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Efficiency and Convergence of Bisection, Secant, and Newton Raphson Methods in Estimating Implied Volatility

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ABSTRACT

This study aims to estimate volatility prices based on the black-Scholes model (BSM) function with research data taken during the COVID-19 pandemic. The estimates of the volatility values are obtained by using three numerical methods, namely the bisection, secant, and Newton Raphson methods. The numerical processes that produce some iteration results in the three methods are then analyzed and the best convergence is sought. As a result, Newton Raphson method produces the smallest number of iterations, which stops at the 3rd iteration and gets a volatility value of 0.500451 with an absolute error value of 0.000388. However, the method requires an initial approximation which lies only in two intervals on the axis σ which are close to the true root. Meanwhile, for the other two methods, namely Bisection and Secant, this limitation does not apply, as long as there is an interval that guarantees the existence of roots. In this case, bisection method requires11 iterations to converge with volatility value of 0.500342 and error value of 0.000878. Whereas secant method requires 4 iterations to converge with a volatility value of 0.500449 and error value of 1.68938E-05. This suggests, that in some cases the use of Newton method, should be initialized with the use of bisection or secant method, to ensure successful iteration and accelerate the rate of convergence.

***Keywords*:** Black-Scholes model, volatility, volatility implied, bisection, secant, Newton Raphson

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1. INTRODUCTION

Volatility is a statistical measure of fluctuation in stock or foreign exchange prices over a certain period. High volatility is a price that goes up quickly and then suddenly falls quickly. The volatility values can be used to estimate the level of risk of loss and even the benefits of making a decision. There are two kinds of ways to determine the value of volatility, namely that by using historical data, or that with implied volatility.

According to Canina [1], the value given by implied volatility is better and more appropriate to use in making a decision option, than the one generated from the calculation of historical data. This is due to the use of real time data applied by implied volatility. There have been many studies that describe models for determining implied volatility. The most popular and widely used one is the black-Scholes model (BSM) published in 1973 [2, 3]. Meanwhile, a recent research discusses the test of the predictability of the implied volatility of the stock market using the autoregressive model (AR) [4-6]. Although many new models are available, we are still interested in using BSM to estimate the implied volatility price.

This is in line with a research emerged in 2020 designing a high-order numerical approach for efficient solutions of the Black-Scholes fractional equation [7]. This study succeeded in proving the stability and convergence of BSM. Our study aims to estimate prices of volatility based on the BSM function with research data taken during the COVID-19 pandemic. The Covid-19 pandemic has lasted for a long time and at the same time it has given social and economic impacts widespread.

Monthly production data at US companies show that the COVID-19 shock caused a peak decline in industrial production by 12-19% [8]. This decline resulted in doubts and affected stock market players in the world [9, 10]. Hence, the volatility of the stock market experiences uncertainty in determining market policy. On the other hand, many investors prefer safe-haven and flight-to-safety behavior.

This means that investors at all the capital markets are more focused on the energy and precious metal markets, which are not significantly affected by COVID-19 [11]. Based on this, we tried to use stock data obtained from international precious metal companies to be used as secondary research data. The numerical methods used to find a solution to the BSM equation are Bisection, Secant, and Newton Raphson methods, which with their results a comparative analysis among the three methods was carried out.

2. MATERIALS AND METHODS

2. 1. Materials

This study uses secondary data from the *Freeport-McMoRan Inc.* (FCX) stock market obtained at *http://www.finance.yahoo.com*. The data taken are the current stock price (S_0), the strike price (K), and the Call Option Observation price (C_{obs}) with a maturity date (T) of 0.25 years. The share price used is the daily closing share price on November 19, 2020. The interest rate (r) used is the average *World Center Bank* interest rate obtained from *https://www.investing.com/*

The tool used to find the numerical solutions of the volatility values is software R Programming, which is also designed to display numerical solutions based on the iterations that can be shown.

2. 2. Black-Scholes Function

According to Corbet [12], the option price in the capital market is the same as the theoretical price calculated using the Black-Scholes formula which can be written as follows,

$$C_{obs} = C_{bs}(\sigma), \tag{1}$$

Meanwhile, the theoretical call option price ($C_{bs}(\sigma)$) with volatility (σ) from the Black-Scholes formula can be defined as [2, 11]:

$$C_{bs}(\sigma) = S_0 N(d_1) - Ke^{-r(T-t)} N(d_2), \tag{2}$$

with $N(d_i)$ is a function of the cumulative normal distribution of values d_i following,

$$d_1 = \frac{ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \tag{3}$$

$$d_2 = d_1 - \sigma\sqrt{T} \tag{4}$$

Based on equation (1), the BSM function can be formed, namely,

$$f(\sigma) = C_{obs} - C_{bs}(\sigma), \tag{5}$$

or

$$f(\sigma) = C_{obs} - \left[S_0 N(d_1) - Ke^{-r(T-t)} N(d_2)\right], \tag{6}$$

2. 3. Numerical Methods

The numerical methods used to estimate the volatility value (6) are the bisection method, the secant method, and the Newton Raphson method. Their algorithms are respectively presented as follows.

Bisection Method

Step 1: Define the initial approximations σ_{i-1} and σ_i, also set the error tolerance value $\varepsilon_{tol} = 10^{-4}$

Step 2: Calculate $f(\sigma_{i-1})$ dan $f(\sigma_i)$

Step 3: Check whether the function f changes sign on an interval $[\sigma_{i-1}, \sigma_i]$, this can be checked with $f(\sigma_{i-1})f(\sigma_i) < 0$. If accepted, the initial approximation values can be used for the iteration, but if not, select new initial approximation values.

Step 4: Define $c = \frac{\sigma_{i-1}+\sigma_i}{2}$

Step 5: Calculate the value $f(c)$

Step 6: Perform an evaluation to determine within which subinterval the root of the function lies. If $f(c)f(\sigma_i) < 0$ then $\sigma_{i-1} = c$. Otherwise, set $\sigma_i = c$

Step 7: Calculate$|\varepsilon| = \frac{\sigma_i - \sigma_{i-1}}{\sigma_i}$

Step 8: Checking, if $|\varepsilon| < \varepsilon_{tol}$ with $i = 1, 2, \dots, n$, then the iteration is stopped with c as the estimate of the solution σ of the volatility function $f(\sigma)$, but if $|\varepsilon| > \varepsilon_{tol}$, with $i = 1, 2, \dots, n$, then the process is continued back to step 4.

Secant Method

Step 1: Define the initial approximations σ_{i-1} and σ_i, also set the error tolerance value $\varepsilon_{tol} = 10^{-4}$

Step 2: Calculate $f(\sigma_{i-1})$ and $f(\sigma_i)$

Step 3: Calculate the new approximation using $\sigma_{i+1} = \sigma_i - \frac{f(\sigma_i)(\sigma_i - \sigma_{i-1})}{f(\sigma_i) - f(\sigma_{i-1})}$

Step 4: calculate $|\varepsilon| = \frac{\sigma_i - \sigma_{i-1}}{\sigma_i}$

Step 5: Checking, if $|\varepsilon| < \varepsilon_{tol}$ with $i = 1, 2, \dots, n$, then the iteration is stopped with σ_{i+1} as the estimate of the solution σ of the volatility function $f(\sigma)$, but if $|\varepsilon| > \varepsilon_{tol}$, with $i = 1, 2, \dots, n$, then the process is continued back to step 1 by making σ_i as σ_{i-1} and σ_{i+1} as σ_i.

Newton Raphson Method

Derivation of the formula for Newton Raphson method can be obtained geometrically or with the help of the Taylor series. If σ_{i-1} is the initial approximation, then the next approximation can be calculated by the following equation,

$$\sigma_i = \sigma_{i-1} - \frac{f(\sigma_{i-1})}{f'(\sigma_{i-1})}, f'(\sigma_{i-1}) \neq 0 \qquad (7)$$

The derivative of the volatility function (5) can be defined as follows,

$$f'(\sigma_{i-1}) = -\frac{\partial C_{BS}(\sigma_{i-1})}{\partial \sigma_{i-1}} \qquad (8)$$

or,

$$f'(\sigma_{i-1}) = -S_0\sqrt{T-t}\frac{1}{\sqrt{2\pi}}e^{-\frac{{d_1}^2}{2}} \qquad (9)$$

An algorithm for Newton Raphson method is given as follows,

Step 1: Define the initial approximations σ_{i-1} and the error tolerance value $\varepsilon\varepsilon_{tol} = 10^{-4}$

Step 2: Calculate the value $f(\sigma_{i-1})$ and $f'(\sigma_{i-1})$

Step 3: Determine the next approximate value (7), that is σ_i which lies at the intersection of the tangent going through $\left(\sigma_{i-1}, f(\sigma_{i-1})\right)$ with axes σ,

Step 4: Calculate $|\varepsilon| = \frac{\sigma_i - \sigma_{i-1}}{\sigma_i}$

Step 5: Checking, checking, if $|\varepsilon| < \varepsilon_{tol}$ dengan $i = 1, 2, \dots, n$, then the iteration is stopped with σ_i as the estimate of the solution σ of the volatility function $f(\sigma)$, but if $|\varepsilon| > \varepsilon_{tol}$, then the process is continued back to step 1.

3. RESULTS AND DISCUSSION

Based on the obtained data, the volatility function (6) can be written as:

$$f(\sigma) = 2{,}18 - 21{,}11. N(d_1) + 21. e^{-0{,}0030625} N(d_2) \quad (10)$$

where,

$$d_1 = 2{,}0166012\sigma^{-1} + 0{,}25\sigma \quad (12)$$

$$d_2 = 2{,}0166012\sigma^{-1} \quad (13)$$

and

$$f'(\sigma) = -21{,}11 \times 0{,}5 \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1{}^2}{2}} \quad (14)$$

when $f(\sigma) = 0$, then it satisfies equation (1) or (6) and has a unique volatility value or values.. This condition can be seen in Figure 1 where the curve intersects with the axis σ in the intervals $0 < \sigma < 1$.

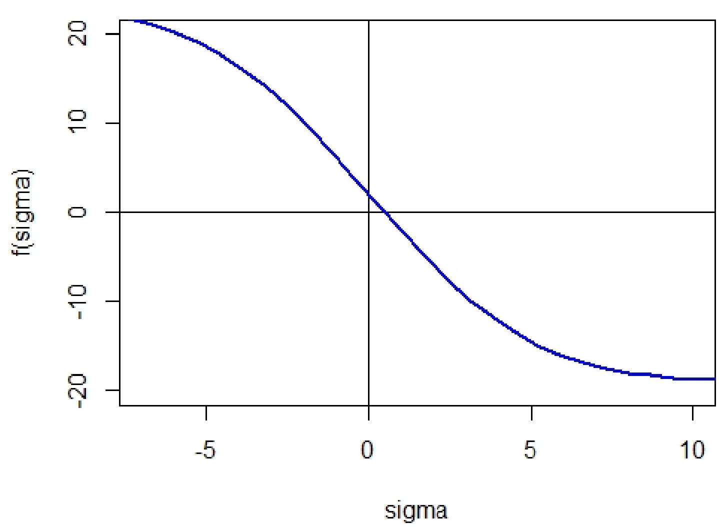

Figure 1. Plot of the volatility function

The iteration results of estimating the root of $f(\sigma)$ using the bisection, secant, and Newton Raphson methods, are respectively shown in Tables 1, 2, and 3. The initial approximation values used in bisection and tangent methods are $\sigma_{i-1} = 0.1$ and $\sigma_i = 1$ respectively, with the tolerance error 0.0001. Whereas, the initial approximation used in Newton's method is $\sigma_i = 0.1$ with the same tolerance error.

Table 1. Iterations of bisection method in finding the volatility value.

i	σ_{i-1}	$f(\sigma_{i-1})$	σ_i	$f(\sigma_i)$	σ_{i+1}	$f(\sigma_{i+1})$	$\lvert\varepsilon_r\rvert$
1	0.1	1.674629	1	-2.05716	0.55	-0.20603	0.818182
2	0.1	1.674629	0.55	-0.20603	0.325	0.731913	0.692308
3	0.325	0.731913	0.55	-0.20603	0.4375	0.262202	0.257143
4	0.4375	0.262202	0.55	-0.20603	0.49375	0.027881	0.113924
5	0.49375	0.027881	0.55	-0.20603	0.521875	-0.08913	0.053892
6	0.9375	0.027881	0.521875	-0.08913	0.507813	-0.03064	0.027692
7	0.49375	0.027881	0.507813	-0.03064	0.500781	-0.00138	0.014041
8	0.49375	0.027881	0.500781	-0.00138	0.497266	0.013249	0.00707
9	0.497266	0.013249	0.500781	-0.00138	0.499023	0.005934	0.003523
10	0.499023	0.005934	0.500781	-0.00138	0.499902	0.002276	0.001758
11	0.499902	0.002276	0.500781	-0.00138	**0.500342**	0.000447	**0.000878**

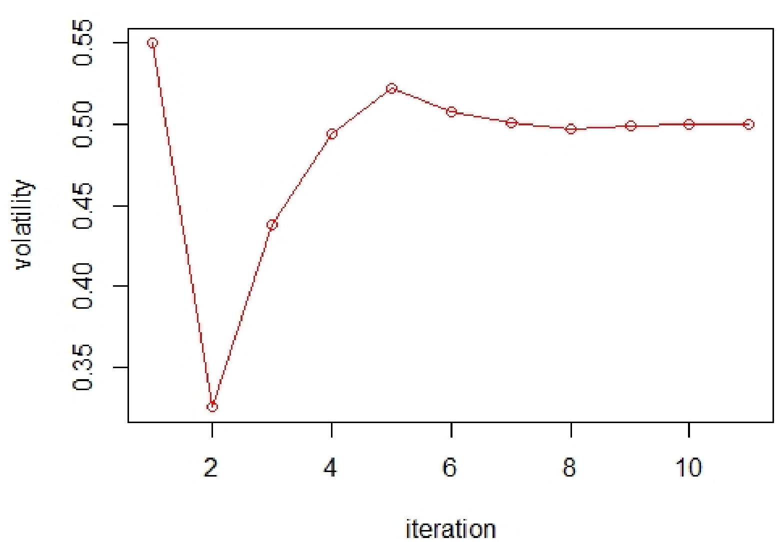

Figure 2. Volatility value per iteration using bisection method

It can be seen (see Table 1) that the iteration stops at the 11th iteration and gets a volatility value of 0.500342 with an absolute error value of 0.000878. Meanwhile, the results of estimating of the volatility value from the first to the last iteration did not significantly experience a trend. From the first to the fifth iteration, it is seen that the dynamics of fluctuating curves are very erratic. The convergence began to appear in the 8th iteration (see Figure 2). The Bisection method has a linear convergence, the error is reduced by at least ½ of the previous error. This can be seen in the results shown in Table 1.

Table 2. Iterations of secant method in finding the volatility value.

i	σ_{i-1}	$f(\sigma_{i-1})$	σ_i	$f(\sigma_i)$	σ_{i+1}	$\lvert\varepsilon_r\rvert$
1	0.1	1.67462931	1	-2.0571645	0.503872	0.9
2	0.503872	-0.0142412	0.1	1.67462931	0.500466	0.403871831
3	0.500466	-7.03E-05	0.503872	-0.0142412	0.500449	0.003405603
4	0.500449	3.8281E-09	0.500466	-7.03E-05	**0.500449**	**1.68938E-05**

Based on Table 2, it can be seen that the iteration stops at the 4th iteration and gets a volatility value of 0.500449 with an absolute error value of 1.68938E-05. Meanwhile, the results of estimating of the volatility value from the first to the last iteration experienced an upward trend which immediately converged to the true value (see Figure 3). In this case, the number of iterations and the absolute error value obtained by the secant method are smaller than those of the bisection method. This is consistent with the theory that the Secant method has better convergence rate than the Bisection method, where the rate of convergence for secant method is of order 1.6.

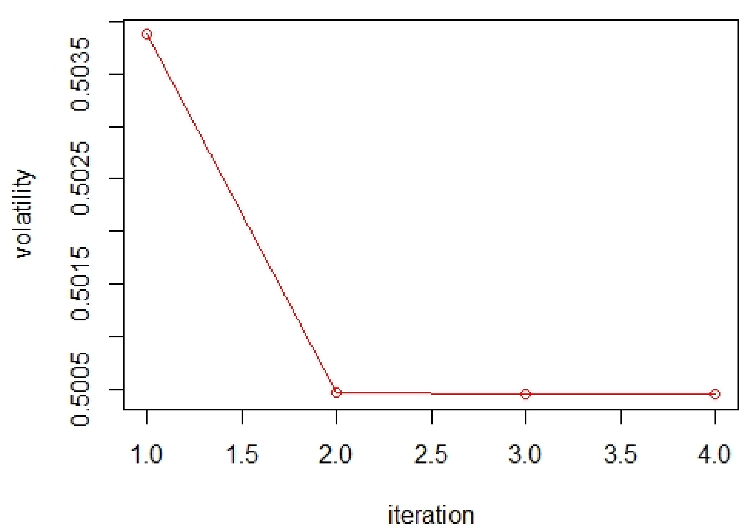

Figure 3. Volatility value per iteration using secant method

Table 3. Iterations of newton raphson method in finding the volatility value.

i	σ_{i-1}	$f(\sigma_{i-1})$	$f'(\sigma_{i-1})$	σ_i	$\lvert\varepsilon_r\rvert$
1	0.1	1.674629	-3.925683	0.526583	0.810096
2	0.526583	-0.108703	-4.129087	0.500257	0.052626
3	0.500257	0.000802	-4.131593	**0.500451**	**0.000388**

Based on Table 3, it can be seen that the iteration stops at the 3rd iteration and gets a volatility value of 0.500451 with an absolute error value of 0.000388. In this case, the number of iterations and the absolute error value obtained by the newton method are smaller than those of the bisection and the secant method. The problem with this method lies in the difficulty of finding an appropriate initial approximation value which is sometimes close enough to the root. In addition, Equation (7) requires that $f'(\sigma_{i-1}) \neq 0$. Otherwise, is undefined. However, the Newton method has a quadratic convergence, which means the error decreases by a factor of 2. This is also shown in Table 3.

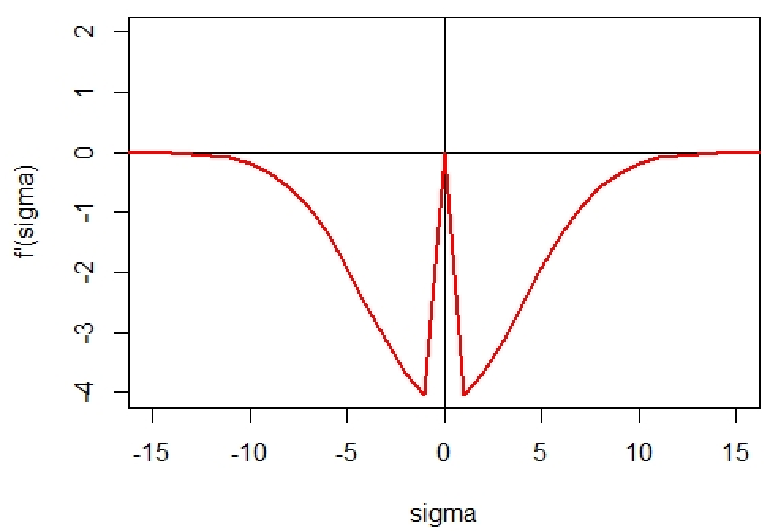

Figure 4. The plot of volatility function derivative

Setting the initial value to zero causes the value of σ_i immediately undefined in the first iteration. Another problem is that there exists an undefined value of σ_i at certain iterations. That case occurs when the initial approximation used, returns the value of $f'(\sigma_{i-1})$ being close to zero, Such examples occur when we use initial approximation values $\sigma_{i-1} = -10$, and $\sigma_{i-1} = 0.001$. It can be seen from Figures 5 and 6, that the former makes the value of σ_i undefined in the 3rd iteration, and the latter makes the value of σ_i undefined in the 2nd iteration. This is due to the value of $f'(\sigma_{i-1}) \approx 0$.

```
> newton.raphson(Snol,K,COBS,t,r)
Iteration  1
sigma(i-1) =  -10
f(sigma(i-1)) = 22.8547
f'(sigma(i-1)) = -0.1834007
sigma(i) =  114.6162
absolute Error =  1.087248

Iteration  2
sigma(i-1) =  114.6162
f(sigma(i-1)) = -18.93
f'(sigma(i-1)) = -2.141845e-178
sigma(i) =  -8.838174e+178
absolute Error =  1

Iteration  3
sigma(i-1) =  -8.838174e+178
f(sigma(i-1)) = 23.11579
f'(sigma(i-1)) = 0
sigma(i) =  Inf
absolute Error =  NaN
```

Figure 5. Iterations of the Newton Raphson method with an initial approximation of $\boldsymbol{\sigma_{i-1}}$ = -10

```
> newton.raphson(Snol,K,COBS,t,r)
Iteration  1
sigma(i-1) =  0.001
f(sigma(i-1)) = 2.005786
f'(sigma(i-1)) = -2.466933e-265
sigma(i) =  8.130686e+264
absolute Error =  1

Iteration  2
sigma(i-1) =  8.130686e+264
f(sigma(i-1)) = -18.93
f'(sigma(i-1)) = 0
sigma(i) =  -Inf
absolute Error =  NaN
```

Figure 6. Iterations of the Newton Raphson method with an initial approximation of $\boldsymbol{\sigma_{i-1}}$ = 0.001

By using trial and error operated in the R programming software, we come to the conclusion that the appropriate initial approximation for the Newton Raphson method lies in the interval $\{-5.3233 \leq \sigma_{i-1} \leq$-0.0159} or $\{0{,}0157 \leq \sigma_{i-1} \leq 5.8663\}$ (with rounding to 5 digit numbers).

The estimation of volatility value using the Newton Raphson method has taken a smaller number of iterations. This is due to the rate of convergence of this method is quadratic. Using $\sigma_{i-1} = 5.8663$ as the initial approximation (see Figure 7) Newton method takes 12 iterations to converge.

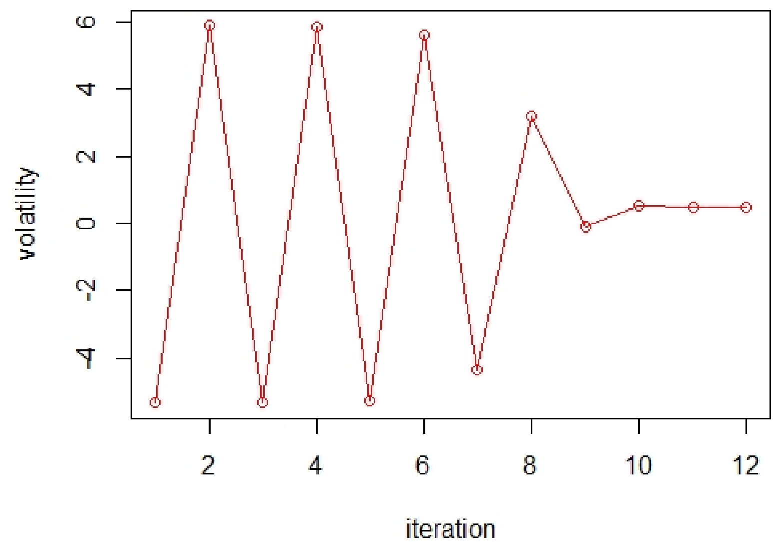

Figure 7. The furthest iteration in understanding the volatility value using the Newton Raphson method

Very high deviation of fluctuation of volatility values resulted by newton method occurs in iterations 1-7. In subsequent iterations, the deviation starts to decrease, hence it convergences to the true volatility value.

4. CONCLUSIONS

The estimates of the volatility values obtained by using the bisection, secant, and Newton Raphson methods can be seen in Table 4.

Table 4. Volatility approximation value using bisection, secant, and newton raphson method.

Method	Total iterations	Volatility	Absolute error
Bisection	11	50.0342%	0.0878%
Secant	4	50.0449%	1.68938E-03%
Newton Raphson	3	50.0451%	0.0388%

We can see from Table 4, the method with the smallest number of iterations is achieved by the Newton Raphson method with a total of 3 iterations. Even though the secant method has the smallest absolute error compared to the other 2 methods, however, with the same number of iteration, such as 4, Newton Raphson has the smallest absolute error. The problem with the Newton Raphson method lies in the difficulty of finding an appropriate initial approximation that must be in the range $\{-5.3233 \leq \sigma_{i-1} \leq$-0.0159$\}$ or $\{0,0157 \leq \sigma_{i-1} \leq 5.8663\}$ in this case. The farther the estimate from the actual volatility value, the more iterations needed.

Therefore, to take advantage of having high rate of convergence owned by the Newton Raphson method, its use is generally combined with other simple, guaranteed convergence methods, for example Bisection. Bisection is usually used as initialization providing an appropriate initial approximation for Newton's method. In this way, hopefully a solution is guaranteed to exist, and convergence is being faster.

The largest number of iterations that the Newton Raphson method can achieve to calculate this volatility is 12 iterations. The maximum iteration occurs when the initial estimate used is σ_{i-1} = 5.8663.

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