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Fractional Integral Approximation and Caputo Derivatives with Modification of Trapezoidal Rule

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ABSTRACT

In classical calculus, a function can be derived or integrated as many as natural numbers. Then a question arises regarding the fractional order of derivatives and integrals. There is a development of classical calculus called fractional calculus. Fractional calculus may be a department of science that amplifies the orders of derivatives and integrals into the order of rational numbers or even real numbers. The difficulty of finding solutions analytically for a complicated function of fractional integrals or fractional derivatives often occurs. In this paper, we will solve Rieman Liouville's fractional integral and Caputo's fractional derivative analytically using the trapezoidal rule modification method. Trapezoidal method is an approximation method that is resulted from the linear interpolation function. In this paper, we will find numerical simulations with modified trapezoidal method, to estimate some functions, and the results will be compared with previous research related to the Rieman Liouville fractional integral approximation and the Caputo fractional derivative. The result from simulation find that modified trapezoidal can approximate Caputo fractional derivative by replace α with $-\alpha$ and Quadratic schemes method is the best method to approximate Rieman Liouville fractional integral and Caputo fractional derivative.

***Keywords*:** Modified trapezoidal rule, Fractional integral, Rieman Liouville fractional integral, Caputo fractional derivative

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1. INTRODUCTION

Fractional derivatives have played a vital part in analyzing the behavior of physical wonders through various domains of science and engineering. A few of the spearheading commitments in this concern are in the field of biology [1], viscoelasticity [2-4], biotechnology [5]. Some of the more recent applications of derived fractions also exist in mathematical biology [6-9] and flow of heat and fluids [10]. Because the impact from fractional derivatives is quite significant, fractional derivatives have received a lot of consideration in recent years and this is could also be the result due to their broader essence compared to traditional integer derivatives [1]. Furthermore, in addition to fractional derivatives there is some other well-known as fractional integral.

Numerical integration could be an essential instrument for getting approximate values for exact integrals where analytical integration is troublesome to assess. Previous research shows that numerical integration for fractional integrals has more significant in evolve algorithms to deal with applied problems that are determined using fractional derivatives. Hence, fractional derivative is anti fractional integral [1]. A quadratic approximation scheme for fractional differential equations solved using Adams-Bashforth-Moulton method already reasearched [11-13]. Odibat [14] presented a modified trapezoidal rule algorithm, for fractional integral approximation and Caputo's fractional derivative, in that research Odibat obtained the error estimate. [15] Agrawal discusses the limited component approach and fractional power series solutions to fractional variational problems. [16] Talks about the comparative ponder of distinctive numerical strategies such as direct, quadratic and quadratic-linear plans for fathoming variational issues of fractions defined in terms of common derivatives. Recently [17] presented solving the approximation of Rieman Liouville's fractional integral and Caputo's fractional derivative by quadratic and cubic methods.

Returning to the problem of how to estimate a fractional integral and derivative, specifically Rieman Liouville's fractional integral and Caputo's fractional derivative, we will first describe the relationship between fractional integrals and fractional derivatives. This relationship will be applied in the estimation of integral values and fractional derivatives using the trapezoidal rule modification method. Calculations of the approximations are carried out in the python programming language to simplify calculations. In this paper, we will also discuss the accuracy in estimating the fractional derivative of caputo fractional with the trapezoidal modification method resulting from the analogy of relations between fractional derivative and integrals. The method of the derivative of caputo fractional can be found in [14] and methods of quadratic and cubic functions in [17-20].

2. PREMILINARIES

In this section, we will explain some of the basic formulas and techniques that are essential for understanding this entire journal. This section will start from the gamma function.

2. 1. Gamma Function

Gamma function has an important role for solve Rieman Liouville fractional integral, following the definition of gamma function;

Definition 2.1. *If* $z \in \mathbb{C}$*, we can defined the Gamma function with* [15]

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

This integral converges to $\mathrm{Re}(z) > 0$.
One of the properties of the Gamma function is [15]

$$\Gamma(z+1) = z\Gamma(z). \tag{1}$$

To prove these properties, through Definition 2.1 we get a part

$$\Gamma(z+1) = \int_0^\infty e^{-t} t^z dt = [-e^{-t}\, t^z]_{t=0}^{t=\infty} + z\int_0^\infty e^{-t} t^{z-1} dt$$

where, if ignoring the first part and then the second part is equal to $z\Gamma(z)$, thus following equation (1). If $\Gamma(1) = 1$ and if using equation (1) it is obtained,

$$\begin{gathered} \Gamma(2) = 1.\Gamma(1) = 1 = 1! \\ \Gamma(3) = 2.\Gamma(2) = 2.1! = 2! \\ \Gamma(4) = 3.\Gamma(3) = 3.1! = 3! \\ \vdots \\ \Gamma(n+1) = n.\Gamma(n) = n.(n-1)! = n! \end{gathered}$$

hence, it can be seen that $\Gamma(n+1) = n!$ for all $n \in \mathrm{N}$.

2. 2. Beta Function

In some cases, Gamma function is less pleasant than Beta function [15]. Since the beta function is concerned with calculating the solution of the fractional integral, the Beta function is also described in this section.

Definition 2.2. *if* $z, w \in \mathbb{C}$*, then defined function Beta* [15]

$$\mathrm{B}(z,w) = \int_0^1 \tau^{z-1}(1-\tau)^{w-1} d\tau$$

for $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(w) > 0$. Gamma function can be expressed Beta function, after using the Laplace transform for convolution the Beta function [15],

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} \tag{2}$$

and through equation (2) it is found that

$$B(z,w) = B(w,z). \tag{3}$$

3. RESULT AND DISCUSSION

3. 1. Integral Fraksional Riemann-Liouville

The Riemann-Liouville integral difference is obtained by combining derivatives and integrals of whole number orders. First, to get the Cauchy function, we must generalize the definition of the integral. If $f(\tau)$ is integrable in every interval (a, t), hence integral

$$f^{-1}(t) = \int_a^t f(\tau)d\tau$$

exists. Then, for two integrals:

$$f^{-2}(t) = \int_a^t d\tau_1 \int_a^{\tau_1} f(\tau)d\tau = \int_a^t f(\tau)d\tau \int_\tau^t d\tau_1$$
$$= \int_a^t (t-\tau)f(\tau)d\tau.$$

if the last equation is integrated, then this gives three integrals of $f(t)$

$$f^{-3}(t) = \int_a^t d\tau_1 \int_a^{\tau_1} d\tau_2 \int_a^{\tau_2} f(\tau)d\tau = \int_a^t d\tau_1 \int_a^t (t-\tau)f(\tau)d\tau.$$
$$= \frac{1}{2}\int_a^t (t-\tau)^2 f(\tau)d\tau.$$

Then, we can get the Cauchy formulation

$$f^{-n}(t) = \frac{1}{\Gamma(n)} \int_a^t (t-\tau)^{n-1} f(\tau)d\tau. \tag{4}$$

If n in Cauchy's formulation in equation (4) is replaced by the real number p, we will get an integral for any order.

Definition 3.1. *Riemann-Liouville fractional integral of the order $p \in \mathbb{R}_{>0}$ give* [17]

$${}_aD_x^{-\alpha}(f(x)) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt \tag{5}$$

One of important property of Riemann-Liouville fractional integral that given in Definition 3.1 represented with equation [18]

$${}_aD_x^{-p}\left({}_aD_x^{-q} f(t)\right) = {}_aD_x^{-q}\left({}_aD_x^{-p} f(t)\right) = {}_aD_x^{-p-q} f(t). \tag{6}$$

Equation (6) can be called the compositional rule of the Riemann-Liouville fractional integral. To prove equation (6) can use the definition of the Riemann-Liouville fractional integral itself [18]

$$
\begin{aligned}
{}_aD_x^{-p}\left({}_aD_x^{-q}f(t)\right) &= \frac{1}{\Gamma(p)}\int_a^t (t-\tau)^{p-1}\left({}_aD_x^{-q}f(\tau)\right)d\tau \\
. \qquad &= \frac{1}{\Gamma(p)}\int_a^t (t-\tau)^{p-1}\left(\frac{1}{\Gamma(q)}\int_a^\tau (\tau-\xi)^{q-1}f(\xi)d\xi\right)d\tau \\
&= \frac{1}{\Gamma(p)\Gamma(q)}\int_a^t\int_a^\tau (t-\tau)^{p-1}(\tau-\xi)^{q-1}f(\xi)d\xi d\tau.
\end{aligned}
$$

hence by exchanging orders from integration it gives the equation,

$$
{}_aD_x^{-p}\left({}_aD_x^{-q}f(t)\right) = \frac{1}{\Gamma(p)\Gamma(q)}\int_a^t f(\xi)\int_\xi^\tau (t-\tau)^{p-1}(\tau-\xi)^{q-1}d\tau d\xi.
$$

Make substitutions $\frac{\tau-\xi}{t-\xi}=\varsigma$, hence $d\tau=(t-\xi)d\varsigma$ and the new interval [0,1] for an integral. Now the final statement will rewrite as,

$$
\begin{aligned}
{}_aD_x^{-p}\left({}_aD_x^{-q}f(t)\right) &= \frac{1}{\Gamma(p)\Gamma(q)}\int_a^t f(\xi)\left((t-\xi)^{p+q-1}\int_0^1 (1-\zeta)^{p-1}\zeta^{q-1}d\zeta\right)d\xi \\
&= \frac{1}{\Gamma(p)\Gamma(q)}\int_a^t f(\xi)\left((\tau-\xi)^{p+q-1}B(p,q)\right)d\xi
\end{aligned}
$$

where, in the final formulation using the Beta function given in Definition 2.2. if using equation (2) then the Beta function can be expressed by the Gamma function as follows [18],

$$
\begin{aligned}
{}_aD_x^{-p}\left({}_aD_x^{-q}f(t)\right) &= \frac{1}{\Gamma(p)\Gamma(q)}\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\int_a^t f(\xi)(t-\xi)^{p+q-1}d\xi \\
&= \frac{1}{\Gamma(p)\Gamma(q)}\int_a^t (t-\xi)^{p+q-1}f(\xi)d\xi \\
&= {}_aD_x^{-p-q}f(t).
\end{aligned}
$$

Example 1. Application of the Rieman Liouvile integral to fractional integrals, when $x=1, \alpha=\frac{1}{2}$

$${}_0I_1^{\frac{1}{2}}(x^3) = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^1 (x-t)^{-\frac{1}{2}} t^3 dt \text{, where Beta function: } \mathrm{B}(\alpha,\beta) = \int_0^1 (1-t)^{\alpha-1} t^{\alpha-1} dt$$

$${}_0I_1^{\frac{1}{2}}(x^3) = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^1 t^{4-1}(1-t)^{\frac{1}{2}-1} dt$$

$${}_0I_1^{\frac{1}{2}}(x^3) = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\mathrm{B}\left(4,\frac{1}{2}\right)$$

$${}_0I_1^{\frac{1}{2}}(x^3) = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\frac{\Gamma(4)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(4+\frac{1}{2}\right)}$$

$${}_0I_1^{\frac{1}{2}}(x^3) = \frac{3!}{\frac{105\sqrt{\pi}}{16}}$$

Properties [19]:

P1. $\quad {}_aI_x^{\alpha}(cf(x)) = c\,{}_aI_x^{\alpha}(f(x))$

P2. $\quad {}_aI_x^{\alpha}(f(x) \pm g(x)) = {}_aI_x^{\alpha}(f(x)) \pm {}_aI_x^{\alpha}(g(x))$

P3. $\quad {}_aI_x^{\alpha}({}_aI_x^{\beta}(f(x))) = {}_aI_x^{\alpha+\beta}(f(x))$

or, we consider the Abel-Liouville integral of the order α (also known as the Riemann-Liouville integral),

$$I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{a-1} f(s)ds \qquad (x \geq 0)\text{, for } \alpha > 0$$

where Γ denotes Euler's gamma function. $I^{\alpha}f(x)$ depends analytically on α (to be sure f and x. If f is a continuous time k it is differentiated at $[0, x]$, it can be analytically continuous at α with a negative Real through, $I^{\alpha}f(x) = \frac{d^k}{dx^k} I^{a+k} f(x)$, untuk $\alpha > -k$ if $-k \leq \alpha < 0$ and $f^{(j)}(0) = 0$ for $j = 0,1, \dots, k-1$ and $y(x) = I^{\alpha}f(x)$ Abel is Abel's first integral solution.

$$\frac{1}{\Gamma(-\alpha)}\int_0^x (x-s)^{-\alpha-1} y(s)ds = f(x) \ (x \geq 0).$$

For whole numbers α, I^{α} is a recurring integral or differentiation [18]:

$$I^k f(x) = \int_0^x \int_0^{x_k} \dots \int_0^{x_2} f(x_1) dx_1 dx_2 \dots dx_k,$$

$$I^0 f(x) = f(x),$$

$$I^{-k} f(x) = \frac{d^k}{dx^k} f(x).$$

I^α is called a fractional integral for the order α and is also denoted by $D^{(-\alpha)}$, the fractional derivative of the order $-\alpha$.

3. 2. Caputo fractional derivatives

Let $f:[a,b] \to R$ on the interval $[a,b]$ be continuous function and α is a positive real number.

Definition 3.2. The left and right sided Caputo fractional derivatives of order α , respectively by [17]

$${}^C D_{\alpha+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\tau)^{-\alpha} f'(\tau) d\tau \qquad (t > a), \tag{7}$$

and

$${}^C D_{b-}^{\alpha} f(t) = \frac{-1}{\Gamma(1-\alpha)} \int_t^b (\tau - t)^{-\alpha} f'(\tau) d\tau \qquad (t < b), \tag{8}$$

where the Gamma function is defined in Definition 2.1

The ELT of left and right sided Caputo fractional derivatives can be stated:

$$\mathfrak{L}_\alpha \left\{ {}^C D_{\alpha+}^{\alpha} f(t) \right\} = \mathfrak{L}_\alpha \left\{ \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\tau)^{-\alpha} f'(\tau) d\tau \right\}$$

$$= L\left\{ \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \right\} . \mathfrak{L}_\alpha \left\{ f'(t) \right\}$$

$$= s^{\alpha-1} \left[s\mathfrak{I}(s) - f(t).e^{-st} \right]_{t=a}$$

$$= s^{\alpha} \mathfrak{I}(s) - s^{\alpha-1} e^{-\alpha s} f(a),$$

and

$$\mathfrak{L}_b \left\{ {}^C D_{b-}^{\beta} f(t) \right\} = \mathfrak{L}_b \left\{ \frac{-1}{\Gamma(1-\beta)} \int_t^b (\tau - t)^{-\beta} f'(\tau) d\tau \right\}$$

$$=(-1)^{1-\beta} L\left\{\frac{t^{-\beta}}{\Gamma(1-\beta)}\right\}.\mathfrak{L}_{\beta}\left\{f^{'}(t)\right\}$$

$$=(-1)^{1-\beta} s^{\beta-1}\left[s\Im(s)-f(t).e^{-st}\right]_{t=b}$$

$$=(-1)^{1-\beta}.[s^{\beta}\Im(s)-s^{\beta-1}e^{-bs}f(b)].$$

The connections between the Caputo fractional derivative (7-8) and the Riemann-Liouville derivatives are given by the relations [17],

$$^{C}D_{\alpha+}^{\alpha}f(t) = {}^{RL}D_{\alpha+}^{\alpha}\left[f(t)-\sum_{k=0}^{n-1}\frac{y^{k}a}{k!}(t-a)^{k}\right](x) \tag{9}$$

and

$$^{C}D_{\alpha-}^{\alpha}f(t) = {}^{RL}D_{\alpha-}^{\alpha}\left[f(t)-\sum_{k=0}^{n-1}\frac{y^{k}b}{k!}(b-t)^{k}\right](x) \tag{10}$$

respectively.

3. 3. Modified trapezoidal rule

Theorem 2. Subdivided interval $[0,a]$ *into* k subintervals $[x_j, x_{j+1}]$, by using the nodes $x_j = jh$ is equal width $h = a/k$, *for* $j = 0,1,...,k$. We can state the modified trapezoidal rule

$$T(f,h,\alpha) = ((k-1)^{\alpha+1}-(k-\alpha-1)k^{\alpha})\frac{h^{\alpha}f(a)}{\Gamma(\alpha+2)} + \sum_{j=1}^{k-1}((k-j+1)^{\alpha+1}-2(k-j)^{\alpha+1}+(k-j-1)^{\alpha+1})\frac{h^{\alpha}f(x_j)}{\Gamma(\alpha+2)} \tag{11}$$

is an approximation to fractional integral [14]

$$(J^{\alpha}f(x))(a) = T(f,h,\alpha)-E_T(f,h,\alpha),\ a>0,\ \alpha>0.$$

Furthermore, there is a constant $C_{\alpha}^{'}$ if $f(x)\in C^{2}[a,b]$, only depending on α so that the error term $E_T(f,h,\alpha)$ can be find with, [14]

$$\left|E_T(f,h,\alpha)\right| \le C_{\alpha}^{'}\left\|f^{''}\right\|_{\infty} a^{\alpha}h^{2} = \mathbf{O}(h^{2}).$$

Proof. We have [14]

$$(J^{\alpha}f(x))(a) = \frac{1}{\Gamma(\alpha)}\int_{0}^{a}(a-\tau)^{\alpha-1}f(\tau)d\tau$$

If $\tilde{f}_k$ is the piecewise linear interpolant for f whose nodes are chosen at the nodes $x_j, j = 0,1,2,...,k$ then, we obtain

$$\int_0^a (a-\tau)^{\alpha-1} \tilde{f}_k(\tau)d\tau = \frac{h^\alpha}{\alpha(\alpha+1)} \cdot \begin{cases} \left((k-1)^{\alpha+1} - (k-\alpha-1)k^\alpha\right) f(0) + f(a) \\ +\sum_{j=1}^{k-1}\left((k-j+1)^{\alpha+1} - 2(k-j)^{\alpha+1} + (k-j-1)^{\alpha+1}\right) f(x_j) \end{cases}$$

and

$$\left| \int_0^a (a-\tau)^{\alpha-1} f(\tau)d\tau - \int_0^a (a-\tau)^{\alpha-1} \tilde{f}_k(\tau)d\tau \right| \le C_\alpha \left\| f^{''} \right\|_\infty \alpha^\alpha h^2.$$

It is evident that the behavior of method is independent of the parameter α which it carries on in a way that's exceptionally comparative to the classical trapezoidal rule. The modified trapezoidal rule can being trapezoidal rule, if. The idea of piecewise linear interpolation that is the background of the product trapezoidal method for the Riemann-Liouville integral has also been used successfully for the Riemann-Liouville derivative [18].

Eqs. (7) and (8) can be useful to approximate Riemann Liouville derivative, from the corresponding expressions for the Riemann-Liouville integral, *viz.* Eqs. (11), or in a nutchell by replacing α with $-\alpha$.

From [18] we know the relation between Caputo fractional derivaive (7-8) and the Riemann-Liouville derivatives. In this paper, we will find approximation the Caputo fractional derivative from modified trapezoidal rule, for the approximation quality of this method, we have Lemma 2.2 of [18] and Theorem 2.3 of [19].

Example 2. If we have function $f(x) = \sin x$, we will approximate the fractional integral $\left(J^\alpha f(x)\right)(1)$ with modified trapezoidal rule to for different values of α, Tables 1 - 3 shown the result along with error.
E_{IQ} (f, h, 0.5)

Table 1. Result from modified trapezoidal rule for $(J^{0.5} \sin x)(1)$

n	h	IQ (f,h,0.5)	T (f,h,0.5)	E_{IQ} (f, h, 0.5)	E_{IT} (f, h, 0.5)
10	0,05	0.6696838267	0.669178250	4,33E-7	1,304E-4
20	0,025	0.6696842212	0.669553853	3,83E-8	3,33E-5
40	0,0125	0.6696842561	0.669650982	3,39E-9	8,44E-6
80	0,00625	0.6696842592	0.669675822	3,01E-10	2,13E-6

Table 2. Result from modified trapezoidal rule for $(J^1 \sin x)(1)$

n	h	IQ (f,h,1)	T (f,h,0.5)	E_{IQ} (f, h, 1)	E_{IT} (f, h, 1)
10	0,05	0,4596977	0.4593145	1,60E-8	9,58E-5
20	0,025	0,4596976	0.4596019	9,98E-10	2,39E-5
40	0,0125	0,4596976	0.4596737	6,24E-11	5,99E-6
80	0,00625	0,4596976	0.4596917	3,90E-12	1,50E-6

Table 3. Result from modified trapezoidal rule for $(J^{1,5} \sin x)(1)$

n	h	IQ (f,h,1,5)	T (f,h,1.5)	E_{IQ} (f, h, 1.5)	E_{IT} (f, h, 1.5)
10	0,05	0,2823225	0.2820860	1,21E-7	5,89E-5
20	0,025	0,2823223	0.2822634	7,67E-9	1,47E-5
40	0,0125	0,2823223	0.2823076	4,85E-10	3,68E-6
80	0,00625	0,2823223	0.2823187	3,06E-11	9,1911E-7

Example 3. Assume we have function $f(x) = \sin x$, we will approximate the fractional derivative $\left(D^{\alpha} f(x)\right)(1)$ we use the modified trapezoidal rule for $\alpha = -0.5$. Tables 4 shown the result along with error.

Table 4. Result from modified trapezoidal rule for $(D^{0.5} \sin x)(1)$

n	h	DQ(f,h,0.5)	C(f,h,0.5)	T(f,h,0.5)	E_C (f, h, 0.5)	E_{DQ} (f, h, 0.5)	E_T (f, h, 0.5)
10	0,05	0.846057377	0.845382	0.851541	6,7379E-4	5,912E-7	1,706E-5
20	0,025	0.846056841	0.845886	0.848078	1,706E-5	5,46E-8	4,3054E-5
40	0,0125	0.846056791	0.846013	0.846791	4,305E-6	5,96E-9	1,0836E-6
80	0,00625	0.846056787	0.846045	0.846321	2,722E-7	4,50E-10	2,7222E-7

4. CONCLUSION

In this paper it is proven that the use of modified trapezoidal rule by changing the order value to negative can be used to estimate fractional derivatives. In the case of example in this paper, comparing the modified trapezoidal rule method with quadratic schemes to estimate a fractional integral function, it is found that the approximation is better when using the Quadratic schemes method, because it has a higher level of accuracy, and hence experiences convergence faster than the modified trapezoidal method. Meanwhile, to estimate the fractional derivative, three methods were used, and the Quadratic schemes method was still the best, with a smaller error value, and finally we estimate the caputo fractional derivative with trapezoidal modification, using $-\alpha$. The results are not much different from the modification method of Caputo fractional derivative itself which was carried out by Odibat.

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