x}[2C(2a + 3\sqrt{x})^2 - (3a^2 + 4a\sqrt{x})]}{2(a + \sqrt{x})(2a + 3\sqrt{x})^2}, \tag{24} \]

\[ \Delta = \frac{2\alpha C\sqrt{x}}{a + \sqrt{x}}, \tag{25} \]

\[ E^2 = \frac{C(-2a^2 - 2a\sqrt{x} + 3x) + Bx(a^2 + 2a\sqrt{x}) - \alpha\sqrt{x}(7a^2 + 22a\sqrt{x} + 18x)}{\sqrt{x}(a + \sqrt{x})(2a + 3\sqrt{x})^2} - \frac{\alpha\sqrt{x}(7a^2 + 22a\sqrt{x} + 18x)}{(a + \sqrt{x})(2a + 3\sqrt{x})^2} \tag{26} \]

Interestingly, by setting \( \alpha = 0 \), we regain the 1st class of charged isotropic solutions of [12]. If we further set \( a = 0 \), we obtain

\[ e^{2\nu} = A^2 cr^2, \quad e^{2\lambda} = \frac{3}{1 - Br^2}, \quad \rho = \frac{1}{2r^2} + B, \quad p_r = p_t = \frac{1}{6r^2} - B, \quad E^2 = \frac{1}{3r^2} \tag{27} \]

which is the quark stellar model of [11]. By setting \( B = 0 \) in (27), we regain the [22] solution.

The physical features of the solutions (27) were studied by Mak and Harko [11] and shown to be consistent with the interior of a quark star with charged material. This corresponds to a single stable quark configuration with radius \( R = 9.46 \, km \) and mass \( M = 2.86M_\odot \); these figures are consistent with values obtained using numerical methods by other researchers.

Consequently, the more general class of solutions (20)-(26) is likely to produce charged quark models consistent with stellar evolution and observational data. However, even though the gravitational potentials remain well behaved for the obtained class of solutions as in previously found solutions of [12, 13]; the matter variables and the electric field suffer from singularity in this case.

3.2. The case \( n = 1 \)

For this case (17) gives the first metric function

\[ y(x) = (a + x)^2 \]
Equation (19) can be integrated to give the second metric function

\[
Z = \left[9(35a^3 + 35a^2x + 21ax^2 + 5x^3) - \frac{2Bx}{C}(105a^3 + 189a^2x + 135ax^2 + 35x^3) + 2ax^2(63a^2 + 90ax + 35b^2x^2)\right] \times \frac{1}{315(a + x)^2(a + 3x)}
\]

The subsequent solution and matter variables are given as:

\[
e^{2\mu} = A^2(a + x)^4, \quad \text{(28)}
\]

\[
e^{2\lambda} = \frac{315(a + x)^2(a + 3x)}{9(35a^3 + 35a^2x + 21ax^2 + 5x^3) - f(x)}, \quad \text{(29)}
\]

\[
\rho = \frac{6C(70a^4 + 217a^3x + 159a^2x^2 + 75ax^3 + 15x^4) + 2B[3(35a^5 + 133a^4x + 246a^3x^2) + 5(254a^2x^3 + 209ax^4 + 63x^5)]}{35(a + x)^3(a + 3x)^2} - \frac{ax(126a^4 + 207a^3x - 535a^2x^2 - 835ax^3 - 315x^4)}{105(a + x)^3(a + 3x)^2}, \quad \text{(30)}
\]

\[
p_r = \frac{2C(70a^4 + 217a^3x + 159a^2x^2 + 75ax^3 + 15x^4)}{35(a + x)^3(a + 3x)^2} - \frac{2B[3(35a^5 + 497a^4x + 1854a^3x^2) + 5(1678a^2x^3 + 1177ax^4 + 315x^5)]}{315(a + x)^3(a + 3x)^2}, \quad \text{(31)}
\]

\[
p_t = \frac{2C(70a^4 + 217a^3x + 159a^2x^2 + 75ax^3 + 15x^4)}{35(a + x)^3(a + 3x)^2} - \frac{2B[3(35a^5 + 497a^4x + 1854a^3x^2) + 5(1678a^2x^3 + 1177ax^4 + 315x^5)]}{315(a + x)^3(a + 3x)^2} - \frac{ax(126a^4 + 207a^3x - 535a^2x^2 - 835ax^3 - 315x^4)}{315(a + x)^3(a + 3x)^2}, \quad \text{(32)}
\]

\[
\Delta = \frac{2\alpha C}{a + x}, \quad \text{(33)}
\]
\[ E^2 = \frac{Cx(196a^3 + 1452a^2x + 1356ax^2 + 420x^3)}{35(a + x)^3(a + 3x)^2} \]

\[- \frac{Bx \left( 56a^4 + 432a^3x + 2176a^2x^2 + \frac{7280}{3}ax^3 + 840x^4 \right)}{105(a + x)^3(a + 3x)^2} \]

\[- \frac{2ax \left( 84a^4 + 633a^3x + 1699a^2x^2 + \frac{4865}{3}ax^3 + 525x^4 \right)}{105(a + x)^3(a + 3x)^2} , \] (34)

where \( f(x) = \frac{2Bx}{C}(105a^3 + 189a^2x + 135ax^2 + 35x^3) + 2ax^2(63a^2 + 90ax + 35b^2x^2) \)

It is to be stressed here that this particular solution is a generalization of the second class of solutions obtained earlier by Komathiraj and Maharaj [12] which can be regained by setting \( \alpha = 0 \). The exact model (28)-(34) constitutes a new family of analytical solutions for quark star with charged and anisotropic matter. The gravitational potentials \( e^{2\mu} \) and \( e^{2\lambda} \) in (28)-(29) have the advantage of having a simple analytic form, and they are written in terms of rational functions. The matter variables and the measure of anisotropy have a simple analytic representation with vanishing electric field and measure of anisotropy at the centre of the star.

The finiteness of \( e^{2\mu}, e^{2\lambda}, \rho, p_r, p_t, \Delta \) and \( E \) at the origin \( x = 0 \) is a very welcome feature which is absent in the previous class of solutions (20)-(26). Consequently the class of solutions found in this section are good candidates to produce charged anisotropic stars with physically reasonable interior distributions.

4. PHYSICAL ANALYSIS

By utilizing the matching conditions, regularity conditions and other physical requirements [23], let us now find the appropriate bounds on the model parameters for the particular solution (28)-(34):

C1. The gravitational potentials \( e^{2\lambda} \) and \( e^{2\mu} \) should remain positive throughout the stellar interior. From equations (28) and (29), we note that \( e^{2\mu}(r = 0) = A^2a^4, (e^{2\mu})'(r = 0) = 0 \) and \( e^{2\lambda}(r = 0) = 1, (e^{2\lambda})'(r = 0) = 0 \). The results show that the gravitational potentials are regular at the centre \( r = 0 \).

C2. The energy density and pressure should be non-negative inside the stellar interior. From (30), we obtain the central density \( \rho_0 = \rho(r = 0) = \frac{12C}{a} + 2B \). Using (31), we have \( p_r(r = 0) = p_t(r = 0) = \frac{4C}{a} - \frac{2B}{3} \). These results imply that the energy density and the two pressures will be non-negative at the centre if the following condition is satisfied: \( \frac{C}{a} > \frac{B}{6} \).

C3. The interior metric should be matched to the exterior Reissner-Nordstrom metric at the boundary of the star \( r = R \). Using this conditions, the constant \( A \) is obtained in terms of model parameters and the boundary radius.
C4. The requirement \( p_r(r = 0) = 0 \) yields the bag constant \( B \) in terms of model parameters and the boundary radius.

C5. For a realistic star, it is expected that the gradient of density, radial pressure and the tangential pressure should be decreasing functions of the radial parameter \( r \) i.e., \( \frac{d\rho}{dr} \leq 0, \frac{dp_r}{dr} \leq 0 \) and \( \frac{dp_t}{dr} \leq 0 \). Using equations (30)-(32) these nature can be shown.

C6. The causality condition demands that the radial and the tangential sound speeds should not exceed the speed of light i.e., \( 0 < \frac{dp_r}{d\rho} < 1, \ 0 < \frac{dp_t}{d\rho} < 1 \). In this model we have \( 0 < \frac{dp_r}{d\rho} < \frac{1}{3} \).

By choosing the model parameters appropriately, we can show that the requirement \( 0 < \frac{dp_t}{d\rho} < 1 \) is also fulfilled in this model.

C7. For a realistic model, the following energy conditions are to be satisfied: (i) The Weak Energy Condition (WEC) \( \rho - p_r \geq 0 \) and \( \rho - p_t \geq 0 \). (ii) The Strong Energy Condition (SEC) \( \rho - 3p_r \geq 0 \) and \( \rho - 3p_t \geq 0 \). (iii) The Trace Energy Condition (TEC) \( \rho - p_r - 2p_t \geq 0 \). Since \( \rho, p_r \) and \( p_t \) are non-negative quantities, the energy condition(s) are satisfied in this model.

C8. For a stable configuration, it is expected that the adiabatic index \( \Gamma = \frac{\rho + p_r}{p_r \frac{dp_r}{d\rho}} \) should be greater than 4/3 [24, 25]. The above requirement is fulfilled in our model as can be seen from equation (30)-(31).

We have proved that the second class of solution (28)-(34) obtained in this paper is regular and well-behaved. Since the solution has been obtained by assuming the bag model EOS for a quark star, one can use the solution to model compact stellar objects like Her X-1 and SAX and J1808.4-3658, among others, which have been claimed to be good strange star candidates in the recent past.

Note that the model contains four constants namely, \( a, C, B \) and \( \alpha \). The constants \( \alpha \) appear in the potential \( y \) given in equation (28); the constant \( C \) has been utilized in the transformation [21]; \( B \) is the bag constant given in equation (1) and \( \alpha \) corresponds to the anisotropic factor given in (33). Three of these parameters do get fixed by the matching conditions at the boundary, namely matching of the interior solution to the Schwarzschild exterior metric at \( r = R \) and imposition of the requirement that pressure must vanish at the boundary i.e., \( p_r(r = R) = 0 \). The parameters \( \alpha \) fixes the extent of anisotropy (\( \alpha = 0 \) implies isotropic configuration). Thus, for a given bag constant within the stability window we have a physically reasonable and well-behaved model.

5. CONCLUSIONS

To summarize, in this work, we have been able to provide a couple of new solutions for an anisotropic stellar configuration couched on the Reissner-Nordström background spacetime.
We obtained solutions for a charged strange quark star with anisotropic matter described by the MIT bag model for the metric functions

\[ y(x) = a + \sqrt{x} \]
\[ y(x) = (a + x)^2 \]

The model reduces to an integration of a first order differential equation. The first solution generalized the model of Mak and Harko [11] for a quark star in an electromagnetic field with the absence of anisotropic factor. The second solution has the advantage of not containing any singularities at the stellar centre. We have demonstrated that there exist particular values of the model parameters for which a particular class of solution (28)-(34) satisfies the requirements of a physically reasonable stellar model. Since the solution has been obtained for a composition admitting a bag model EOS, the solution might be useful for the description of compact strange star candidates. Hopefully, our results will contribute to the rich class of exact solutions to the Einstein-Maxwell system of field equations. It is to be stressed that we have been able to generate solutions for parameter values (i) \( n = 1/2 \); and (ii) and \( n = 1 \), only. It will be interesting to check what other values of the model parameters can yield solution which are regular, well behaved and can describe realistic stars. Such possibilities, however, will be taken up elsewhere.

References


