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An Interval-Valued Intuitionistic Fuzzy Matrices Based on Hamacher Operations

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ABSTRACT

The objective of this paper is to apply the concept of fuzzy matrices to interval-valued intuitionistic fuzzy matrices. In this paper, we introduce the Hamacher operations of interval-valued intuitionistic fuzzy matrices and prove some desirable properties of these operations, such as commutativity, idempotency and monotonicity. Further, we prove De Morgan's laws over complement for these operations. Then we constructe the scalar multiplication $(n._h A)$ and exponentiation $(A^{\wedge_h n})$ operations of interval-valued fuzzy intuitionistic matrices and investigates the algebraic properties.

***Keywords*:** Fuzzy matrix, Interval-valued intuitionistic fuzzy matrix, Hamacher sum, Hamacher product, Scalar multiplication, Exponentiation operations

1. INTRODUCTION

Fuzzy matrices arise in many applications, one of which is as adjacency matrices of fuzzy relations and fuzzy relational equations have important applications in pattern classification and in handing fuzziness in knowledge based systems. Several authors presented a number of results on fuzzy matrices. In [15] Thomason initiated the study on convergence of powers of fuzzy matrix. The theory of fuzzy matrix was developed by Kim and Roush [2] as an extension of Boolean matrix, with max-min operation over the fuzzy algebra. Ragab and Emam [10]

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presented some properties on determinant and adjoint of a square fuzzy matrix. Ragab and Emam [11] presented some properties of the min-max composition of fuzzy matrices. We deal with interval valued fuzzy matrices (IVFM) that is, matrices whose entries are intervals and all the intervals are subintervals of the interval [0,1]. Recently the concept of IVFM a generalization of fuzzy matrix was introduced and developed by shymal and pal (2006), by extending the max. min operations on fuzzy algebra F = [0,1], for elements, $a, b \in F, a + b = \max(a, b)$ and $a - b = \min(a, b)$.

In [6], Meenakshi and Kaliraja have represented and IVFM as an interval matrix of its lower and upper limit fuzzy matrices. Anita and Pal [9] introduced intervalvalued fuzzy matrices (IVFMs) as the generalization of interval-valued fuzzy sets. Pal [8] a new type of interval-valued fuzzy matrices (IVFMs) are called interval-valued fuzzy matrices with interval-valued fuzzy rows and columns (IVFMFRCs) are defined and several important properties are investigated. Khan and Pal [4] introduced an interval valued intuitionistic fuzzy matrices (IVIFMs) and the interval-valued intuitionistic fuzzy determinant is also defined. Hamacher [1] proposed a more generalized t-norm and t-conorm. Hamacher operation includes the Hamacher product and Hamacher sum [13, 14] we defined two new operations called Hamacher sum and Hamacher product of fuzzy matrices and investigate the algebraic properties and extended to Hamacher operations of intuitionistic fuzzy matrices. This paper we extend Hamacher operations of interval-valued intuitionistic fuzzy matrices.

The paper is organized in three sections. In section 2, the definitions and operations on interval-valued intuitionistic fuzzy matrices and Hamacher operations of fuzzy matrices. In section 3, we introduce the Hamacher operations of interval-valued intuitionistic fuzzy matrices and focusing on its properties. The De Morgan's law for the Hamacher operations are established in section 4. In section 5, we constructed Hamacher scalar multiplication $(n._h A)$ and Hamacher exponentiation $(A^{\wedge_h n})$ operations of interval-valued intuitionistic fuzzy matrix A and investigated their algebraic properties. Section 6 concludes the paper with some future directions.

2. PRELIMINARIES

In this section, some basic concepts of interval-valued intuitionistic fuzzy matrices and Hamacher operations of fuzzy matrices is given below.

Definition 2.1. [4]

An interval-valued intuitionistic fuzzy matrix of order $m \times n$ is defined as, $A = (a_{ij\mu,} a_{ij\nu})_{m\times n}$, where $a_{ij\mu} = [a_{ij\mu L}, a_{ij\mu U}]$ and $a_{ij\nu} = [a_{ij\nu L}, a_{ij\nu U}]$ is the ij^{th} elements of A, represents the membership value. All the elements of an IVIFM are intervals and all the intervals are the subintervals of the interval [0,1], which maintaining the condition $a_{ij\mu L} + a_{ij\mu U} \leq 1$ and $a_{ij\nu L} + a_{ij\nu U} \leq 1$.

In the IVIFM, the elements are the membership grade of some attributes, they not crisp number, so naturally some new operations are needed to handel such matrices. Before applying binary or unary operations between IVIFMs we need clear idea about the similar operations between the elements.

Definition 2.2. [4]

An IVIFM is said to be a null IVIFM if all its elements are zero, *i.e.*, all elements are <[0,0],[1,1]> and the matrix is denoted by O.

Definition 2.3. [4]

An IVIFM $A = (a_{ij\mu,} a_{ij\nu})_{m\times n}$, of order $n \times n$ is called unit IVIFM or identity IVIFM if all the diagonal entries of A are [1,1] and all other entries are [0,0]. It is denoted by I_n.

Definition 2.4. [4]

For any two IVIFMs $A = [< [a_{ij\mu L}, a_{ij\mu U}], [a_{ij\nu L}, a_{ij\nu U}] >]$ and $B = [< [b_{ij\mu L}, b_{ij\mu U}], [b_{ij\nu L}, b_{ij\nu U}] >]$ of order $m \times n$ in the following.

$(i)\ A \vee B$

$$= [< (\max\{a_{ij\mu L}, b_{ij\mu L}\}, \max\{a_{ij\mu U}, b_{ij\mu U}\}), (\min\{a_{ij\nu L}, b_{ij\nu L}\}, \min\{a_{ij\nu U}, b_{ij\nu U}\}) >]$$

$(ii)\ A \wedge B$

$$= [< (\min\{a_{ij\mu L}, b_{ij\mu L}\}, \min\{a_{ij\mu U}, b_{ij\mu U}\}), (\max\{a_{ij\nu L}, b_{ij\nu L}\}, \max\{a_{ij\nu U}, b_{ij\nu U}\}) >]$$

$(iii) A^C = [< [a_{ij\nu L}, a_{ij\nu U}]\ [a_{ij\mu L}, a_{ij\mu U}] >]$

$(iv) A^T = [< [a_{ji\mu L}, a_{ji\mu U}]\ [a_{ji\nu L}, a_{ji\nu U}] >]$(the transpose of A)

$(v) A \leq B$ if and only if $a_{ijL} \leq b_{ijL}$ and $a_{ijU} \leq b_{ijU}$ for all i,j.

Definition 2.5. [13]

Let $A = (a_{ijL}, a_{ijU})$ and $B = (b_{ijL}, b_{ijU})$ of same size, then the Hamacher operations of interval-valued fuzzy matrices are defined as follows:

(i) The Hamacher sum of A and B is defined by $A \oplus_H B = (c_{ijL}, c_{ijU})$,

$$\text{where } (c_{ijL}, c_{ijU}) = \begin{cases} 1, & if \quad a_{ijL}, a_{ijU} = 1, b_{ijL}, b_{ijU} = 1 \\ \left(\frac{a_{ijL}+b_{ijL}-2a_{ijL}b_{ijL}}{1-a_{ijL}b_{ijL}}, \frac{a_{ijU}+b_{ijU}-2a_{ijU}b_{ijU}}{1-a_{ijU}b_{ijU}}\right), & otherwise \end{cases}$$

(ii) The Hamacher product of A and B is defined by $A \otimes_H B = (c_{ijL}, c_{ijU})$,

$$\text{where } (c_{ijL}, c_{ijU}) = \begin{cases} 0, & if \quad a_{ijL}, a_{ijU} = 0, b_{ijL}, b_{ijU} = 0 \\ \left(\frac{a_{ijL}b_{ijL}}{a_{ijL}+b_{ijL}-a_{ijL}b_{ijL}}, \frac{a_{ijU}b_{ijU}}{a_{ijU}+b_{ijU}-a_{ijU}b_{ijU}}\right), & otherwise \end{cases}$$

respectively.

Lemma 2.6. [13]

$$For\ any\ a, b \in [0,1], \frac{ab}{a+b-ab} \leq \frac{a+b-2ab}{1-ab}.$$

3. HAMACHER OPERATIONS OF IVIFMS.

In this section, based on the Definition 2.5, we define some operations for any two IVFMs $A = [< [a_{ij\mu L}, a_{ij\mu U}], [a_{ij\nu L}, a_{ij\nu U}] >]$, $B = [< [b_{ij\mu L}, b_{ij\mu U}], [b_{ij\nu L}, b_{ij\nu U}] >]$ of order $m \times n$ in the following.

Definition 3.1.

Let $A = [< [a_{ij\mu L}, a_{ij\mu U}], [a_{ij\nu L}, a_{ij\nu U}] >]$, $B = [< [b_{ij\mu L}, b_{ij\mu U}], [b_{ij\nu L}, b_{ij\nu U}] >]$ of same size, then the Hamacher operations of interval-valued intuitionistic fuzzy matrices are defined as follows:

(i) The Hamacher sum of A and B is defined by $A \oplus_H B = [< (c_{ij\mu L}, c_{ij\mu U}), (c_{ij\nu L}, c_{ij\nu U})>]$ where

$$[< (c_{ij\mu L}, c_{ij\mu U}), (c_{ij\nu L}, c_{ij\nu U}) >] = \begin{cases} 1, & if\ \ a_{ij\mu L}, a_{ij\mu U} = 1, b_{ij\mu L}, b_{ij\mu U} = 1,\ \ a_{ij\nu L}, a_{ij\nu U} = 1, b_{ij\nu L}, b_{ij\nu U} = 1 \\ \left[\left\langle \begin{array}{c} \left(\frac{a_{ij\mu L} + b_{ij\mu L} - 2a_{ij\mu L}b_{ij\mu L}}{1 - a_{ij\mu L}b_{ij\mu L}}, \frac{a_{ij\mu U} + b_{ij\mu U} - 2a_{ij\mu U}b_{ij\mu U}}{1 - a_{ij\mu U}b_{ij\mu U}}\right), \\ \left(\frac{a_{ij\nu L}b_{ij\nu L}}{a_{ij\nu L} + b_{ij\nu L} - a_{ij\nu L}b_{ij\nu L}}, \frac{a_{ij\nu U}b_{ij\nu U}}{a_{ij\nu U} + b_{ij\nu U} - a_{ij\nu U}b_{ij\nu U}}\right) \end{array} \right\rangle\right], & otherwise \end{cases}$$

(ii) The Hamacher product of A and B is defined by $A \otimes_H B = [< (c_{ij\mu L}, c_{ij\mu U}), (c_{ij\nu L}, c_{ij\nu U})>]$
where

$$[< (c_{ij\mu L}, c_{ij\mu U}), (c_{ij\nu L}, c_{ij\nu U}) >] = \begin{cases} 1, & if\ \ a_{ij\mu L}, a_{ij\mu U} = 1, b_{ij\mu L}, b_{ij\mu U} = 1,\ \ a_{ij\nu L}, a_{ij\nu U} = 1, b_{ij\nu L}, b_{ij\nu U} = 1 \\ \left[\left\langle \begin{array}{c} \left(\frac{a_{ij\mu L}b_{ij\mu L}}{a_{ij\mu L} + b_{ij\mu L} - a_{ij\mu L}b_{ij\mu L}}, \frac{a_{ij\mu U}b_{ij\mu U}}{a_{ij\mu U} + b_{ij\mu U} - a_{ij\mu U}b_{ij\mu U}}\right), \\ \left(\frac{a_{ij\nu L} + b_{ij\nu L} - 2a_{ij\nu L}b_{ij\nu L}}{1 - a_{ij\nu L}b_{ij\nu L}}, \frac{a_{ij\nu U} + b_{ij\nu U} - 2a_{ij\nu U}b_{ij\nu U}}{1 - a_{ij\nu U}b_{ij\nu U}}\right) \end{array} \right\rangle\right], & otherwise \end{cases}$$

respectively.

We present some basic properties of IVIFMs. The commutative, associative and identity laws are valid for IVIFMs under the operations addition $\oplus_H$ and multiplication $\otimes_H$.

Property 3.2. For any IVIFMs, $A = (a_{ijL}, a_{ijU})$ and $B = (b_{ijL}, b_{ijU})$ of same size, then $A \otimes_H B \leq A \oplus_H B$.

Proof. By using Lemma 2.6,

$$\left[\left\langle \begin{matrix} \left(\frac{a_{ij\mu L} b_{ij\mu L}}{a_{ij\mu L} + b_{ij\mu L} - a_{ij\mu L} b_{ij\mu L}} \leq \frac{a_{ij\mu L} + b_{ij\mu L} - 2a_{ij\mu L} b_{ij\mu L}}{1 - a_{ij\mu L} b_{ij\mu L}}\right), \\ \left(\frac{a_{ij\mu U} b_{ij\mu U}}{a_{ij\mu U} + b_{ij\mu U} - a_{ij\mu U} b_{ij\mu U}} \leq \frac{a_{ij\mu U} + b_{ij\mu U} - 2a_{ij\mu U} b_{ij\mu U}}{1 - a_{ij\mu U} b_{ij\mu U}}\right) \end{matrix} \right\rangle\right]$$

,

$$\left[\left\langle \begin{matrix} \left(\frac{a_{ij\nu L} + b_{ij\nu L} - 2a_{ij\nu L} b_{ij\nu L}}{1 - a_{ij\nu L} b_{ij\nu L}} \geq \frac{a_{ij\nu L} b_{ij\nu L}}{a_{ij\nu L} + b_{ij\nu L} - a_{ij\nu L} b_{ij\nu L}}\right), \\ \left(\frac{a_{ij\nu U} + b_{ij\nu U} - 2a_{ij\nu U} b_{ij\nu U}}{1 - a_{ij\nu U} b_{ij\nu U}} \geq \frac{a_{ij\nu U} b_{ij\nu U}}{a_{ij\nu U} + b_{ij\nu U} - a_{ij\nu U} b_{ij\nu U}}\right) \end{matrix} \right\rangle\right]$$

for all i,j.
Hence, $A \otimes_H B \leq A \oplus_H B$.

Property 3.3. For any IVIFM A,

$(i) A \oplus_H A \geq A,$

$(ii) A \otimes_H A \leq A.$

Proof. (i)$A \oplus_H A = \left(\frac{2a_{ij\mu L} - 2a_{ij\mu L}^2}{1 - a_{ij\mu L}^2}, \frac{2a_{ij\mu U} - 2a_{ij\mu U}^2}{1 - a_{ij\mu U}^2}\right)$
$= \left(\frac{2a_{ij\mu L}}{1 + a_{ij\mu L}}, \frac{2a_{ij\mu U}}{1 + a_{ij\mu U}}\right)$
$\geq (a_{ij\mu L}, a_{ij\mu U})$ and similarly we can prove that, $(a_{ij\nu L}, a_{ij\nu U})$.
=A. Since $a_{ij\mu L} \leq \frac{2a_{ij\mu L}}{1 + a_{ij\mu L}}$, $a_{ij\mu U} \leq \frac{2a_{i\mu jU}}{1 + a_{ij\mu U}}$ for all i,j.

Therefore $A \oplus_H A \geq A$.
(ii) $A \otimes_H A = \left(\frac{a_{ij\mu L}^2}{2a_{ij\mu L} - a_{ij\mu L}^2}, \frac{a_{ij\mu U}^2}{2a_{ij\mu U} - a_{ij\mu U}^2}\right) = \left(\frac{a_{ij\mu L}}{2 - a_{ij\mu L}}, \frac{a_{ij\mu U}}{2 - a_{ij\mu U}}\right) \leq (a_{ij\mu L}, a_{ij\mu U})$
and similarly we can prove that, $(a_{ij\nu L}, a_{ij\nu U})$ for all i,j.
=A.

Therefore $A \otimes_H A \leq A$.

The operations $\oplus_H$ and $\otimes_H$ are commutative as well as associative and identity and with respect to $\oplus_H$ and $\otimes_H$ are exist.
The following properties are obvious.

Property 3.4. For any IVIFMs, $A = (a_{ijL}, a_{ijU}), B = (b_{ijL}, b_{ijU})$ and $C = (c_{ijL}, c_{ijU})$of same size, then

$(i) A \oplus_H B = B \oplus_H A,$

$(ii)(A \oplus_H B) \oplus_H C = A \oplus_H (B \oplus_H C),$

$(iii) A \otimes_H B = B \otimes_H A,$

$(iv)(A \otimes_H B) \otimes_H C = A \otimes_H (B \otimes_H C).$

Property 3.5 For any IVIFMs, $A = (a_{ijL}, a_{ijU})$ and $B = (b_{ijL}, b_{ijU})$ of same size, then

$(i) A \oplus_H B \geq (A \vee B),$

$(ii) A \otimes_H B \leq (A \vee B),$

$(iii) A \oplus_H B \geq (A \wedge B),$

$(ii) A \otimes_H B \leq (A \wedge B).$

Property 3.6. For any IVIFM A,

$(i) A \oplus_H O = O \oplus_H A = A,$

$(ii) A \otimes_H J = J \otimes_H A = A,$

$(iii) A \oplus_H J = J,$

$(iv) A \otimes_H O = O.$

Thus $(F_{mn}, \oplus_H)$ and $(F_{mn}, \otimes_H)$ forms a commutative monoid.

The operators $\oplus_H$ and $\otimes_H$ do not obey the De Morgan's laws over transpose.

Property 3.7. For any IVIFMs, $A = (a_{ijL}, a_{ijU})$ and $B = (b_{ijL}, b_{ijU})$ of same size, then

$(i)(A \oplus_H B)^T = A^T \oplus_H B^T,$

$(ii)(A \otimes_H B)^T = A^T \otimes_H B^T.$ Where A^T is the transpose of A.

Property 3.8. For any IVIFMs, $A = (a_{ijL}, a_{ijU})$ and $B = (b_{ijL}, b_{ijU})$ of same size, if $A \leq B$, then $A \otimes_H C \leq B \otimes_H C$.

Proof. Let $a_{ijL} \leq b_{ijL}$ and $a_{ijU} \leq b_{ijU}$ for all i,j.

$a_{ij\mu L} c_{ijL}^2 \leq b_{ij\mu L} c_{ij\mu L}^2$

$a_{ij\mu L} c_{ij\mu L}^2 + a_{ij\mu L} b_{ij\mu L} c_{ij\mu L}(1 - c_{ij\mu L}) \leq b_{ij\mu L} c_{ij\mu L}^2 + a_{ij\mu L} b_{ij\mu L} c_{ij\mu L}(1 - c_{ij\mu L})$

$a_{ij\mu L} c_{ij\mu L}^2 + a_{ij\mu L} b_{ij\mu L} c_{ij\mu L} - a_{ij\mu L} b_{ij\mu L} c_{ij\mu L}^2 \leq b_{ij\mu L} c_{ij\mu L}^2 + a_{ij\mu L} b_{ij\mu L} c_{ij\mu L} - a_{ij\mu L} b_{ij\mu L} c_{ij\mu L}^2$

$a_{ij\mu L} c_{ij\mu L}(c_{ij\mu L} + b_{ij\mu L} - b_{ij\mu L} c_{ij\mu L}) \leq b_{ij\mu L} c_{ij\mu L}(c_{ij\mu L} + a_{ij\mu L} - a_{ij\mu L} c_{ij\mu L})$

$$\left(\frac{a_{ij\mu L} c_{ij\mu L}}{a_{ij\mu L} + c_{ij\mu L} - a_{ij\mu L} c_{ij\mu L}}\right) \leq \left(\frac{b_{ij\mu L} c_{ij\mu L}}{b_{ij\mu L} + c_{ij\mu L} - b_{ij\mu L} c_{ij\mu L}}\right)$$

and similarly we prove that,

$$\left(\frac{a_{ij\mu U}c_{ij\mu U}}{a_{ij\mu U}+c_{ij\mu U}-a_{ij\mu U}c_{ij\mu U}}\right)\leq\left(\frac{b_{ij\mu U}c_{ij\mu U}}{b_{ij\mu U}+c_{ij\mu U}-b_{ij\mu U}c_{ij\mu U}}\right)$$

and

$$\left(\frac{a_{ij\nu L}+c_{ij\nu L}-2a_{ij\nu L}c_{ij\nu L}}{1-a_{ij\nu L}c_{ij\nu L}}\right)\leq\left(\frac{b_{ij\nu L}+c_{ij\nu L}-2b_{ij\nu L}c_{ij\nu L}}{1-b_{ij\nu L}c_{ij\nu L}}\right)$$

and similarly we prove that,

$$\left(\frac{a_{ij\nu U}+c_{ij\nu U}-2a_{ij\nu U}c_{ij\nu U}}{1-a_{ij\nu U}c_{ij\nu U}}\right)\leq\left(\frac{b_{ij\nu U}+c_{ij\nu U}-2b_{ij\nu U}c_{ij\nu U}}{1-b_{ij\nu U}c_{ij\nu U}}\right)$$

Property 3.9. For any IVIFMs, $A=(a_{ijL},a_{ijU})$, $B=(b_{ijL},b_{ijU})$ and $C=(c_{ijL},c_{ijU})$ of same size, if $A\leq B$, then $A\oplus_H C\leq B\oplus_H C$

Proof. Let $a_{ijL}\leq b_{ijL}$ and $a_{ijU}\leq b_{ijU}$ for all i,j.

$a_{ij\mu L}(1-c_{ij\mu L})^2\leq b_{ij\mu L}(1-c_{ij\mu L})^2$

$\rightarrow a_{ij\mu L}(1-2c_{ij\mu L}+c_{ij\mu L}^2)\leq b_{ij\mu L}(1-2c_{ij\mu L}+c_{ij\mu L}^2)$

$\rightarrow a_{ij\mu L}-2a_{ij\mu L}c_{ij\mu L}+a_{ij\mu L}c_{ij\mu L}^2\leq b_{ij\mu L}-2b_{ij\mu L}c_{ij\mu L}+b_{ij\mu L}c_{ij\mu L}^2$

$\rightarrow a_{ij\mu L}-2a_{ij\mu L}c_{ij\mu L}+a_{ij\mu L}c_{ij\mu L}^2+(c_{ij\mu L}-a_{ij\mu L}b_{ij\mu L}c_{ij\mu L}+2a_{ij\mu L}b_{ij\mu L}c_{ij\mu L}^2)\leq b_{ij\mu L}-2b_{ij\mu L}c_{ij\mu L}b_{ij\mu L}c_{ij\mu L}^2+(c_{ij\mu L}-a_{ij\mu L}b_{ij\mu L}c_{ij\mu L}+2a_{ij\mu L}b_{ij\mu L}c_{ij\mu L}^2)$

$\rightarrow a_{ij\mu L}+c_{ij\mu L}-2a_{ij\mu L}c_{ij\mu L}-a_{ij\mu L}b_{ij\mu L}c_{ij\mu L}-b_{ij\mu L}c_{ij\mu L}^2+2a_{ij\mu L}b_{ij\mu L}c_{ij\mu L}^2\leq b_{ij\mu L}+c_{ij\mu L}-2b_{ij\mu L}c_{ij\mu L}-a_{ij\mu L}b_{ij\mu L}c_{ij\mu L}-a_{ij\mu L}c_{ij\mu L}^2+2a_{ij\mu L}b_{ij\mu L}c_{ij\mu L}^2$

$\rightarrow a_{ij\mu L}+c_{ij\mu L}-2a_{ij\mu L}c_{ij\mu L}-b_{ij\mu L}c_{ij\mu L}(a_{ij\mu L}+c_{ij\mu L}-2a_{ij\mu L}c_{ij\mu L})\leq b_{ij\mu L}+c_{ij\mu L}-2b_{ij\mu L}c_{ij\mu L}-a_{ij\mu L}c_{ij\mu L}(b_{ij\mu L}+c_{ij\mu L}-2b_{ij\mu L}c_{ij\mu L})$

$\rightarrow (a_{ij\mu L}+c_{ij\mu L}-2a_{ij\mu L}c_{ij\mu L})(1-b_{ij\mu L}c_{ij\mu L})\leq(b_{ij\mu L}+c_{ij\mu L}-2b_{ij\mu L}c_{ij\mu L})(1-a_{ij\mu L}c_{ij\mu L})$

$$\left(\frac{a_{ij\mu L}+c_{ij\mu L}-2a_{ij\mu L}c_{ij\mu L}}{1-a_{ij\mu L}c_{ij\mu L}}\right)\leq\left(\frac{b_{ij\mu L}+c_{ij\mu L}-2b_{ij\mu L}c_{ij\mu L}}{1-b_{ij\mu L}c_{ij\mu L}}\right)$$

and similarly we prove that,

$$\left(\frac{a_{ij\mu U}+c_{ij\mu U}-2a_{ij\mu U}c_{ij\mu U}}{1-a_{ij\mu U}c_{ij\mu U}}\right)\leq\left(\frac{b_{ij\mu U}+c_{ij\mu U}-2b_{ij\mu U}c_{ij\mu U}}{1-b_{ij\mu U}c_{ij\mu U}}\right)$$

and

$$\left(\frac{a_{ij\nu L}c_{ij\nu L}}{a_{ij\nu L}+c_{ij\nu L}-a_{ij\nu L}c_{ij\nu L}}\right)\leq\left(\frac{b_{ij\nu L}c_{ij\nu L}}{b_{ij\nu L}+c_{ij\nu L}-b_{ij\nu L}c_{ij\nu L}}\right)$$

and similarly we prove that,

$$\left(\frac{a_{ij\nu U}c_{ij\nu U}}{a_{ij\nu U}+c_{ij\nu U}-a_{ij\nu U}c_{ij\nu U}}\right)\leq\left(\frac{b_{ij\nu U}c_{ij\nu U}}{b_{ij\nu U}+c_{ij\nu U}-b_{ij\nu U}c_{ij\nu U}}\right)$$

for all i,j.

Property 3.10. For any IVIFMs, $A=(a_{ijL},a_{ijU})$ and $B=(b_{ijL},b_{ijU})$ of same size, then

$(i)(A\wedge B)\oplus_H(A\vee B)=A\oplus_H B,$

$(ii)(A\wedge B)\otimes_H(A\vee B)=A\otimes_H B,$

$(i)(A\oplus_H B)\wedge(A\otimes_H B)=A\otimes_H B,$

$(ii)(A\oplus_H B)\vee(A\otimes_H B)=A\oplus_H B.$

Proof. We prove (i) and (ii), other can be proved analogously.

$(i)(A\wedge B)\oplus_H(A\vee B)$

$$=\left(\frac{a_{ij\mu L}+b_{ij\mu L}-2a_{ij\mu L}b_{ij\mu L}}{1-a_{ij\mu L}b_{ij\mu L}},\frac{a_{ij\mu U}+b_{ij\mu U}-2a_{ij\mu U}b_{ij\mu U}}{1-a_{ij\mu U}b_{ij\mu U}}\right)$$

Similarly we can prove that,

$$\left(\frac{a_{ij\nu L}b_{ij\nu L}}{a_{ij\nu L}+b_{ij\nu L}-a_{ij\nu L}b_{ij\nu L}},\frac{a_{ij\nu U}b_{ij\nu U}}{a_{ij\nu U}+b_{ij\nu U}-a_{ij\nu U}b_{ij\nu U}}\right)$$

for all i,j.

$.=A\oplus_H B$.

$(ii)(A\wedge B)\otimes_H(A\vee B)$

$$=\left(\begin{array}{c}\frac{min(a_{ij\mu L},b_{ij\mu L})max(a_{ij\mu L},b_{ij\mu L})}{(min(a_{ij\mu L},b_{ij\mu L})+max(a_{ij\mu L},b_{ij\mu L})-min(a_{ij\mu L},b_{ij\mu L})max(a_{ij\mu L},b_{ij\mu L}))},\\ \frac{min(a_{ij\mu U},b_{ij\mu U})max(a_{ij\mu U},b_{ij\mu U})}{(min(a_{ij\mu U},b_{ij\mu U})+max(a_{ij\mu U},b_{ij\mu U})-min(a_{ij\mu U},b_{ij\mu U})max(a_{ij\mu U},b_{ij\mu U}))}\end{array}\right)$$

$$=\left(\frac{a_{ij\mu L}b_{ij\mu L}}{a_{ij\mu L}+b_{ij\mu L}-a_{ij\mu L}b_{ij\mu L}},\frac{a_{ij\mu U}b_{ij\mu U}}{a_{ij\mu U}+b_{ij\mu U}-a_{ij\mu U}b_{ij\mu U}}\right)$$

Similarly we can prove that,

$$\left(\frac{a_{ij\nu L}+b_{ij\nu L}-2a_{ij\nu L}b_{ij\nu L}}{1-a_{ij\nu L}b_{ij\nu L}},\frac{a_{ij\nu U}+b_{ij\nu U}-2a_{ij\nu U}b_{ij\nu U}}{1-a_{ij\nu U}b_{ij\nu U}}\right)$$

for all i,j.
$= A \otimes_H B.$

The following Properties are obvious.

Property 3.11. For any IVIFMs, $A = (a_{ijL}, a_{ijU}),\ B = (b_{ijL}, b_{ijU})$ and $C = (c_{ijL}, c_{ijU})$ of same size, then

$(i) A \oplus_H (B \vee C) = (A \oplus_H B) \vee (A \oplus_H C),$
$(ii) A \otimes_H (B \vee C) = (A \otimes_H B) \vee (A \otimes_H C),$
$(iii) (A \vee B) \oplus_H C = (A \oplus_H C) \vee (B \oplus_H C),$
$(iv) (A \wedge B) \oplus_H C = (A \oplus_H C) \wedge (B \oplus_H C).$

Property 3.12. For any IVIFMs, $A = (a_{ijL}, a_{ijU}),\ B = (b_{ijL}, b_{ijU})$ and $C = (c_{ijL}, c_{ijU})$ of same size, then

$(i) A \oplus_H (B \wedge C) = (A \oplus_H B) \wedge (A \oplus_H C),$
$(ii) A \otimes_H (B \wedge C) = (A \otimes_H B) \wedge (A \otimes_H C),$
$(iii) (A \vee B) \otimes_H C = (A \otimes_H C) \vee (B \otimes_H C),$
$(iv) (A \wedge B) \otimes_H C = (A \otimes_H C) \wedge (B \otimes_H C).$

4. SOME RESULTS ON COMPLEMENT OF IVIFMS.

In this section, the complement of on IVIFM is used to analysis the complement nature of any system. For example, if A represents the crowdness of the roads at a particular time period then its complement A^C represents the clearness of the roads. Using the following results we can study the complement nature of a system with the help of original on IVIFMs.

he operator complement obey the De Morgan's law for the operator $\oplus_H$ and $\otimes_H$. This is established in the following property.

The following property is obvious.

Property 4.1. For any IVIFMs, $A = (a_{ijL}, a_{ijU})$ and $B = (b_{ijL}, b_{ijU})$ of same size, then

$(i) (A \oplus_H B)^C = A^C \otimes_H B^C,$
$(ii) (A \otimes_H B)^C = A^C \oplus_H B^C,$
$(iii) (A \oplus_H B)^C \leq A^C \oplus_H B^C,$
$(iv) (A \otimes_H B)^C \geq A^C \otimes_H B^C.$

5. HAMACHER SCALAR MULTIPLICATION AND EXPONENTIATION OPERATIONS OF IVIFMS.

We defined the following operations over Hamacher operations of IVIFMs. In this section, we construct Hamacher scalar multiplication $(n._h A)$ and Hamacher exponentiation $(A^{\wedge_h n})$ operations of IVIFM A and investigate their algebraic properties.

Based on the Definition 3.1, Hamacher sum and Hamacher product over two IVIFMs A and B are further indicated as the following operations.

Theorem 5.1 If n is any positive integer and A is a IVIFM, then the Hamacher scalar multiplication operation $(._h)$ is

$$n._h A = A \oplus_h \underbrace{\ldots}_{n} \oplus_h A$$

$$= \left[\langle\left(\frac{na_{ij\mu L}}{1+(n-1)a_{ij\mu L}}, \frac{na_{ij\mu U}}{1+(n-1)a_{ij\mu U}}\right), \left(\frac{na_{ij\nu L}}{1+(n-1)a_{ij\nu L}}, \frac{na_{ij\nu U}}{1+(n-1)a_{ij\nu U}}\right)\rangle\right]. \quad (5.1)$$

Proof. Mathematical induction can be used to prove that the above equation (5.1) holds for all positive integer *n.* The equation (5.1) is called *P(n).* Using the above Definition 3.1 of Hamacher sum (i), $A \oplus_h B$ we have:

$$A._h A = \left(\frac{a_{ij\mu L} + a_{ij\mu L} - 2a_{ij\mu L}a_{ij\mu L}}{1 - a_{ij\mu L}a_{ij\mu L}}, \frac{a_{ij\mu U} + a_{ij\mu U} - 2a_{ij\mu U}a_{ij\mu U}}{1 - a_{ij\mu U}a_{ij\mu U}}\right),$$

$$= \left(\frac{2a_{ij\mu L} - 2a_{ij\mu L}^2}{1 - a_{ij\mu L}^2}, \frac{2a_{ij\mu U} - 2a_{ij\mu U}^2}{1 - a_{ij\mu U}^2}\right),$$

$$= \left(\frac{2a_{ij\mu L}(1 - a_{ij\mu L})}{1 - a_{ij\mu L}^2}, \frac{2a_{ij\mu U}(1 - a_{ij\mu U})}{1 - a_{ij\mu U}^2}\right),$$

$$= \left(\frac{2a_{ij\mu L}}{1 + a_{ij\mu L}}, \frac{2a_{ij\mu U}}{1 + a_{ij\mu U}}\right),$$

$$= \left(\frac{2a_{ij\mu L}}{1 + (2-1)a_{ij\mu L}}, \frac{2a_{ij\mu U}}{1 + (2-1)a_{ij\mu U}}\right),$$

since $a_{ij\mu L} = (2-1)a_{ij\mu L}$, $a_{ij\mu U} = (2-1)a_{ij\mu U}$.

P(n) holds.

Suppose that equation (5.1) holds for $n = m$,

$$i.e., m._h A = A \oplus_h \underbrace{\ldots}_{m} \oplus_h A = \left(\frac{m.a_{ij\mu L}}{1+(m-1)a_{ij\mu L}}, \frac{m.a_{ij\mu U}}{1+(m-1)a_{ij\mu U}}\right)$$

Then, $(m+1)._h A = ((m._h A) \oplus_h A)$

$$= \left(\begin{array}{c} \dfrac{\dfrac{m.a_{ij\mu L}}{1+(m-1)a_{ij\mu L}} + a_{ij\mu L} - 2\dfrac{m.a_{ij\mu L}}{1+(m-1)a_{ij\mu L}}.a_{ij\mu L}}{1-\dfrac{m.a_{ij\mu L}}{1+(m-1)a_{ij\mu L}}.a_{ij\mu L}}, \\ \dfrac{\dfrac{m.a_{ij\mu U}}{1+(m-1)a_{ij\mu U}} + a_{ij\mu U} - 2\dfrac{m.a_{ij\mu U}}{1+(m-1)a_{ij\mu U}}.a_{ij\mu U}}{1-\dfrac{m.a_{ij\mu U}}{1+(m-1)a_{ij\mu U}}.a_{ij\mu U}} \end{array} \right),$$

$$= \left(\frac{a_{ij\mu L}(m+1)(1-a_{ij\mu L})}{(1+m.a_{ij\mu L})(1-a_{ij\mu L})}, \frac{a_{ij\mu U}(m+1)(1-a_{ij\mu U})}{(1+m.a_{ij\mu U})(1-a_{ij\mu U})} \right),$$

$$= \left(\frac{(m+1)a_{ij\mu L}}{1+m.a_{ij\mu L}}, \frac{(m+1)a_{ij\mu U}}{1+m.a_{ij\mu U}} \right),$$

$$= \left(\frac{(m+1)a_{ij\mu L}}{1+[(m+1)-1]a_{ij\mu L}}, \frac{(m+1)a_{ij\mu U}}{1+[(m+1)-1]a_{ij\mu U}} \right).$$

So, when *n=m+1,*

$$n._h A = A \oplus_h \underbrace{\ldots}_{n} \oplus_h A = \left(\frac{na_{ij\mu L}}{1+(n-1)a_{ij\mu L}}, \frac{na_{ij\mu U}}{1+(n-1)a_{ij\mu U}} \right)$$

Similarly we can prove that,

$$\left(\frac{na_{ij\nu L}}{1+(n-1)a_{ij\nu L}}, \frac{na_{ij\nu U}}{1+(n-1)a_{ij\nu U}} \right)$$

also holds.
Using the induction hypothesis that *P(n)* holds for any positive integer *n.*

Theorem 5.2. If n is any positive integer and A is a IVIFM, then the Hamacher exponentiation operation $(\wedge_h)$ is

$$A^{\wedge_h n} = A \otimes_h \overbrace{\ldots}^{n} \otimes_h A$$
$$= \left[\left\langle \left(\frac{a_{ij\mu L}}{n-(n-1)a_{ij\mu L}}, \frac{a_{ij\mu U}}{n-(n-1)a_{ij\mu U}} \right), \left(\frac{a_{ij\nu L}}{n-(n-1)a_{ij\nu L}}, \frac{a_{ij\nu U}}{n-(n-1)a_{ij\nu U}} \right) \right\rangle \right] \quad (5.2)$$

Proof. Mathematical induction can be used to prove that the above equation (5.2) holds for all positive integer n. The equation (5.2) is called *P(n).* Using the above Definition 3.1 of Hamacher product (ii), $A \otimes_h B$ we have:

$$A^{\wedge_h A} = \left(\frac{a_{ij\mu L}^2}{2a_{ij\mu L} - a_{ij\mu L}^2}, \frac{a_{ij\mu U}^2}{2a_{ij\mu L} - a_{ij\mu U}^2}\right),$$

$$= \left(\frac{a_{ij\mu L}^2}{a_{ij\mu L}(2-a_{ij\mu L})}, \frac{a_{ij\mu U}^2}{a_{ij\mu U}(2-a_{ij\mu U})}\right),$$
$$= \left(\frac{a_{ij\mu L}}{2-a_{ij\mu L}}, \frac{a_{ij\mu U}}{2-a_{ij\mu U}}\right),$$
$$= \left(\frac{a_{ij\mu L}}{2-(2-1)a_{ij\mu L}}, \frac{a_{ij\mu U}}{2-(2-1)a_{ij\mu U}}\right), \text{ since } a_{ij\mu L} = (2-1)a_{ij\mu L}, a_{ij\mu U} = (2-1)a_{ij\mu U}.$$
$$A^{\wedge_h n} = \left(\frac{a_{ij\mu L}}{n-(n-1)a_{ij\mu L}}, \frac{a_{ij\mu U}}{n-(n-1)a_{ij\mu U}}\right),$$

P(n) holds.

Suppose that equation (5.2) holds for $n=m$,

$$i.e., A^{\wedge_h m} = A \otimes_h \overbrace{\ldots}^{m} \otimes_h A = \left(\frac{a_{ij\mu L}}{m-(m-1)a_{ij\mu L}}, \frac{a_{ij\mu U}}{m-(m-1)a_{ij\mu U}}\right).$$

So, when $n = m+1$,

$$A^{\wedge_h m+1} = \left(\frac{a_{ij\mu L}}{m+1-[(m+1)-1]a_{ij\mu L}}, \frac{a_{ij\mu U}}{m+1-[(m+1)-1]a_{ij\mu U}}\right),$$
$$A^{\wedge_h n} = A \otimes_h \overbrace{\ldots}^{n} \otimes_h A = \left(\frac{a_{ij\mu L}}{n-(n-1)a_{ij\mu L}}, \frac{a_{ij\mu U}}{n-(n-1)a_{ij\mu U}}\right)$$

Similarly we can prove that,

$$\left(\frac{a_{ij\nu L}}{n-(n-1)a_{ij\nu L}}, \frac{a_{ij\nu U}}{n-(n-1)a_{ij\nu U}}\right)$$

also holds.
Using the induction hypothesis that P(n) holds for any positive integer n.

Property 5.3. For any two IVIFMs A and B of same size, then positive integer n.

$(i)\ (n_1 \cdot_h A) \oplus_h (n_2 \cdot_h A) = (n_1 + n_2) \cdot_h A,$
$(ii)\ (n \cdot_h A) \oplus_h (n \cdot_h B) = n \cdot_h (A \oplus_h B),$
$(iii)\ A^{\wedge_h n_1} \otimes_h A^{\wedge_h n_2} = A^{\wedge_h (n_1+n_2)},$
$(iv)\ A^{\wedge_h n} \otimes_h B^{\wedge_h n} = (A \otimes_h B)^{\wedge_h n},$
$(v)\ n_2 \cdot_h (n_1 \cdot_h A) = (n_1 n_2) \cdot_h A,$
$(vi)\ (A^{\wedge_h n_1})^{\wedge_h n_2} = A^{\wedge_h (n_1 n_2)}.$

Proof. The proof is follows from Theorem 5.1, 5.2 & Definition 3.1.

Property 5.4. For any two IVIFMs A and B of same size, then positive integer n.

$(i)\, n\cdot_h (A \wedge B) = (n\cdot_h A) \wedge (n\cdot_h B),$
$(ii)\, n\cdot_h (A \vee B) = (n\cdot_h A) \vee (n\cdot_h B),$
$(iii)\ (A \wedge B)^{\wedge_h n} = A^{\wedge_h n} \wedge B^{\wedge_h n},$
$(iv)\ (A \vee B)^{\wedge_h n} = A^{\wedge_h n} \vee B^{\wedge_h n}.$

Proof. The proof is follows from Theorem 5.1, 5.2 & Definition 2.4.

6. CONCLUSIONS

The work has expanded the Hamacher operations results under an interval-valued intuitionistic fuzzy matrices. In this paper, we have developed the Hamacher operations of IVIFMs and researched their algebraic properties. We likewise demonstrated that the arrangement of all IVIFMs concerning Hamacher sum and Hamacher product frames a commutative monoid. An investigation of the mathematical structure of IVIFMs as for Hamacher activities gives us a profound knowledge into the applications. At that point, De Morgan's laws are checked.

Besides, we developed Hamacher scalar multiplication and Hamacher exponentiation operations an IVIFMs and examined their arithmetical properties. we may apply these activities in the field of various zones, for instance, dynamic choice and accord, business and advertising the board, structure, designing and assembling, data innovation and systems administration applications, HR the executives, military applications, vitality the executives, geographic data framework applications, and so forth. It is worth to call attention to that the proposed Hamacher operations over IVIFMs will be applied to aggregating interval-valued fuzzy information in the future.

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Biography

I. Silambarasan received his Ph.D. degree in Mathematics from the Department of Mathematics, Annamalai University, Tamil Nadu, India, in 2020. He has published 20 research articles in various international journals (Scopus and web of science). He has presented research papers in international conferences and attended many conferences, workshops, and seminars held in India. His research interests are Fuzzy Mathematics.

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