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Splitting 4D spacetime into 3D space and 1D time involves operationally needed presence of two 3D spaces enclosed inside two 4D paired dual reciprocal spatial structures

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ABSTRACT

The anticipated separation of the 4D spacetime into 3D length-based space and distinct abstract 1D space of elapsing time, which would vary independently of the usual variables of the length-based space, is not reasonable in the formerly unspoken single space reality (SSR) framework, because the separation would lead to inadmissible identification of multiplicatively inverse/reciprocal derivative taken with respect to elapsing time parameter with the derivative taken with respect to the 1D time variable. It is thus unacceptable because $1/(\partial/\partial t) = \partial t/\partial$ is a contravariant expression posing as derivative (with respect to some unspecific variable) in the SSR setting and as such cannot be equated to the regular covariant derivative $\partial/\partial t$ taken with respect to the independently varying parameter of the elapsing time t. Nevertheless, the separation of spacetime is feasible within the multispatial reality (MSR) framework that allows the ensuing two separated spaces to reside in twin distinct spatial structures each of which is equipped with different orthogonal homogeneous basis, provided the twin spatial structures are equidimensional 3D spaces that are mutually paired to form a hierarchical hyperspatial structure comprising mutually dual reciprocal spaces. The conclusion is supported by special relativity. In a sense thus, the process of splitting of spacetime apparently demands a paradigm shift: from the formerly unspoken and thus unchallenged SSR paradigm to a certain MSR paradigm.

***Keywords*:** Splitting heterogeneous spacetime, inverse/reciprocal relativistic Lorentz functional, paired dual reciprocal spaces, equidimensional spatial hyperstructures

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1. INTRODUCTION

Traditional mathematics was developed and operated under ~~the~~ auspices of formerly unspoken (and thus never explicitly questioned) single space reality (SSR) paradigm, which assumed that the abstract infrastructure underlying mathematical reality can always be equated with a single universe that is set-theoretical space, whose subspaces were identified with sets understood (in a set-theoretical sense) as just selections made from the universal set. Notice that the definite article 'the' is purposely crossed out above because the guidance and/or prognostication of the possible impact of directives supplied or imposed by paradigms in mathematical sciences is neither definite nor a common knowledge and therefore presence of definite article might have inadvertently created a false impression implying that we can always determine all possible outcomes of each mathematical paradigm we have inherited and espoused. Since the first two conditions do not apply to the essentially primitive (i.e. undefined in their scope and of unforeseeable impacts) fundamental mathematical paradigms, and no repeated use of the noun/phrase could possibly turn it into "common knowledge", the use of definite article would be unwarranted, especially in the case of formerly unrecognized mathematical paradigms, at least not at the present stage of their still unfinished development. Hence, I am not violating the mindless interpretation of the linguistic rule of definiteness (namely: either common knowledge or definite consequences/implications or its contextual repeated use) but I am actually upholding the rule, despite possible uninformed appearances to the contrary. Note that emergence of numerous tacitly veiled inconsistencies as well as outright operational and less outright (or merely indefinite) structural nonsenses in the framework of the SSR paradigm, substantiates my claim that the paradigm's indefinite implications were extremely difficult to pinpoint and thus tough to recognize. Otherwise, the traditional mathematics might have already rectified them instead of just tacitly covering them up in defiance of logic, and frequently with open disregard for scientific common sense.

As long as the supreme goal of genuinely scientific papers is unambiguous presentation of accurate and quite unequivocally communicated ideas, the – often thoughtless – adherence to well-trodden linguistic phraseology could be discarded. Having said that the conventional phraseology could be disregarded when it sows confusion, I am not advocating disrespect for linguistic or other rules. On the contrary. The use of definite article is acceptable when the noun or phrase has either definite meaning, or falls under the semiotic umbrella of common knowledge, or when it was already used in the text before, provided its repeated use has really contributed something of semiotic importance to clarification of the phrase's meaning, which is not the case with mathematical primitive notions or unrecognized subliminal paradigms.

As the style of my communications is concerned, I tend to follow the "Style Guide" of the American Institute of Physics, one of whose advisories included the recommendation that hyphenated words should be avoided, as well as several other features facilitating computer-friendly search and retrieval. For we don't really need a Cartesian product of all possible combinations of 'self' and all the other words that could be associated with 'self' when we search for 'self-evident', for example, using some well-known yet sometimes ill-designed, if not utterly infantile, search engines. Other than that, I must abide by the rules instituted and enforced by editors and publishers of scientific journals.

In summary: Although it is admissible to say "the SSR paradigm" because it certainly is definite statement that only introduces the name of the paradigm without saying anything else of semiotic importance, I shall avoid putting the definite article in front of such phrase as "auspices of the SSR paradigm" in order not to imply that all the unforeseeable (even if not quite feasible to implement, or perhaps not really possible even to fathom yet) outcomes of the paradigm are definitely known at present. For they are not. By the same token, however, the use of definite article in the phrase "the implications of the SSR paradigm" could be acceptable only after the pertinent implications have already been listed before the phrase is going to be used. In my experience with mathematical theories, the situation in which 'auspices' or any other consequential descriptor of fundamental mathematical paradigms involving prognostication might become an entirely definite part of the so-called "common knowledge", or whatever the latter phrase could mean, will never materialize, because I have already demonstrated that the operational mathematics that links differentials (i.e. rates of change) to evaluations of mathematical laws (or perhaps of the operational rules that are underlying physical laws) is expandable indefinitely [1]. Hence the mathematical laws are fixed only to the extent – or at the given depth of inquiry into the physical phenomena governed by the laws, which could be expanded as the present level of inquiry deepens – that their prospective impacts can remain unaltered.

Therefore, I think that colloquial linguistic usage should not dictate how reasonings in mathematics or in mathematized sciences shall be presented, especially in the cases where some semiotically significant reasonings could be inadvertently compromised through sticking to obviously inconsiderate though customary manners of linguistic expressions coined in contexts whose exactness does not matter. The nontraditional mathematical reasonings that I frequently offer are not just for those readers who can think quite independently but also for those students who dare to think outside the traditional confines of the former axiomatized abstract mathematics, regardless of whether the confines are constricted (i.e. unnecessarily restrained, not merely restricted) by unchallenged traditions rooted either in blindsided linguistical habits or in mathematical idiosyncrasies. When one consults a dictionary, all these terms might appear as being linguistically equivalent, but one could also perceive some slight though semiotically significant nuances in their contextual connotations, which usually are indiscernible in casual presentations. The distinctions are not necessary to be taken under consideration, if the reader's first objective is just to grasp the ideas being presented. But for those trying to advance the development of mathematics via building of new concepts upon nontraditional ideas, the nuances are of utmost importance.

There is yet another aspect of one's choice of linguistic and mathematical expressions. If I would try to submit for publication the paper "Flawed fundamentals of tensor calculus" [2] in exactly the same form a few decades ago, it would be flatly rejected before I have rectified the – faulty as it was – former theory of differential operators [3], as well as related papers and especially [4], in which I have demonstrated the feasibility of unrestricted division by zero via multiplication by an operationally sound notion of infinity. The use of expressions in scientific papers presenting new ideas is like a tightrope balancing act. For I must balance the words I prefer to use (with their prospective importance in mind), against the words that the editors/referees/publishers of scientific journals would not consider as being objectionable, given the present *status quo* of mathematical and mathematized sciences.

Moreover, every single scientific theory I have tried to learn has its problems. The goal of sciences should not be the strive to keep at all costs their problems tacitly concealed but rather to reveal them in earnest attempts to eventually rectify them. If not we, then perhaps the alert students, who want to read honestly presented scientific papers despite the often stinging them fuzzy feeling of being fooled by some somewhat controversial feats found in traditional presentations of vividly unreasonable mathematical reasonings, could be able to eventually correct the still undisclosed yet conceptually problematic scientific issues.

The gravest danger to sciences comes from making existential postulates, which is clear for such postulates can create nonexistent entities by decree. Less obvious danger comes from excessive axiomatization. Gustave Choquet pointed out that Euclid-Hilbert axiomatization of geometry, which was based on the notions of length, angle and triangle, so marvelously concealed the underlying vector space that the concept of vector remained unrecognized for ages [5]. Though taken from experience, axioms are often abstracted without deference to experiments implying presence of different reality. In absence of formal mathematics in Greece prior to Euclid, his compilation of achievements of philosophers under axiomatic umbrella started pyramidlike buildup of all sciences. Yet once the apex of that pyramid was reached and there is no way to go up, even though the Euclid's deductive model can still be applied to erect some other, previously nonexistent science-pyramids, the foundations of his axiomatic pyramid should be nonpostulatively enlarged via inductive synthetic reasonings.

In reference to Axiom of Choice (AC), for example, R.L. Goodstein has pointed out that it does not create the choice set, for no axiom, no declaration of existence can create a real object. The AC is [just] a limitation on the use of the word set in a formalized set theory and thus the acceptance or rejection of the axiom of choice is a decision about the use of a word [6]. The importance of awareness of unrecognized dangers of axiomatization and of making existential postulates disguised as axioms is that they can tacitly contribute to concealing of an abstract mathematical- and physical reality even quite inadvertently. Thus, I prefer to adhere to constructivist and intuitionistic trends in the philosophy of mathematics.

2. THEORETICAL POSSIBILITY OF MULTISPATIAL INFRASTRUCTURE OF THE ABSTRACT MATHEMATICAL REALITY WE DEAL WITH

Having investigated several previously quite unanticipated results of unbiased physical experiments I was able to explain and reconcile them either fully or just partly when the data collected was insufficient. Nevertheless, my mathematical analyses of certain recorded experimental data hinted at the possibility that the physical reality we live in (and thus also the abstract mathematical infrastructure of the physical reality revealed in those experiments) may form a hierarchical quasispatial structure. Therefore, I have proposed the multispatial reality (MSR) paradigm as replacement for the formerly unspoken, and thus unchallenged even though clearly unjustified, SSR paradigm. The term 'multispatial' does embrace considerably large variety of formerly totally unrecognized structural (i.e. geometric and/or quasigeometric, as opposed to just algebraic and/or operational, including differential, in general) features, which shall be further explained elsewhere.

The theoretical possibility of actual presence of an intricate multispatial infrastructure of the abstract mathematical realm that underlies the physical reality we live in, has several roots:

- Conceptual: inspired by curious inability of some theoretically well-founded mathematical theories of physics to reconcile or explain previously unanticipated results of unbiased physical experiments
- Experimental: inspired by either discrepant or previously unanticipated results of childishly simple (yet previously unreconciled by traditionally accepted theories of former physics) unbiased experiments

- Theoretical: inspired by reevaluations of both mathematical and physical theories, which may be further subdivided into:
    - Operational: inspired mainly by differential as well as algebraic operations some of which have been tacitly compromised in traditional mathematics
    - Structural: inspired by investigations of unpostulated geometric and/or quasigeometric structures whose presence is inferred from operational algebraic structures rather than upon postulates declaring their existence, which can create nonexistent entities
    - Constitutional: inspired by reevaluations of unsubstantiated laws usually decreed upon guiding principles sometimes disguised as scientific laws
    - Synthetic: derived/synthetized from abstract confrontations of apparently contradictory yet realistic and often experimentally corroborated or merely conjectural assertions
    - Predictive: inspired by their apparent ability to make some prospective yet formerly quite unforeseeable scientific predictions.

However, some abstract mathematical concepts are rather difficult to fathom and it can be challenging to visualize them or even to express them in easy to comprehend terms. Among structural concepts, the multispatial infrastructure of the mathematical reality is very difficult to grasp, indeed. Among operational concepts, the abstract notion of neverending ascending real (as in the real numbers domain $\mathbb{R}$) infinity that is necessary for operating on multispatial structures can also be hard to grasp and even more tough to conceptualize. In these cases, it may be helpful to perform algebraic and/or differential operations whose outcomes could then be used to formulate distinctive predictions, which then could be actually tested in physical experiments at least in principle, or perhaps in abstract thought experiments.

Giordano Bruno, who was burned at the stake of Inquisition for his philosophical insistence that infinities can actually exist in reality, among his other then-controversial yet based on contemporary science assertions, admitted that although the infinite cannot be the object of sense perception, the one who would deny the existence of a thing merely because it cannot be apprehended by the senses nor is visible, would presently be led to the denial of his own substance and being – compare [7]. For existence of imperceptible by senses concepts could perhaps be recognized by their observed effects.

L'Hôpital has devised mathematical rule to avoid any explicit use of infinity understood as the limit of a tendency [8]. Then Georg Cantor, who further formalized the notion of infinity in set-theoretical terms, realized nonetheless, that it can cause paradoxes [9]. The culture of avoidance continued with impunity and indifference to such obviously questionable operation as: $x+\infty=\infty+x=\infty$ for any real number $x\in\mathbb{R}$, which was proposed in [10], [11], and perpetuated in other academic publications, despite the fact that the inconsiderate traditional approach to infinity generates such nonsenses as $1=0$ see [12]. Since infinity and zero are naturally reciprocal, I have demonstrated that unrestricted division by zero is feasible if it is implemented via multiplication by operational infinity [4], [13], [14] and references therein. However, unrestricted division by zero demanded truly operational notion of the ascending infinity, which is quite reasonable requirement, indeed – see [15].

These theoretical developments confirmed my scientific belief that for mathematics to become realistic, it needs not only feedback from physical experiments but also continuous guidance from the hints that are supplied by curious results of some previously unanticipated experiments. For if mathematics is left alone, it is likely to degenerate again. The conceptual root of my suspicion that a multispatial mathematical infrastructure of the physical reality we live in could indeed exist, emerged from the conclusion drawn by George Szekeres upon his analysis of some experiments conducted by Sadeh and his teams, which I have already discussed in [16] and in fairly simple terms in [17]. Szekeres concluded in AD 1968 that no relevant theories of then contemporary physics could explain, not to mention reconcile, the curious experimental results [18]. His undeniably honest conclusion virtually amounts to slap in the face of overconfident theoretical physics as well as of the haughty traditional mathematics upon which the physics had been developed. Notice that there was no revolutionary breakthrough in physics since AD 1968 (i.e. nothing comparable to scientific revolutions due to works of Planck, Einstein or de Broglie, to name just a few), at least not in the realm of the pertinent relativistic theories (i.e. gravitational and electromagnetic).

Although both devastating and mindboggling to me at the time I reviewed it, the Szekeres' conclusion sparked my interest in redefining traditional theories of mathematics and physics so that they could make predictions in line with the formerly quite unanticipated results of the unbiased curious experiments, which I was eventually able to explain and reconcile in [16] and casually explained in [17]. But the successful reconciliation indicated that many traditional theories of mathematics and physics – among those which I have studied – may need profound conceptual and theoretical improvements.

3. EXPERIMENTAL HINTS AT POSSIBILE PRESENCE OF A MULTISPATIAL INFRASTRUCTURE OF THE ABSTRACT MATHEMATICAL REALITY WE EVIDENTLY LIVE IN AND THUS HAVE TO DEAL WITH

Galileo Galilei's conclusion that composition of matter, which is conventionally measured by specific gravity, is immaterial for then-observed radial gravitational phenomena, was instrumental for Newton's and then Einstein's theory of gravitation, which goes by the name of general theory of relativity (GTR). Nonetheless, several experiments, which I too have investigated, cast some doubts on that hasty Galilei's conclusion. Among them, three very ingenious experiments conducted by Sadeh and his teams, revealed to me something quite to the contrary, even though Sadeh did not discount the Galilei's claim. Yet he compared the observed frequency decrease in rays coming from Taurus A that passed near our Sun before they were intercepted on the Earth [19], to similar frequency decrease found in radio waves emitted by stationary atomic clock, when the waves have been intercepted by atomic clock riding on a truck driven on the interstate highway I-95 that runs along the US East Coast [20].

Their further search for an effect of mass on frequency during a close (within 5 angular degrees) approach of a pulsar to our Sun showed no evidence of such an effect when they measured the pulse arrival time [21]. This last result means that the previously observed frequency decrease is a tiny close to surface effect. Therefore, I presumed that the decrease in frequency ν (which determines energy $E = h\nu$ where h denotes Planck's constant) should not depend only on the masses of the respective celestial bodies (namely our Sun and Earth) virtually extorting energy from the rays from Taurus A and the radio waves that happen to pass too close to the respective bodies on practically almost tangential paths along their surfaces. Further indication that I was right came from the fact that our Sun exerted 3.9 times larger effect on the rays than the Earth exerted on the radio waves [20]. The ratio 3.9 can be obtained also from comparison of the respective densities of matter of the Earth (5.52) and of our Sun (1.42) [22]. Yet because less dense Sun exerted 3.9 larger effect on the rays than much denser Earth exerted on the radio waves, I realized that density of matter, not that of mass, which is inertial attribute and thus can be altered in motion, as the special theory of relativity (STR) predicted, must be placed into denominator of the extra formula, which I have proposed [16] for describing the extra nonradial effects of gravitational force fields that I have discovered mathematically upon my analyses of the aforesaid (and few other) ingenious physical experiments. As the reader can see, my explanation and reconciliation of these experiments was pretty simple, because besides our Sun (and Earth, in the case of radio waves) there was no other culprit to blame for the observed frequency decreases. But the mathematics to justify the nonradial effects of the radial/center-bound gravitational force fields of our Sun and Earth was not only nonexistent at that time, but its prospective development was rather difficult then to accept because it contradicted many traditional views and even some theories of mathematics as well as physics.

None of our contemporary theories of gravitation is undermined by the new theory of nonradial effects of gravitation because all previously developed theories were purely radial. The theory of nonradial effects complements all radial theories of gravitation without ever contradicting any of them. Einstein himself freely admitted that "deviations to be expected from the radial gravitational effects [due to some other than purely radial effects] are much too slight to be noticeable on the earth's surface." see [23] p.161. Thus, he did not preclude the possibility of presence of some nonradial gravitational effects that would make the GTR's effects deviate from his purely radial predictions, but merely ignored their possible impact as being then undetectable on the Earth. However, the two Sadeh's experiments [19] and [20] demonstrated that some nonradial effects of gravity fields can be detected indeed.

Since he has made his remark in reference to the possibly changing gravitational potential [23] p.159, the remark could also be attributed to any other than the purely radial (hence nonradial) effects of the – even if only slightly – varying scalar potential, which could include also some other nonradial effects, i.e. tangential and/or binormal in the parlance of differential geometry. I have showed that the tensor calculus upon which the GTR relies is fundamentally flawed because it tacitly discarded any other than purely radial effects [2] of the usual radial/center-bound gravitational force fields. I have also showed that equipotential energy exchange depends inversely on density of matter besides its directly proportional dependence on mass [24], which also contributes to emergence of some nonradial effects of purely radial fields of gravity, because on equipotential surfaces there is no change in the usual radial effects of gravity (that depend on changing gravitational potential). For purely radial effects alone are of main interest to all traditional theories of gravitation. Thus, like all others, the Einstein's theory has a limited domain of applicability [25].

The rays coming from Taurus A was the Nature's counterpart of the reported Galilei's experiment at the leaning tower of Pisa [17]. Instead of dropping balls, which are too small and of insignificant mass that cannot be meaningfully compared with the massive Earth, the Nature dropped rays, whose rest mass is insignificant, by our Sun. And the radio waves that are a sort of rays too, pushed by atomic clocks along the Earth's surface, supplied data for comparing the Earth with our Sun, which is feasible. The rays and radio waves were just tools for comparison of the impact of our Sun with the impact of the Earth on the respective rays. The prospective mathematics that must be devised to handle the impact of mass, which also equates to energy via the famous Einstein's formula $E=mc^2$, and the inversely proportional (i.e. reciprocal) impact of density of matter on the energy of rays/waves are being superposed, which fact has already been confirmed [16], demands a certain multiplicatively inverse (i.e. truly reciprocal) mathematical approach. That is the chief reason why I have envisioned the presence of geometric or quasigeometric multispatial structures established via pairing of dual reciprocal spaces and then developed operational mathematics to handle the structures.

It is known that energy cannot be localized in the GTR that was created and operates in the SSR framework. This is because depicting reciprocal variables or functions cannot be unambiguously accomplished within the very same single space while allowing both types of magnitudes (i.e. primary and their reciprocals) to vary quite independently of each other. This fact was not known, neither to Einstein nor to anybody else accepting the SSR paradigm. That is why the traditional mathematics wrote extensively about operations involving commuting variables with just a few marginal exceptions, but was almost silent on operations involving noncommuting variables, because primary magnitudes do not commute with their truly multiplicatively inverse reciprocals. Some authors recognized the dilemma but declared that the source of our difficulties evidently lies in the use of general curvilinear coordinates [26]. Recall that scale factors, which form Jacobian in spherical coordinates, are reciprocal in the traditional gradient [27] p.21f. Therefore it is surely important for operational reasons to explore the possibility of splitting the heterogeneous quasispatial structure of 4D spacetime into two distinct homogeneous spatial structures: 3D usual geometric space and 1D time, for in that situation no reciprocal scale factors would appear explicitly due to the instituted homogeneity and orthogonality of the ensuing spatial and temporal representations.

However, such disassembling of 4D spacetime gets rid of the truly reciprocal scale factors that are necessary in curvilinear coordinaters at the expense of having established paired dual reciprocal spatial structures. Hence we are not loosing anything mathematically speaking, because the reciprocity is preserved. But instead of operational reciprocity we are going for structural reciprocity. For we are just replacing the troublesome operational reciprocity, which frequently generated previously unsurmountable problems, with the – corresponding to the, often difficult to assuage differential operations performed under the former SSR paradigm – structural/geometric reciprocity, which we could handle pretty well, provided we work in the multispatial framework under the MSR paradigm. In other words: we are not abandoning the inherent reciprocity, but we just want to make it reliably manageable via the new methods that I have already developed for unambiguous operations performed on the paired dual reciprocal spatial structures. In my professional opinion it is fairly good tradeoff.

4. SOME THEORETICAL HINTS FROM CLASSICAL PHYSICS IN FAVOR OF MULTISPATIALITY

It seems to me that the multispatial product differentiation rule MSPDR1, which emerged from matching partly covariant and partly contravariant evaluation of an integral kernel according to the – proven and operationally mandatory – regular scalar product differentiation rule (RSPDR1) for compounded scalar-valued functions u(x) and v(x) that depend on the same single independently varying variable x [28], can also be compared to certain equations specifying abstract conditions for evaluation of entropy

$$\frac{\partial U(x,t)}{\partial t} + \frac{\partial}{\partial x} F(U) = 0 \quad where\ U(x,0) = V(x) \qquad (1)$$

which have been concisely discussed in [29]. As one can see, the formula (1) implies the possibility that, in general, the potential U could be a function split between two distinct representations, each of which demanding its very own spatial structure if homogeneous bases are required. The mathematics demonstrated in the eq. (1) is conceptually important for this presentation. However, this particular venue is of interest mainly to physical applications and thus shall be further explored elsewhere.

It has been found that the behavior of eigenfunctions within the domain of complex eigenvalues is very unusual, for the logarithm of the wave function at different coordinates fluctuates strongly just like the position of Brownian particle fluctuates in time, even though the wave functions are strongly localized [30]. That unusual kind of behavior could virtually suggest – at least to me – the real possibility that space- and time domains can be considered as composing a certain hyperspatial structure comprising paired dual reciprocal spaces.

Hence the 4D spacetime may be regarded as virtual quasispatial structure in which the elapsing time plays the role of a hinge or pivot connecting the two naturally reciprocal geometric domains when they are viewed as quasigeometric spaces. Similarly, in the 4D timespace that corresponds to the spacetime, the spatial trajectory curve may be virtually pivoting the whole temporal reciprocal space [31]. This issue shall be further discussed elsewhere. Now let us review a certain traditional approach to splitting the 4D spacetime into its subcomponents, which was performed in the framework of the SSR paradigm.

5. SPLITTING THE HETEROGENEOUS QUASIGEOMETRIC 4D SPATIAL STRUCTURE OF SPACETIME INTO TWO DISTINCT HOMOGENEOUS SPATIAL STRUCTURES OF 3D GEOMETRIC SPACE AND 1D TIME

It is logical and quite natural expectation that if something has been assembled from few independently existing components then it should be possible to disassemble it. However, when it comes to spacetime, those who ever tried to do that might have found that it is not as trivial as it may appear at first glance. If a car mechanic would pull out a part of your car's engine from under the hood of your car while saying that now it shall run better, then you might be a little bit suspicious of the mechanic's credentials if the allegedly extraneous removed part was supplied and/or authorized for use by the vehicle's manufacturer. By the same token, if an abstract mathematical disassembling the 4D spacetime seemingly leaves something else besides the 3D space and the 1D elapsed time that were used to establish the compound quasispatial structure of 4D spacetime, then one might rightly suspect that perhaps there is more to the abstract mathematical reality than we realized. In other words: The Nature that seemingly plays the role of manufacturing assembly of the physical reality as well as of the underlying it mathematics, apparently put into them something extra that we were not aware of before we disassembled the spacetime. The allegedly extraneous part has suddenly just fallen out during the disassembling process to our big surprise is an embarrassment to former mathematics. Just as that mechanic probably did, the traditional mathematics was trying to cover up its conceptual indolence by pretending that nothing has ever fallen out.

Let us review a certain controversial conclusion made by Eberhard Freitag and retrace his informal though mathematically impeccable abstract reasoning, which – despite its logical correctness and perfectly formalized chain of deductions – generated rather unacceptable mathematical conclusion. The emergence of his questionable conclusion cannot be blamed on him, however. It was caused by his adherence to the formerly unspoken SSR paradigm that he – just as almost everyone else – inherited from the traditional mathematical education and virtually espoused, presumably without ever questioning its fundamentals.

Nevertheless, incidentally he has also proved wrong certain very fundamental principles that support the – essentially set-theoretical – foundations of the traditional mathematics, even if quite inadvertently. Through his very nuanced and sophisticated (both conceptually and mathematically) abstract reasoning he has accidentally uncovered tacitly veiled contradiction between the allegedly selfevident logical expectation (that it should be possible to disassemble a compound concept such as spacetime into its constituent subcomponents, i.e. the usual length-based space and the elapsing time) and the operational (i.e. algebraic as oposed to structural/geometric) outcome of his reasonings that the elapsing time ensuing from the decomposition is inverted. Although he did not address the cause of the contradiction, it is easily perceptible to me, for one, that the pernicious controversy between the processes of composition and decomposition emerged due to the conceptually defective SSR paradigm.

For there would be no contradiction in the MSR framework, in which pairing of dual reciprocal spaces ensures automatically the inverse relationship existing between the paired spaces, because mathematical reciprocity equates to multiplicative inversion by definition. Had he worked in the multispatial MSR setting, he might have attained conceptually significant breakthrough instead of the previously unrecognized pesky contradiction. That is why I consider his mode of presentation as well as his operational reasonings exceptional, despite the fact that the SSR paradigm apparently caused him to draw an erroneous and thus operationally and structurally unacceptable conclusion. For he has equated the inverse functional relationship $t^{-1}(t)$ (in his notation [32] p.81) in reference to a submanifold, with its improper, contravariant derivative operator $t^{-1}(t)$ mapped to the proper covariant derivative $\partial/\partial t$ which mapping I regard as definitely inadmissible and thus rather confusing substitution, both operationally and structurally (i.e. conceptually in general). I am not trying to put the proverbial horse before the cart, but to make it easier for the reader to see what is going on in his mode of presentation once his flawless formal reasonig is derailed (or adversely affected, if you will) by the – SSR-infected – operationally unacceptable conclusion.

Although he did not write it explicitly but just alluded to equating of primary and nverse derivarives with respect to time, his conclusion could be virtually equated to the mappping: $\{1/\partial/\partial t = \partial t()/\partial\} \rightarrow \partial/\partial t$ which attempts to map the contravariant/inverse derivative $\partial t()/\partial$ onto the covariant drivative $\partial/\partial t$. Although the latter mapping is acceptable in the MSR setting as mapping of frequency to time interval, for example, it is unacceptable in the SSR setting, because frequency understood as inverse time interval is not an actively varying function but functional (i.e. as if "frozen" or fixed/constant functional t[...] that does not actively vary anymore) and thus must not be differentiated within the same space as the function t(...) from which the functional t[...] emerged, for derivative of a constant function amounts to zero. Recall that in the MSR setting two paired dual reciprocal spaces are naturally inverse with respect to each other, because reciprocity means multiplicative inverse by definition. Hence if taken at its face value, the controversial mapping amounts to putting the inverse relationship upside down, which is commonly overlooked and unadvertised feature of the SSR paradigm.

However, the still unanswered question is what could go wrong if everything he wrote (with the exception of the aforesaid operationally inadmissible substitution that is disguised as mapping) is right? The definitely wrong thing is his SSR-induced operational conclusion, of course. But it has certain inadmissible structural consequences too. For as Theodore Frankel remarked "[…] it does not seem to be possible to differentiate a contravariant vector field to obtain a tensor field." – see [33].

Please watch how E. Freitag was virtually compelled/forced by the SSR paradigm to make the unfortunate conclusion. For there are only two possible culprits to blame. If not him then the traditional mathematics is definitely guilty of clearly tacit misrepresentation of some of its most fundamental – yet previously unchallenged even though unwarranted – presumably entirely subliminal abstract assumptions. I shall ask the reader to be the judge of his admittedly ingenious reasonings that, nevertheless, had fallen in the unrecognized conceptual trap of the unspoken SSR paradigm under which he – as well as virtually all traditionally minded mathematicians – operated.

Starting with few prerequisites he stated that a curve u in a Lorentzian manifold X is called future oriented if all tangent vectors [i.e. the derivatives du(t)/dt that correspond to the instantaneous velocities entertained in physics] are in the causal future [of the conventional Minkowski spacetime diagram] – see [32] p.80. A reference system on some part U of a Lorentzian manifold is a vector field A on U such that A_a is a future timelike for all $a \in A$ and an observer is a future timelike worldline [32] p.81.

He then explicitly assumed that the given differentiable Lorentzian manifold X is of [sufficient] dimension n and $t: X \to I$ is a differentiable map onto some open interval I. Then assuming also that the tangent maps $T_aX \to T_{t(a)}I = \mathbb{R}$ are all surjective (i.e. "onto") he then concluded that $X_t = t^{-1}(t)$ is a hypersurface (i.e. a submanifold) so that the kernel $T_aX \to \mathbb{R}$ can be identified with the tangent space of X_t. Now if the manifold X carries a pseudoriemannian metric g then one can restrict it to a positive definite tensor g_t (on X_t), in which case the orthogonal complement W (with respect to the metric g) of T_aX_t in T_aX is actually 1D and thus $T_aX = T_aX_t \oplus W$. Hence in the case when g is negative definite in W it maps isomorphically to $T_{t(a)}I$.

Then he wrote: "We take the inverse image of d/dt with respect to this isomorphism. This gives a vector field on X. For sake of simplicity we denote this vector field in X by $\partial/\partial t$. This vector field is time-like, i.e. $(\partial/\partial t, \partial/\partial t)$ is negative. So (X,g) has been equipped with a structure as Lorentzian manifold." - compare [32] p.81. Hence with all the aforementioned assumptions put in place, the Lorentzian manifold (X,g) admits [formal] separation [of what is commonly called spacetime] into [distinct] time and space if there is a surjective differentiable map $t: X \to I$ onto some open interval, provided that the timelike vector field is future oriented for the parameter t plays the role of [elapsing] time as an instant $x \in X$ takes place at the point of time t(x) for we can consider [the – assumed as differentiable – Lorentzian manifold] X as a family of 3D Riemannian manifolds X_t parametrized by time - compare [32] p.81f. The context suggests overlapping of the two (incompatibly varying) spatial structures onto each other in the universal singlespace reality. Hence it seems to be structural conflict of covariant- versus contravariant representation.

It is clear to me that he was virtually forced to make the unfortunate conclusion, because in the singlespace realm offered under the SSR paradigm there is no other space for placing the inverse differential operator that yields the contravariant derivative, which is operationally improper in the SSR setting but can be equated with (operationally proper) covariant derivative in a dual reciprocal space paired with the given space under the MSR paradigm – see [28] for more comprehensive discussion. For under ~~the~~ auspices of the MSR paradigm, most of which are still unforeseeable at present, an analogous reasoning could produce entirely different yield.

Yet because the artlike traditional mathematics is too unreliable to be taken seriously, we shall restate the controversial conclusion in the MSR framework, but still with direct references to notations used in the traditional mathematics and physics. For the main objective of this presentation is to make it as easily comprehensible as possible.

6. APPLYING SOME FINEGRAINED DIFFERENTIAL OPERATORS WITHIN ORTHOGONAL HOMOGENEOUS 3D SPATIAL STRUCTURES

Already in 1773 AD Lagrange has found that the Newtonian gravitational force **F** equals to the gradient **F** = –**∇**V of a scalar potential function V=1/r that solves Laplace equation [35]. The value of the generic scalar potential V(r), which can be equated to the planar 2D projection of the normal/radial curvature of free space |κ|=|V|=1/r, suggests that it is possible to identify the potential function V(r) with a length- and time-dependent speed function **v**(**r**,t), and the 2nd derivative of the radius r=|**r**| pointing to the given trajectory curve, with radial acceleration **a**(**r**,t) taken with respect to the elapsing time parameter t. Hence, we obtain

$$\mathbf{a}(\mathbf{r},\mathrm{t}) = \frac{\mathrm{d}\mathbf{v}(\mathbf{r},\mathrm{t})}{\mathrm{dt}} = \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{dt}^2} = -\frac{\mathrm{dV}(\mathbf{r})}{\mathrm{dr}} \tag{2}$$

which has already been exposed in [36], provided that we assume |**F**|= – dV/dr given the reciprocity of the magnitude r=|**r**| of the pointing radius and the time parameter t. For attractive gravitational force field, we must assume |**F**|= – dV/dr because the gravitational inertial force is attractive and thus increases towards the center of the field in the direction opposite to that of the outward increasing radius. Thus, the traditional one-way mapping → used above could be replaced with implication ⇒, provided the expressions will be evaluated in terms of operational functions rather than just mappings, or even by the equivalence sign ⇔, if the two unidirectional implications ⇐ and ⇒ are applicable.

It has already been shown that finegrained differential operators introduced in [3] act simultaneously on the primary function within the primary space P|p (i.e. space P equipped with homogeneous orthogonal basis p) and on its dual reciprocal function that is depicted in the dual reciprocal space Q|q (i.e. space Q equipped with homogeneous orthogonal basis q) [37]. Since derivative of a local inverse function is the inverse of the derivative of the primary function [38], we can apply to local functions (depicted in the mutually dual reciprocal yet equidimensional spaces P and Q) the pattern encoded in the interspatial duality principle introduced in [37], namely:

$$\mathrm{v(t)_s} \cdot \mathrm{SGrad(P(s))|p} \Leftrightarrow \mathrm{v(s)_t} \cdot \mathrm{SGrad(Q(t))|q} \tag{3}$$

where SGrad() is the finegrained scalar gradient differential operator defined in [3] and the $v(t)_s$=(dt/ds)=1/$v(s)_t$ is the inverse function of the regular speed function $v(s)_t$ =ds/dt. Thus, in the simple case of local functions, we can conclude that their scalar gradients are formwise equivalent, even though they reside in distinct spaces equipped with different homogeneous algebraic bases.

Hence, if performed within the multispatial framework of the MSR paradigm, the splitting of 4D spacetime into its subcomponents: the 3D length-based usual space and the anticipated 1D timeline, may actually require two equidimensional 3D dual reciprocal spatial structures paired to form hyperspace. I used to call the usual 3D homogeneous length-based space LBS. Thus the – corresponding to the LBS – 3D homogeneous time-based space can be called TBS, if it is equipped with its own native homogeneous basis. Hence in terms of algebraic bases, this inference can be expressed as:

$$\{\mathrm{SGrad}(P(s))|p \Leftrightarrow \mathrm{SGrad}(Q(t))|q\ \} \Rightarrow \left\{p \coloneqq (x, y, z)\ \Leftrightarrow\ q \coloneqq \left(\frac{1}{t_x}\ ; \frac{1}{t_y}\ ; \frac{1}{t_z}\right)\right\} \tag{4}$$

where the reciprocal basis q does not need to be aligned with the directional components of the primary basis p, as long as components of each homogeneous basis are orthogonal and homogeneous i.e. denominated in the same units. I have shown in [28] that an improperly depicted inverse expression pretending to be derivative shown in contravariant representation in the primary space P can be turned into proper derivative in covariant representation within the dual reciprocal space Q that is paired with the primary space P – see Appendix 1 below or [28]. Compare also [39].

This conclusion agrees with the STR wherein elapsing time is essentially a curvilinear parameter whose varying depends on the given locally constant velocity, and thus cannot really be regarded as being straightlinear. Although it is possible to imagine that the straightlinear timeline changes only longitudinally, i.e. along the timeline – as the relativistic dilation of time rates may suggest – the experimentally confirmed phenomena of frequency decrease near massive celestial bodies in conjunction with deflection of light that passes by such bodies, suggest that the timeline could be curved in all three dimensions.

The virtual scalar potential $V_n(\partial x^n(s))$ is prototype of the scalar covariant differential operator SCovar, which is multiplicative inverse of the scalar gradient differential operator:

$$V_n\big(\partial x^n(s)\big) = \frac{\partial x^n}{ds} = 1\ /\ SGrad_n(s) = SGrad_n^{(-1)}(s) \eqqcolon SCovar\big(\partial x^n(s)\big) \tag{5}$$

$$V_n\big(\partial x^n(t)\big) = \frac{\partial x^n}{dt} = 1\ /\ SGrad_n(t) = SGrad_n^{(-1)}(t) \eqqcolon SCovar\big(\partial x^n(t)\big) \tag{6}$$

for arclengths of "spatial" (i.e. length-based) and temporal (i.e. time-based) trajectory paths, respectively. Evidently the temporal trajectory apparently spreads geometrically in a certain temporal representation of the dual reciprocal space just as the length-based trajectory does. Here the components of the scalar differential operator SGrad and SCovar are taken only in a single n^{th} spatial (i.e. length-based) direction or a temporal direction [37].

However, the geometric spread of these 3D finegrained differential operators is based upon a different quasispatial principle that shall be further explored elsewhere, for it requires the deployment of overlapping paired 4D spatial structures. The overlap is necessary because the highest dimensionality of realistic single space is 4D, according to the obvious – yet unrecognized in traditional mathematics – consequences of (otherwise unquestioned) achievements of Abel, Galois, and Lagrange [31], [34]. Moreover, that quasispatial principle demands unrestricted division by zero already explained in [4], which makes operations on mathematically sound infinity unambiguous [15] with theoretical support of interspatial abstract pivoting [40]. Although the Leibniz's notation can be enlightning, some mathematicians see its pitfalls for making conclusions in abstract reasonings; one of them is what some call "second fundamental confusion of calculus" [41] p.189.

This undesirable feature was one of the most compelling reasons for me to redefine differential operators in more finegrained fashion [3].

7. DISASSEMBLING THE 4D SPACETIME WITHIN THE MULTISPATIAL FRAMEWORK

E. Freitag rightly assumed that it should be possible to split the compound structure of spacetime into its subcomponents. This assumption is intuitively clear. His was thus mathematically perfect abstract reasoning though performed in the – somewhat imperfect – SSR-infected realm of traditional mathematics. Nevertheless, his effort was not made in vain. For if we cast it into the MSR framework of paired dual reciprocal spaces P|p and Q|q, where the 3D primary space P equipped with the regular homogeneous algebraic basis p=(x,y,z) is paired with a dual reciprocal 1D space Q equipped with a certain distinct multiplicatively inverse/reciprocal algebraic basis $q=\{t^{-1}(t)\}$, whatever it would eventually evaluate to, such that the hetrogeneous interspatial mappping shown in an informally imposed pattern as

$$\{1/\partial T(x,y,z,[ict])/ic\partial[t]\}:P|p \rightarrow \{\partial T^{*}([x],[y],[z],ict)/ic\partial t\}:Q|q \quad (7)$$

holds, i.e. remains true to the extent that the imposed on it conditions are satisfied within the scope of their validity. There the functionals are enclosed in square brackets as usual whereas the actively varying functions, parameters and unknowns are displayed as freestanding variables. T() and its dual T*() are certain mutually reciprocal functions of the elapsing time t shown in 4D algebraic bases superimposed on the spaces P and Q respectively, even though these bases are heterogeneous 4D bases of the 4D spacetime (ST).

The inverse differential standing in the expression on the left-hand side (LHS) of the interspatial mapping (7) shows (an operationally improper) contravariant expression pretending to be legitimate derivative whereas the derivative on the RHS is shown in formally covariant representation. That is the main reason I have wriiten the formula (7) as mapping rather than as an equation or perhaps an implication, because the LHS does not realy imply the RHS; neither can these sides be equated when written in the pattern form shown in (7) which is just an example of an operational formula. But the inadmissible mapping can serve as a convenient preliminary bridge connecting the SSR and MSR settings. For the time being I shall neglect on purpose the discussion of signatures of the spaces involved therein mainly for the sake of simplicity, because we would actually have twin quasispatial structures to deal with in general: namely the usual 4D spacetime and the corresponding to it 4D timespace [31]. These issues shall be discussed elsewhere, because the two heterogeneous 4D structures partly overlap [31], [34]. The equidimensional separation of 4D spacetime (ST) into 3D LBS and 3D TBS does not preclude the possibility of the anticipated decomposition into 3D LBS and 1D TBS in the particular case when the timeline varies only longitudinally. But in either case the two ensuing spatial structures should be distinct and dual reciprocal.

A more general equidimensional decomposition case $\{4D{:}ST\} \coloneqq \{3D{:}LBS \oplus 3D{:}TBS\}$ apparently requires making split that is indirectly implied by the operational equivalence of their spatial representations. This kind of splitting suggests partly overlapping bases, of course. Furthermore, similar conclusion could be reached upon an abstract decomposition of squared traditional 4D differential operators that ensued from a certain abstract quaternionic principle of duality [14], whose origin stems from algebraic division by zero implemented as multiplication by infinity [13], which was universalized in [4]. The necessity of operationally sound and conceptually unambiguous unrestricted division by zero is of paramount importance to our ability to handle multispatial structures under the MSR paradigm and especially for operating on geometric objects cast within such structures.

8. MINKOWSKIAN TRANSITION FROM THE USUAL HOMOGENEOUS 3D SPACE TO THE ESSENTIALLY HETEROGENEOUS 4D SPACETIME

The formula (4) evidently implies formal equivalence of 3D reciprocal algebraic bases

$$\left\{\{p \coloneqq (x, y, z)\} \;\Leftrightarrow\; \left\{q \coloneqq \left(\frac{1}{t_x}; \frac{1}{t_y}; \frac{1}{t_z}\right) = \left(f_x, f_y, f_z\right)\right\}\right\} \tag{8}$$

where the frequency function f=1/t that is reciprocal of the elapsing time interval spreads along the LBS coordinate axes. I am employing f do denote frequency because similarity of the fonts used in various editors to display v, v, or υ, can lead to undesirable confusion. For in this paper, I must confront various entrenched traditional notations.

The Minkowski proper time integral $\tau=\int d\tau$ taken along the worldline time differential

$$d\tau = \frac{1}{c}\sqrt{\{c^2(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2\}} \tag{9}$$

[42] p.85 induces thus special-relativistic evaluation of the work done W in terms of the potential energy as well as the – corresponding to it – kinetic energy within the 4D spacetime:

$$W = \frac{1}{c^2}\left(\frac{\dot{x}}{\dot{t}}X + \frac{\dot{y}}{\dot{t}}Y + \frac{\dot{z}}{\dot{t}}Z\right) \tag{10}$$

with the acting force components X,Y,Z, parallel to the space axes [42] p.86, and differential of the work done W "is the rate at which work is done by the [given] force at the worldpoint [of the given worldline] divided by [the speed of light in vacuum] c, remains unchanged." see [42] p.87. However, the four differential coefficients of the spacetime are taken with respect to the proper/moving time τ: $\dot{t} = \frac{dt}{d\tau}$; $\dot{x} = \frac{dx}{d\tau}$; $\dot{y} = \frac{dy}{d\tau}$; $\dot{z} = \frac{dz}{d\tau}$ [42] p.85. The differential coefficients are thus components of the 4-velocity u in the given Lorentz reference frame:

$$u^0 = \dot{t} = \frac{dt}{d\tau} = \frac{1}{\sqrt{1-v^2}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \; where \; v = \frac{v}{c} = \beta; \; u^j = \frac{dx^j}{d\tau} = \frac{v^j}{\sqrt{1-v^2}} = \frac{v^j}{\sqrt{1-\frac{v^2}{c^2}}} \; with$$

$$v^j = \frac{dx^j}{dt} \; so \; that \; v^2 = (v^x)^2 + (v^y)^2 + (v^z)^2 \tag{11}$$

where $v^j = \frac{dx^j}{dt}$ is the ordinary 3-velocity with j=1,2,3 along the spatial directions x,y,z, compare [43] p.50. Notice that I am keeping the traditional notation (of Minkowski, Misner & Company) as much as I can with most explanations in Minkowski's [translated] wording.

However, the imaginary unit as well as the customary Minkowskian imaginary ict factor was omitted on purpose, for as Misner-Thorne-Wheeler wrote "[…] no one has discovered a way to make the imaginary coordinate to work in the general curved spacetime manifold." compare [43] p.51. Nonetheless, as true as the latter assertion is, why should the abstract mathematical spacetime be curved? For as Misner-Thorne-Wheeler later wrote: "Spacetime must be asymptotically flat if there is to be any possibility of defining energy and angular momentum." see [43] p.463.

Since that flattening is rather unlikely to happen in general theoretical (as opposed to mere practical) situations, I have decided to use paired dual reciprocal spaces, each equipped with homogeneous orthogonal bases, in whose setting we get rid of the local curved spacetime manifold, which is never curved but remains always flat (in the infinitesimally small local setting anyway) at the expense of having to deal with the two equidimensional dual reciprocal spaces. I am not claiming that I have discovered how to make the imaginary temporal coordinate ict work in the curved spacetime, but I am saying that in free space – that is: in absence of masses – there is nothing material to curve the inherently flat, STR-induced abstract mathematical spacetime in the given local setting and thus the Minkowskian ict factor can be retained until further notice, just for the case that we may need it to explain some differences between the homogeneous 3-vectors used within the paired dual reciprocal 3D spaces and the heterogeneous 4-vectors used within the 4D spacetime. I shall return to this issue later. Unlike the flat abstract Minkowskian spacetime, gravitational field is material because it possesses an energy density [44] and the energy is understood as the measure of activity or at least readiness for activity [45], whereas the spacetime is just a convenient abstract quasigeometric diagram [46] p.52ff, [47]. For an easy introduction to STR see [48], [49].

The Minkowskian formula (10) can be rewritten with separated space and time variables

$$|dW_r| = |W_r dt| = \frac{1}{c^2}|(Xdx + Ydy + Zdz)| = \frac{1}{c^2}|\boldsymbol{F}\bullet d\boldsymbol{r}| \qquad (12)$$

which is true in terms of absolute values. Now it is mathematically clear that the rate of work done dW_r corresponds only to the radial component of the potential energy that pays for the work being done by the force field because only the radius $r=|\mathbf{r}|$ is actively varying there as the differential $d\mathbf{r}$ clearly indicates. Evidently, the pattern formula (12) is operationally incomplete [2], [50], because it was obtained with blatant disregard for the proven [51] traditional product differentiation rule and as such is mathematically inadmissible in general. Recall that the tensor calculus is valid only in purely radial circumstances [2]. Nevertheless, Stephen Hawking voiced objections to Weyl Tensor Hypothesis in the cosmological context, which is usually considered as the domain which is governed exclusively by radial impacts of gravity [52]. For relativistic transformations of force see [53], [54], [55], [56]. It is known that in relativistic generalization mass has vectorial character [57] and thus energy too, as being equivalent to mass, must have vectorial character in general [16].

This vectorial character of energy seems to be contrary to the traditional presumption that energy is always scalar variable, regardless of how it is evaluated, whether as potential energy, or work done or kinetic energy. This unwarranted presumption was supported by the fundamentally flawed tensor calculus [2]. Moreover, the radial rate of change of the work done is obviously affected by the massive celestial body that generates the radial force field. Nevertheless, my analyses of formerly unanticipated results of several unbiased physical experiments indicated that energy is actually a vectorial variable [16] for it is an operationally necessary consequence of mathematically legitimate evaluations of differentials to regard potential energy as a vector [58]. Furthermore, the other, nonradial components of the work done also depend on density of matter (inversely though) of the massive celestial body that generates the local radial force field [24], [59] in addition to the mass that remains practically unaltered even in relativistic framework when its effect is considered in nonradial (i.e. tangential and binormal) directions with respect to the center of the field. But even in purely radial circumstances considered in terms of radial forces, the inertial and gravitational forces acting on a satellite are not in balance for the satellite, because of the fact that Earth is in free fall in our Sun's field, with the [radial] gravitational attraction of the Sun balanced by the inertial force owing to the Earth's revolution [60].

Stephani wrote that over large spatial regions when the gravitational field is properly included, there is no energy balance equation, which is not [considered as] violation of energy conservation because in general, there exists no local covariant quantity [called] 'energy' to which the property of conservation or nonconservation can be ascribed; hence he concluded that "None of the foundations of physics are thereby destroyed; energy is only a (very important) auxiliary for describing interactions, but the interaction of all parts of the Universe is quite essential for the theory of gravitation." see [61]. Yet he admits that in the STR, electrodynamics, thermodynamics, quantum mechanics, and quantum field theory, the quantity 'energy' can be defined, which is constant if the system is isolated [61]. This evocative "explanation" is a conundrum to me. Were it not for the formerly unrecognized fact that the tensor calculus, upon which such conclusions can be derived with impunity, is fundamentally flawed [2], one might be delighted to see even the ancient art of sophistry triumphing in such utterly confounding ostrichlike approach to physics. Tolman has already remarked that the importance of energy makes it very interesting to consider the possibility of ascribing a definite location in space to any given quantity of energy [62].

The radial gravitational frequency shift (also known as redshift) predicted by Einstein was confirmed with 1% accuracy [63]. See also [64]. Nevertheless, Doppler effect will produce an extra blueshift which reduces the redshift predicted by Einstein [65]. It should be noted, however, that the famous Pound-Rebka experiment that confirmed the radial redshift is incompatible with the Minkowski spacetime diagram [66]. While moving away (or up as it is usually depicted) from the center of the given gravitational force field causes redshift (that is frequency decrease), moving towards the center (i.e. down the potential well) causes blueshift by increasing the frequency and thus shortening the original timerate of the falling clocks [67]. In reference to the gravitational redshift Joseph Weber pointed out that [spectral] lineshift measurements on the sun give different results at different points of the surface [68]. It has been shown that this issue is easily explained by presence of nonradial effects of purely radial/center-bound gravitational force field [16].

Now from the relations (11) and (12) we can deduce the following direct implication

$$c^2 dW_r = c^2 W_r dt = \frac{W_r d\tau}{\sqrt{1-v^2}} \Rightarrow \left\{ W_r d\tau = c^2\sqrt{1-v^2} W_r dt \ \& \ W_r dt = \frac{1}{c^2\sqrt{1-v^2}} W_r d\tau \right\} \quad (13)$$

which may be sometimes acceptable in the SSR setting, but not in general. For it requires that the relativistic motion happens along the radial direction, which is not always the case. Hence we cannot follow the Minkowskian line of thought regarding work done or potential energy.

Nevertheless, as ingenious as the conventional Minkowski's 4D spacetime concept is, neither he nor traditional theories of mathematics and physics – including STR and GTR – were able to explain, reconcile or just predict the unavoidable presence of certain nonradial effects (of purely radial/center-bound force fields) that I have found and through the presence of which I have reconciled some previously unanticipated results of the aforementioned and other physical experiments and observations that I have investigated. For compliance with radial-only components of the potential energy that pays for the work done by the given force field implies that a more comprehensive extension of the spacetime should also exist, which could take into account also the – heretofore neglected and not appreciated – nonradial effects of radial gravity that would complement the purely radial ones. One such operationally feasible extension calls for deployment of twin dual reciprocal spatial structures.

9. DUAL RECIPROCAL SPATIAL STRUCTURES ARE INDUCED ALSO BY INVARIANCE INHERENT IN SPECIAL RELATIVITY

From (11) we can obtain the relation between differentials of the elapsing time, namely the regular/resting time differential dt and the proper/moving time differential dτ

$$\left\{u^0 = \dot{t} = \frac{dt}{d\tau} = \frac{1}{\sqrt{1-v^2}}\right\} \;\Rightarrow\; \left\{dt = \frac{d\tau}{\sqrt{1-v^2}} \;\;\&\;\; d\tau = dt\sqrt{1-v^2}\right\} \tag{14}$$

and on evaluation of the special-relativistic Lorentz functional gamma based on (11) we get

$$\gamma = \frac{1}{\sqrt{1-v^2}} = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \;\; where\; \beta = \frac{v}{c} \;\; and\; \gamma^{-1} = \sqrt{1-v^2} = \sqrt{1-\beta^2} \tag{15}$$

so that the formula (14) can be rewritten in more comprehensible contemporary notation as

$$\left\{u^0 = \dot{t} = \frac{dt}{d\tau} = \frac{dt(\tau)}{d\tau} = \frac{1}{\sqrt{1-v^2}} = \gamma\right\} \Rightarrow \{\,\{\,dt(\tau) = \gamma d\tau(t)\}\,\&\,\{d\tau(t) = \gamma^{-1}dt(\tau)\}\,\} \tag{16}$$

where the Lorentz functional gamma: $u^0 = \dot{t} = \frac{dt}{d\tau} = \frac{dt(\tau)}{d\tau} = \frac{1}{\sqrt{1-v^2}} = \gamma$ is actually as if frozen function (that is fixed functional), which can neither be differentiated nor integrated with respect to the constant speed that serves as just a fixed coefficient therein. Insted of the conventional notation that marks variables in moving frames with apostrophe – which could be mistaken for derivative – I prefer to show direct dependence through the use of Leibnitz's notation. For although the moving clocks accumulate their internally elapsing time in motion at the proper time rate dt(τ) one could also ascribe to them an external time rate dτ(t) as it could be observed from the outside, which could be called truly multiplicatively inverse. Notice that now the equivalence cannot be maintained in general, and thus I have used the ampersand sign & instead of the equivalence sign ⇔ on the RHS of (16).

The compelling reason for it is that in (16) we actually have two possible inversions: one of γ and the other of β embedded in γ. Hence we cannot be sure that all foreseable outcomes of these two combined inversions are equivalent. This kind of nuance was routinely disregarded in traditional mathematics, which generated thus numerous inadmissible conclusions stemming from thoughtless generalizations of incompatibly varying magnitudes. Compatible mappings for spaces with two metrics were discussed in terms of fixed points in the context of pairings in

[69]; compare also [70]. For [conceptually] inadmissible generalizations are common to all paradoxes [71] and thus there is no assurance that contradictions would not emerge even in restricted set-theoretical systems or in other branches of mathematics [72]. As a matter of fact, the reciprocity of elapsing time that is visible in (16) was already recognized before [73]. But the declared reciprocity was sometimes mistaken for mere additive reversion.

The four flying atomic clocks in Hafele-Keating experiments accumulated their internal time at different rates than the four control clocks left on the ground to compare the recordings made in flight with the external time elapsing on the Earth's surface that was assumed as being practically at rest. Some authors call 'inverse' what should actually be called 'reverse, when they refer to "inverse Lorentz transformation" as the one with parameter $-\upsilon$ instead of $+\upsilon$ - see [46] p.17. In fact, any reference to a group without specifying whether the group is additive or multiplicative can cause even inadvertent abuse of mathematical language. Fock remarked that although it seems strange that both: the regular and the reverse [not inverse] Lorentz transformations allegedly yield formwise the same relativistic formula ($\partial/\partial t$, $\partial/\partial t$) for both resting and moving (primed) time intervals, which he identified with $(\partial t'/\partial t)=(\partial t/\partial t')$, he too concluded that there is [allegedly] no paradox because Principle of Relativity (PR) allows one to interchange the roles of the two [resting and moving] frames [74]. But he forgot to mention that no principle could ever justify a nonsense, no matter how eloquently it is proclaimed or how elaborately it may be written and endorsed on paper.

In the LBS the elapsing time t is just a parameter whose varying depends on the constant speed υ, and the moving rate of time flow $dt'=dt(\tau)$ accumulated by the moving clock (as observed inside of the moving reference frame) is dilated/expanded according to STR. The dilation of time has been confirmed in numerous experiments and is well documented, especially in the famous Hafele-Keating experiments with atomic clocks flown aboard commercial airplane [75]. Notice that with the exception of take-offs and landings (during which the gravitational frequency shifts contributed the most to the accumulated time) the Hafele-Keating atomic clocks have been flown along practically equipotential surface where the gravitational potential remains unchanged [76]. Thus during those equipotential phases of the flights the relativistic kinematic effects predicted by the STR contributed to the accumulated time the most. Alley's experiments with flying atomic clocks [77] showed similar discrepancies to that of the Hafele-Keating experiments, which I have practically reconciled in full for the nominally Westward flight but only partly (about 2/3 of the still remaining then-unexplained discrepancy) for the Eastward flight [78]; compare also [79].

The multiplicativly inverse Lorentz functional gamma $\gamma^{-1}=1/\gamma$ that is truly reciprocal to the usual relativistic γ could be derived with respect to the relative speed $\beta=\upsilon/c$ as follows:

$$\gamma^{-1}(\beta) \coloneqq \sqrt{1-v^2} = \sqrt{1-\beta^2} = \sqrt{\frac{c^2-\upsilon^2}{c^2}} = \frac{1}{|c|}\sqrt{c^2-\upsilon^2} = \left|\frac{\upsilon}{c}\right|\sqrt{\frac{c^2-\upsilon^2}{\upsilon^2}} = \left|\frac{v}{c}\right|\sqrt{\frac{c^2}{\upsilon^2}-1} =$$

$$= \left|\frac{\upsilon}{c}\right|\sqrt{-\frac{c^2}{\upsilon^2}+1} = \left|\frac{\upsilon}{c}\right|\sqrt{-\left(1-\frac{c^2}{\upsilon^2}\right)} = \left|\frac{\upsilon}{c}\right|\sqrt{i^2\left(1-\frac{c^2}{\upsilon^2}\right)} = \left|\frac{i\upsilon}{c}\right|\sqrt{1-\frac{c^2}{\upsilon^2}} = \left|\frac{i\upsilon}{c}\right|\sqrt{1-\left(\frac{1}{\beta}\right)^2} =$$

$$= -\left|\frac{\upsilon}{ic}\right|\sqrt{1-\left(\frac{1}{\beta}\right)^2} \eqqcolon -\left|\frac{\upsilon}{ic}\right|\gamma^{-1}\left(\frac{1}{\beta}\right) \qquad (17)$$

which is remarkable coincidence in the SSR setting. For the inverse functional gamma shown as functional depending on the functional beta can be equated with truly reciprocal functional gamma shown as functional of the reciprocal functional beta, up to the coefficient $-\left|\frac{\upsilon}{ic}\right|$ that is constant, because once chosen, the magnitude of the speed υ is constant in the STR and so is its value |υ|. Hence we can summarize the step-by-step derivation (17) as

$$\left\{\gamma^{-1}(\beta) = \sqrt{1-\beta^2}\right\} \Leftrightarrow \left\{\gamma^{-1}\left(\frac{1}{\beta}\right) = -\left|\frac{\beta}{i}\right|\sqrt{1-\left(\frac{1}{\beta}\right)^2} = -\left|\frac{\upsilon}{ic}\right|\sqrt{1-\left(\frac{1}{\beta}\right)^2}\right\} \tag{18}$$

were it not for the fact that the latter has to be multiplied by the negative value of the coefficient $\left(-\left|\frac{\beta}{i}\right|\right)$ whose inverse imaginary value indicates that the reciprocal functional gamma of reciprocal beta $\gamma^{-1}\left(\frac{1}{\beta}\right)$ should be placed in a distinctively separate dual reciprocal space that is differently denominated than the primary space. In other words: if the inverse functional $\gamma^{-1}(\beta)$ dwells in a certain primary 3D space P|p whose native homogeneous basis p=(x,y,z) is denominated in length-based units, then the multiplicatively inverse reciprocal functional $\gamma^{-1}\left(\frac{1}{\beta}\right)$ should dwell in a distinct and separate 3D space Q|q whose native homogeneous basis $q = \left(\frac{1}{t_x};\frac{1}{t_y};\frac{1}{t_z}\right)$ is denominated in reciprocal temporal/time-based units – per eq. (8). Notice that the values/magnitudes of the two inverse functionals gamma are formwise the same – per eq. (17), but their placement requires structurally different spaces due to the double reciprocity.

Their equivalent values suggest that pairing of dual reciprocal spaces within a hyperspatial structure is preferable if not unavoidable. An alert reader might see that encapsulating of primary and reciprocal magnitudes, such as β=υ/c inadvertently concealed the beta in such expressions like $\gamma^{-1}(\beta) \coloneqq \sqrt{1-v^2} = \sqrt{1-\beta^2}$ and thus effectively precluded the recognition of the necessary presence of multispatiality of the structural representations and supported the obstinately thoughtless acceptance of the formerly unspoken SSR paradigm at its face value.

Hence from implication (16) we have obtained the following twin relationships:

$$\{\,dt(\tau) = \gamma d\tau(t)\}\,\&\,\{d\tau(t) = \gamma^{-1}dt(\tau)\} \tag{19}$$

whose complementary expressions could also be considered separately as

$$\{\,dt(\tau[\beta]) = \gamma[\beta]d\tau(t[\beta])\,\} \Leftrightarrow \{\,dt(\tau) = \gamma d\tau(t) = \sqrt{1-\beta^2}d\tau(t)\,\} \tag{20}$$

which is the regular relativistic relationship as well as the inverse/reciprocal relationship

$$d\tau(t) = \gamma^{-1}dt(\tau) \tag{21}$$

which can be rewritten explicitly with respect to the functional beta in the primary space P|p

$$\{d\tau(t[\beta]) = \gamma^{-1}[\beta]dt(\tau[\beta])\} \tag{22}$$

or with respect to the multiplicatively inverse/reciprocal beta in the dual reciprocal space Q|q

$$\{ dt(\tau[\beta^{-1}]) = \gamma[\beta^{-1}]d\tau(t[\beta^{-1}])\} \,\&\, \{d\tau(t[\beta^{-1}]) = \gamma^{-1}[\beta^{-1}]dt(\tau[\beta^{-1}])\} \tag{23}$$

where the internal functionals are in square brackets as usual. Hence one should not try to substitute or "generalize" variables through encapsulating compouded yet incompatibly varying magnitudes, regardless of whether they represent functions or functionals. For this kind of abstracting effectively erases one variable's variability attribute by virtually replacing it with the attribute of the other variable, which action is usually imperceptible (i.e. happening automatically as if "behind the scenes" of our conscious mind). It is thus inappropriate in the case of incompatibly varying magnitudes whose attributes we are not accustomed to control.

The equivalence (18) can be rewritten in terms referencing the functional beta directly as

$$\{\gamma^{-1}(\beta)\, ic \} \;\Leftrightarrow\; \left\{-\,\gamma^{-1}\left(\tfrac{1}{\beta}\right) v \equiv -\,\gamma^{-1}(\beta^{-1})\, v\right\} \tag{24}$$

which relates the primary space P to the reciprocal space Q and then could be compared to the source formula (21). Now interposing (21) over the pattern formula (24) we obtain

$$\sqrt{1-\beta^2}\; ic\; d\tau(t) = -\sqrt{1-\left(\tfrac{1}{\beta}\right)^2}\; v\; dt(\tau) \tag{25}$$

and on substituting $-1 = i^2 = ii = i|\hat{\imath}|$ in front of the RHS in (25) we obtain the pattern

$$\sqrt{1-\beta^2}\; ic\; d\tau(t) = |\hat{\imath}|\sqrt{1-\left(\tfrac{1}{\beta}\right)^2}\; iv\; dt(\tau) \tag{26}$$

where $\hat{\imath}$ is an imaginary operator unit vector, whose value is the same as that of the algebraic imaginary unit: $|\hat{\imath}| = |i| = \sqrt{-1}$. Since the term ic is a constant functional we can put it into the differential $d\tau(t)$ of the 4D four-space (4P) that is also known as spacetime (ST). Hence

$$ST = 4P|4p = \{x, y, z, ic[t]\}@[\beta]{:}\left\{\sqrt{1-\beta^2}\; d\tau(ict, [s = (x, y, z)])\right\} \Leftrightarrow$$

$$TS{:}\, 4Q|4q = \{f_x, f_y, f_z, iv[\Lambda]\}@[\beta^{-1}]{:}\left\{|\hat{\imath}|\sqrt{1-\left(\tfrac{1}{\beta}\right)^2}\; d\lambda\big(iv\Lambda, [f = (f_x, f_y, f_z)]\big)\right\} \tag{27}$$

where 4p is heterogeneous basis of the 4D primary space 4P that represents spacetime and 4q is heterogeneous basis of the 4D reciprocal space 4Q that represents timespace (TS). There $\lambda = 1/s(x, y, z)$ is an inverse/reciprocal functional of the arclength s(x,y,z) that is actively varying in the ST, just as the elapsing time parameter functional [t]=1/f is inverse/reciprocal functional of the arcfrequency function $f = (f_x, f_y, f_z)$ that is actively varying in the TS while the cotemporal functionals λ and Λ serve as auxiliary parameters. Notice that wherever the arclength s(x,y,z) is actively varying, the functional τ plays the role of a fixed parameter that can vary only along the arclength curve and vice versa, wherever the arcfrequency curve $f(f_x, f_y, f_z)$ is actively varying, the steadily varying proper/inverse functional λ of the inverse parameter [Λ]=1/s plays the role of a fixed parameter that can vary only along the arcfrequency curve. The unfortunate names of variables shall be changed elsewhere, but for now they should correspond to those used in the nomenclature of traditional physics.

It is known that direct measurements of the transition frequency ν between two quantum states whose energy difference is ΔE=*h*ν are only feasible in the radio wave frequency, microwave and submilimeter wave regions of the whole frequency spectrum. But practical determination of the energy difference in the infrared, the visible and higher frequency regions is based on measurements of the wavelength λ according to the formula ΔE=*h*c/λ= *h*ck [80] p.1532. Since both, the Planck constant *h* and the speed of light in vacuum c are constant, it follows that either the usual temporal frequency or the wavenumber k=1/λ, i.e. the "spatial frequency" that is reciprocal of the wavelength, can determine the energy difference. For energy and momentum are reciprocal [41] p.503. In other words: Energy difference is always proportional to frequency. Whether it is the usual temporal frequency or the spatial frequency is just a matter of practical determination. Since the formula (27) is just a relativistic pattern relating 4D spacetime to 4D timespace, the codependence of λ and Λ and their relation to the conventinal wavenumber k, whose value actually represents magnitude of a vector (hence the imaginary unit operator $\hat{\imath}$), it shall be further discussed elsewhere. For the time being it is sufficient to realize that the twofold pattern formula (27) is not merely mathematically admissible but is physically meaningful as well.

The equivalence (27) shows that the term ct makes the actively varying elapsing time interval t denominated in units of length/distance, just as it was understood in the STR, whereas the imaginary unit in the whole term ict in $d\tau(ict[\beta])$ on the LHS of (27) turns the homogeneous 3D basis (x,y,z) into heterogeneous 4D basis (x,y,z,ict) within the paired 4D dual reciprocal spatial structure. Similarly, the term $d\lambda(i\upsilon\Lambda[\beta^{-1}])$ on the RHS of (27) also turns the inverse/reciprocal arclength parameter $\lambda = \frac{1}{s(x,y,z)}$ that is denominated in inverse meters into arcfrequency curve functional $\Lambda = \frac{\upsilon}{s(x,y,z)}$ that shall become denominated in inverse seconds, just as the other actively varying variables within the 4D reciprocal space Q|q that is spatial structure dual to the primary 4D space P|p. The imaginary unit i present in ict and iυΛ indicates both: the extension of homogeneous basis onto heterogeneous basis as well as the necessity to place them in separate spaces. The structurally necessary presence of both: 4D spacetime (ST) structure and its reciprocal 4D timespace (TS) structure has already been conjectured upon group-theoretical considerations virtually imposing operational limit on orthogonal spatial dimensions [31].

10. TOWARD COMPLETE ESTIMATION OF POTENTIAL ENERGY

Starting from Poincaré transformation Philippe Tourrenc derived fhe formula

$$\frac{1}{c^2}\frac{\partial^2 T'}{\partial t^2} - \frac{\partial^2 T'}{\partial x^2} = 0 \qquad (28)$$

for the transformed time function T' from which Lorentz transformations and thus the whole infrastructure of the classical STR can be inferred [81]. The eq. (28) represents also 1D wave equation [82] spreading in the x-direction in the traditional SSR setting, of course.

Now if the x axis would be chosen in the radial direction of the local radial/center-bound force field then the eq. (28) would be formwise analogous also to the relativistic work done eq.

(13) as well as operationally compatible with the equivalence (27) of the two quasispatial structures of 4D spacetime and its reciprocal 4D timespace regarded as partly overlapping paired dual reciprocal spaces.

Therefore, I may conjecture that although in the framework of the traditional single space reality the work done (that is equated with the local field's potential energy spent on the work being done by the given radial/center-bound force field) appears as scalar magnitude, in the multispatial realm that demonstratedly underlies the mathematical infrastructure of special-relativistic physics {per the equivalence (27)}, the potential energy could appear as an actively varying function of frequency spreading in three dimensions of the quasispatial structure of 4D timespace which corresponds to the usual 4D spacetime. This conjecture is corroborated by my analysis of the frequency decreases observed in the aforementioned experiments conducted by Sadeh and his teams. These issues are important to physics and thus shall be further discussed elsewhere.

11. CONCLUSIONS

Since 4D spacetime was conceived as compound quasispatial structure comprising the 3D usual length-based space (LBS) and 1D elapsing time parameter, it was anticipated that the reverse process of separation of the spacetime into its constituting components should yield the 3D LBS space and the 1D geometric structure that corresponds to elapsing time parameter. But the reviewed reverse process resulted in inadmissible substitution of primary time derivative with it reciprocal under the formerly unspoken – and thus unchallenged – single space reality (SSR) paradigm.

When essentially the same separation process was performed under the umbrella of multispatial reality (MSR) paradigm, the split of 4D spacetime yields two equidimensional spatial structures: 3D LBS and a certain 3D time-based space (TBS). This result suggests operational necessity of presence of a paired dual reciprocal hyperstructure comprising the two equidimensional 3D spaces: LBS and TBS.

The separation of the 4D spacetime performed in the MSR setting yields two covariant derivatives, one in each of the paired dual reciprocal spaces. This highly desirable feature prevents the inadmissible traditional mapping of contravariantly represented derivatives onto covariant derivatives and vice versa.

It has been shown that transforming the special-relativistic Lorentz functional gamma into multiplicativly inverse/reciprocal Lorentz functional gamma actually yields two quite distinct outcomes: one pertaining to the usual four-fimensional spacetime and another pertaining to its equidimensional reciprocal quasispatial structure of timespace, whose presence had been conjectured also via other reasonings. Due to their heterogeneous four-fimensional bases and the cap limiting homogeneous dimensionality the two quasispatial structures partly overlap. The presence of the two structures emerged from operational reasonings without the need to make any existential postulates.

One general conclusion is that the paradigm shift from the traditionally unspoken singlespatial reality paradigm to the multispatial reality paradigm is operationally necessary in order to prevent the avoidable controversial operations exposed above.

In closing, I should say that making mistakes or even proposing unacceptable at present reasonings, is not inadmissible. What is repugnant is the tacitly veiled defense of certain indefensible statements and the silent suppression of attempts aimed at rectification of the mistakes and/or the nonsenses that follow from questionable reasonings.

APPENDIX 1

TURNING INVERSE EXPRESSION FORMERLY PRETENDED TO BE CONTRAVARIANT DERIVATIVE INTO COVARIANT DERIVATIVE

The integral kernel g(x) should be properly evaluated according to the RSPDR1 rule as:

$$\{f(t) \cdot K(t,x)\}'_t \coloneqq t \cdot \frac{dK(t,[x])}{dt} \oplus \frac{1}{K([t],[x])} \cdot \frac{1}{dt} = tK'(t,[x])dt \oplus \frac{1}{K([t],[x]) \cdot dt} \qquad \text{(A1.1)}$$

for if the integral kernel is supposed to be eventually integrated scalarly then it should be formed as proper differential that is legitimately evaluated according to the RSPDR1 rule.
Yet I can turn the contravariant term into legitimate though inverse covariant differential

$$\frac{1}{K([t],[x])} \cdot \frac{1}{dt} \Rightarrow \int\{1 \,/\, \frac{\partial K(t,[x])}{\partial t}\} \cdot \frac{1}{dt} = \int\{\frac{\partial t}{K'(t,[x])dt} \cdot \frac{1}{dt}\} = \int\{\frac{1}{K'(t,[x])dt}\} \qquad \text{(A1.2)}$$

which yields quite legitimate reciprocal (i.e. multiplicative inverse) yet covariant differential suitable for prospective integration, at least in principle. Now, with the simple function f(t)=t, the compound function g(x) in (A1.1) of the integral kernel can now be properly rewritten as:

$$g(x) = K(t,x) \cdot f(t) = \int_0^\infty K(t,x) \cdot f(t) \cdot dt \Rightarrow \int_1^\infty tK'(t,x)dt \oplus \int_0^1 \frac{1}{K'(t,x)dt} \qquad \text{(A1.3)}$$

where the term K'(t,x)dt in the integrands of both integrals is legitimately obtained and properly formed covariant differential of the integral transformation K() with respect to the formally independent variable t. Now the absence of the extra multiplicative compounding separator ▫ between the derivative K'(t,x) and the differential dt can be tolerated as it does not confuse us anymore on the RHS of the formula (A1.3). Notice, that the inverse term standing on the far RHS in (A1.3) is definitely reciprocal, and thus it must not be integrated within the same primary space as the regular term tK'(t,x)dt that stands on the LHS of the resulting implication (A1.3). Hence the interspatial sum sign ⊕ is necessary – compare [28], wherein these issues were discussed in more detail.

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