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Multiplicative counterpart of the essentially additive Borsuk-Ulam theorem as the pivoting gateway to equidimensional paired dual reciprocal spaces

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ABSTRACT

Multiplicative counterpart of the essentially additive Borsuk-Ulam theorem is proposed as abstract guiding principle suitable for explanation of intricate mathematical – that is both operational/algebraic and structural/geometric – relationships existing between spaces of equal dimensionality forming dual multispatial structures, which comprise twin equidimensional dual reciprocal spaces. Under auspices of the multispatial reality paradigm, the multiplicatively inversive type of the Borsuk-Ulam theorem can thus be also interpreted as an interspatial pivoting gateway between paired dual reciprocal spaces.

***Keywords*:** Borsuk-Ulam theorem, multispatial mathematical reality, dual multispatial structures

1. INTRODUCTION

Traditional mathematics was developed under the formerly unspoken – and therefore unquestioned – single space reality (SSR) paradigm, which is essentially an overly simplistic concept rooted in otherwise meaningful set theory. The SSR paradigm virtually assumed that the set-theoretical mathematical universe contains a single universal set that comprises all imaginable elements of the mathematical realm and its subsets are mere selections from the

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single universal set. In the SSR framework spaces can be identified with subsets and thus also with the operational procedures devised to select the subsets. The operational procedures are essentially algebraic processes even if they comprise differentials and functionals (i.e. fixed magnitudes, which could also be the outcomes of integrations).

In the SSR setting, the geometric or quasigeometric structures can also be identified with mere set-theoretic subsets, whose existence is often arbitrarily postulated. In order to avoid accidental creation of actually impossible to exist or construct structures, I have proposed new synthetic approach to mathematics – and mathematized sciences – that demands that the structures should be constructible.

Moreover, each structure should correspond to a certain operational procedure and vice versa. For there is no point in declaring existence of structures which do not fit their operational procedures. Neither should we postulate the presence of operational procedures that cannot really operate on all the entities being constructed within the – corresponding to them – structures. In other words: the operational procedures should be realistic, and the – corresponding to them – structures should be truly constructible. This does not man, however, that we must restrict ourselves to what we already know. On the contrary. We should devise expanded operations to fit prospective structures when the actual experimental results seem to demand them. By the same token, we should invent extensions to the already available geometric or quasigeometric structures if some experimental results seem to suggest the necessity of their presence.

The synthetic approach is not intended to seek safety in what is known but to discover the unknown yet mathematical (and consequently also physical) reality. The bottom line is that we should not be creators of the reality we would like to see, but just discoverers of the actual reality we live in, regardless of whether we like it or not. I am not talking in some unspecific philosophical terms. For example, in none of the physical experiments I have investigated did I ever stopped saying: oops, I cannot go any further because an already known and successful (in other applications) mathematical formula apparently demanded performing algebraic division by zero, which operation was explicitly prohibited by a decree in the traditional mathematics likely because it might lead to operating on previously misunderstood infinity. Instead of just sitting on my hands and lamenting how unfair the demand of physical reality was, I have devised unambiguous division by zero implemented as realistic multiplication by operational infinity.

Similarly, instead of complaining about the alleged impossibility of differentiation contravariantly represented functions, I realized that contravariant functions too can be differentiated, which is required for their proper subsequent integration, provided the contravariant functions are cast within certain dual reciprocal spaces, which are dual to the given primary spaces. However, the dual reciprocal spaces – and spatial or quasispatial structures, in general – demanded presence of some multispatial constructs. Therefore, I felt compelled to extend the single space implicitly assumed under the traditional SSR paradigm onto multispatial structures.

The new synthetic approach adopted the Hegelian method of contrasting contradictions in order to conceptually synthesize the unknown yet but well anticipated prospective mathematical infrastructure of physical reality. Although I am very well aware of the fact that Hegelian philosophy may smell like a skunk to some, I know from experience that even controversial notions have had some rationale behind their introduction.

Hence, I am not discarding contentious ideas right up front, but let them confront each other in order to fit the puzzle underlying the mathematical infrastructure of our physical reality.

The multispatial reality (MSR) paradigm that I have introduced fairly recently, was originally intended as just a multiplicative extension of the SSR paradigm. But my successful applications of the MSR paradigm to my explorations of formerly unanticipated results of certain unbiased physical experiments virtually demanded numerous – quite unforeseeable before – mathematical extensions, which I have made on as needed basis. Although the era of nontraditional approaches to doing mathematics was initiated by various scientists long ago, the trend was accelerated by emergence of nonstandard analysis, theory of distributions (as a generalization of classical functions, especially spurred by theoretically successful and experimentally confirmed explanation of the theory of electrons present in atoms, by Dirac equation) and other endeavors.

Nevertheless, the Euclidean standard of proofs based upon derivations made from axioms (supported by undefined primitive notions) was retained and viewed as the guarantor of validity of rigorously proven theorems. However, some unbiased physical experiments involving quantum entanglement challenged the traditional paradigms underlying our understanding of the physical reality, even though most of the (questioned at present) paradigms have not always been mentioned by name in the past.

On the mathematical side, the traditional belief that scientific theories (and especially the mathematical ones) should have sure foundations, was clearly shattered by works of Gödel, and therefore the value of proof was slightly deprecated. Yet most mathematicians are very reluctant to question their (often unspoken) traditional paradigms, some of which apparently seem so selfevident that they are not even mentioned as capable of influencing axiomatics and especially altering the meaning of the undefined primitive notions, whose definition could obviously depend on various possible interpretations of the paradigms. The problem with overlooking the unspoken paradigms is that their validity is never questioned, of course.

It is important to realize that formal proofs do not ensure the validity of proven theorems but can only confirm that their derivation followed the rules used for deriving them. The final blow to the traditional belief in infallibility of the allegedly rigorous formal proofs came from investigations of some formerly quite unanticipated results of unbiased physical experiments, which have been reconciled despite contradicting previously accepted theories of physics, and consequently, also defying the – previously unchallenged – theories of mathematics upon which the relevant physical theories had been developed in the first place. The latter statement reflects my perception of theoretical physics as just a practical, explanatory comment on the underlying it abstract mathematics, which is not always understood in its entirety before the (physical) comments are made.

Shifting foundational paradigms in the sense of replacing one paradigm with another one, does not necessarily invalidate the theories of mathematics or physics that had been developed upon the previous paradigm, at least not in their totality. But the new paradigm can indeed restrict the previously accepted scope of validity of their applications. Foundational paradigms are not always inconsequential. Their adoption can change the ways of doing things. An example of that is the operationally necessary restriction of the conceptual validity of tensor calculus to only admissible description of purely radial physical phenomena that happen within radial/center-bound force fields [1], which rather embarrassing feature was not divulged in the past and attempts at rectifying it (or just revealing it) were either tacitly squashed or openly discarded.

2. BORSUK-ULAM THEOREM

The Borsuk-Ulam (BU) theorem has several alternative yet quite equivalent formulations in the SSR setting. Two of them pertain to the topics to be discussed in this paper. For integer n the BU theorem can be formulated in the traditional set-theoretical terms as follows:
For n>0 the two statements are equivalent:

1) for every continuous mapping $f: S^n \rightarrow \mathbb{R}^n$ there exist a point $x \in S^n$ such that the following relation holds true: $f(x) = f(-x)$
2) for every antipode-preserving map $f: S^n \rightarrow \mathbb{R}^n$ there is a point $x \in S^n$ satisfying the equation $f(x) = 0$

see [2] where simple proofs are concisely discussed. It is obvious that the BU theorem has additive character for the antipodal points on the given sphere with its origin at zero add to zero: $x + (-x) = 0$ see [3] for intuitive explanation; compare also [4] for antipodal theorem. This latter conclusion means that the antipodal points lie on the straight line segment that passes through the center of the sphere provided that the center is identified with zero, which is assumed as the origin of the given coordinate system. Thus, for antipode-preserving function f() we have: $f(-x) = -f(x)$.

For some abstract generalizations of the BU theorem see [5-12]. For the BU type theorem for multifunctions see [me], and a nonstandard version of the theorem is discussed in [13]; see also [14] and [15].

The clearly additive character of the BU theorem makes its conceptual impact on abstract and applied mathematics fairly simple, just as trivial as the traditional topology. In the sense, the BU theorem supports the formerly unspoken SSR paradigm in general and previous simplistic mathematical reasonings in particular. But prospective applications of realistic mathematics to the new physical theories that emerged from experimentally induced ideas and were developed with hints from some previously quite unanticipated yet unbiased experimental results in mind, seemingly prefer – and sometimes demand – the conceptual transition from the SSR paradigm to the conceptual framework of the MSR paradigm.

At the present stage of its theoretical development, the multispatial theory (that originated under auspices of the MSR paradigm) require slightly different conceptual approach. We cannot rely anymore on the simplistic concept of mappings that was the workhorse of generalized abstract reasonings in the traditional theoretical mathematics; neither morphisms are unambiguous in the MSR setting at present.

This more restrictive yet down to earth approach might subsequently change, of course. But the oversimplified reasonings in the SSR setting must not be carried over to the MSR framework at present, if we want to avoid generating of all those conceptual and operational nonsenses that have plagued traditional mathematics without most former mathematicians ever realizing that.

However, until we accomplish the development of the new and truly realistic mathematics that is capable of explaining many currently unexplained physical phenomena, it is safer to stick to the unassailable operational methods and forget – for the time being – about the irks to hastily generalize the topics which may not yet be quite understood in their entirety.

3. INVERSIVE GEOMETRY IN THE TRADITIONAL SSR SETTING

The effect of geometric inversion in the domain ℝ of real numbers is to change a cross ratio into its conjugate [16] p.44. Thus, the inversions map $0 \rightarrow 1$, $1 \rightarrow 0$, $\infty \rightarrow \infty$ [16] p.39 in the traditional SSR setting. The inversion of a given plane with central point O and radius r is the transformation determined by the following rule: A point X different from the points O and O_∞ is carried into the point X' on the ray OX which satisfies the equation $OX' = r^2/OX \Leftrightarrow OX'/r = r/OX$ [17]. One can see thus that the equivalence also equates the operation of 3D geometric inversion with algebraic reciprocation, and vice versa, the algebraic/operational reciprocation equates with geometric inversion even in the traditional SSR framework. Furthermore, truly geometric inversion should be definitely a multiplicative operation.

Nevertheless, all the aforementioned inversions were presented under the unspoken SSR paradigm. Complex inversion transformation $w = 1/z$ is discussed in [18], for instance. Yet complex, quaternionic and hypercomplex inverses are mixed: multiplicative when it comes to modulus and additive when it comes to argument/angle. Therefore, the set of complex numbers ℂ and that of hypercomplex numbers ℍ as well as the set of 4D quaternions ℚ are – unfortunately – unsuitable for truly geometric or quasigeometric inversions, both of which must not be merely algebraic (or operational, in general) features but also structural, which would require definitely multiplicative (or multiplicatively reciprocal) inverses, even though the algebraic sets are often depicted in quasigeometric terms. Inverse images of vector bundles and differential 1-forms are concisely discussed in the traditional SSR setting in [19].

Although there is nothing wrong with having mixed inverses when operating on objects portrayed in the essentially algebraic domains ℂ, ℚ, or ℍ, the prospective deployment of totally unrestricted division by zero, which is absolutely necessary for ensuring operationally and structurally unambiguous dual reciprocity of geometric objects depicted in spatial and/or quasispatial abstract structures, clearly demands the deployment of multispatial structures and consequently prompts us to make the shift from the conceptually unwarranted traditional SSR paradigm to the MSR paradigm.

Having analyzed several previously unanticipated results of some unbiased physical experiments I have discovered theoretical necessity of presence of certain other than purely radial (that is definitely nonradial, i.e. tangential and/or binormal in the parlance of differential geometry) effects of the usual purely radial/center-bound gravitational force fields [20]; see also more casual, explanatory report in [21].

Moreover, I have realized that some functions whose magnitudes were traditionally considered as scalars, such as that of potential energy within radial/center-bound force fields, could actually be represented as certain vectors but depicted in a separate space that is distinct from the usual length-based space of motion. If this is really the case, however, then our traditional mathematics should be expanded onto some new differential operations [22] and its scope should be extended onto new quasispatial structures governed by slightly distinct structural laws and different operational rules [23].

That the required expansion and extensions are indeed possible was demonstrated in [24]. In the latter case, however, the traditional reliance on purely algebraic approach is rather untenable. Nonetheless, both algebraic and differential operations are needed in order to constrain the envisioned geometric and quasigeometric multispatial structures, both of which are of importance for this paper.

To grasp the topic to be discussed in this paper it is advised to gain at least cursory understanding – or just transitory comprehension for those not interested in the subject matter presented in this paper – of multiplicative inversion shown in terms of differential and integral operations are outlined in the Appendix A, which is recommended as concise prerequisite.

For more details on that see [23, 25-27], and references therein.

4. MULTISPATIAL IMPLEMENTATION OF MULTIPLICATIVE INVERSION OPERATIONS SHALL INVOLVE BOTH ZERO AND INFINITY

The most abstract representation $\mathfrak{R}$ of the intuitively clear and operationally natural reciprocal relationship between the real zero and the ascending neverending infinity was assumed in [26] as being formally equivalent due to homogeneity of their distinct respective algebraic bases. This assumption is enshrined in the following conceptual pattern formula

$$\mathfrak{R}[\![\text{R}]\!]0 \Leftrightarrow \mathfrak{R}[\![\text{ʁ}]\!]\frac{1}{\infty} \tag{1}$$

which means that the ascending setvalued infinity does not belong in the same space as the primary space in which the singlevalued zero dwells and here it is natively represented [26]. The upside-down symbol ʁ is the generalized symbolic algebraic reciprocal basis when it is shown in double brackets. It also emphasizes the operational fact that the chosen primary native basis ⟦R⟧ – and the primary space P denominated by the primary basis – is reciprocal to the reciprocal space Q (or U) with its native reciprocal algebraic or geometric basis ⟦ʁ⟧.

The real zero $0 = 1/\infty$ is thus represented natively (i.e. housed) within the primary space P whereas the ascending infinity ∞ is housed (i.e. represented natively) within the reciprocal space Q. If the real zero would need to be hosted in the reciprocal space Q for a certain operation involving entities/objects residing in the space Q then the zero should be converted to infinity according to the multiplicative inversion formula $0=1/\infty$; or perhaps if the infinity would need to be hosted in the primary space P for a certain operation involving entities/objects residing in the space P, then the infinity should be converted to zero according to the multiplicative inversion (reciprocal) formula $\infty = 1/0$ see [26]. All objects should be represented in the native basis of the space operated on regardless of whether the objects are native to (i.e. housed in) the space or just hosted in it for the duration of the operation.

The choice of the space in which the desired operations are to be performed depends only on the basis in which we choose to denominate the result of the operations. The decision is up to us; it is not a dictate. But the point I am trying to make is that all the algebraic entities associated with the geometric (or quasigeometric) objects involved in the desired operations must reside in the same space for the duration of the operations. For there is no justifiable reason to operate on entities or objects that are not represented directly or indirectly in the space operated on. Either the entities are housed (i.e. represented natively/permanently) within the space of the operations to be performed, or they must be hosted, that is represented temporarily through conversion from a foreign basis to the native basis of the chosen space. Whether the conversion is made explicitly or in memory, it does not matter. But in order to successfully operate in multispatial framework, all algebraic entities to be operated on simultaneously must be gathered into the same space, or operational basket, if you will. In other words: bananas and apples must be placed in the very same proverbial blender if you want to drink the blended "banapple shake" craved for. Although it is possible to slice or crash the fruits in separate dishes, this procedure can give you only ingredients for making a fruity salad, but not a juicy "fruitshake". Performing unambiguous operational procedures shall resemble blending ingredients rather than simply mixing them without their possible operational interference. Slicing is certainly needed in both procedures but mainly to prevent rotting fruits (or operationally inadmissible terms) from being blended or mixed in.

Many alleged impossibilities, which emerged in traditional ways of doing mathematics, are actually due to attempted operating on compound entities whose representations were not quite operationally compatible. Perhaps that is why algebraic bases of nominally scalar magnitudes were routinely omitted in the past. Some magnitudes previously presumed as scalars, such as potential energy, for example, might actually have vectorial nature, if the potential energy were accumulated along mutually orthogonal directions (i.e. in radial, tangential and/or binormal directions, in parlance of differential geometry), as some previously unanticipated results of certain unbiased physical experiments and fairly precise observations indicated [20, 21]. That choice is up to the Nature, not up to us to make.

Both, the proper multispatial evaluation of genuine integral kernels (which should have been legitimately differentiated before they could be rightfully integrated) expressed as:

$$k(x) = \int_0^\infty d\{K(t,x) \cdot f(t)\} = \int_1^\infty t \cdot dK(t,[x])dt \oplus \int_0^1 \frac{1}{K'(t,[x])dt} \tag{2}$$

compare (27), as well as the root expression of evaluation of unrestricted division by zero

$$N_{n\uparrow}^{n\cdot\infty} \coloneqq n \cdot \infty = \frac{\mathrm{n}}{0} \eqqcolon N_{n\uparrow}^{n/0} \Leftrightarrow \sum_0^n \{[\int_1^\infty tv'([x],t)dt] \,|\, \text{ȸ} \oplus \left[\int_0^1 [\frac{1}{v'([x],t)dt}]\right] |\varphi\} \tag{3}$$

that was implemented as multiplication by infinity [28], indicated twin structure of abstract expressions involving both real zero and the ascending neverending infinity ∞ in the real domain $\mathbb{R}$. Compare also the Appendix A.

From the intuitively obvious algebraic evaluation of the operation of simple addition (or perhaps compounding) of two formally identical ascending infinities ∞ in the real domain $\mathbb{R}$

$$\infty + \infty = 2 \cdot \infty = \frac{2}{0} = \frac{1}{0} + \frac{1}{0} \tag{4}$$

from which – compare [27] – in conjunction with the pattern formula for multispatial evaluation of legitimately differentiated integral kernel k(x) in (2), and the core pattern formula (3), an interspatial addition $\infty \oplus \frac{1}{0}$ can be implied so that we can eventually obtain

$$\infty + \infty = 2 \cdot \infty = \frac{2}{0} = \frac{1}{0} + \frac{1}{0} \Rightarrow \infty \oplus \frac{1}{0} \tag{5}$$

where the symbol $\oplus$ denotes an interspatial addition performed within twin set of paired dual reciprocal spaces. If the twin paired spaces are equidimensional then we can conjecture also equivalence of an interspatial addition when it is implemented in the real numbers' domain $\mathbb{R}$

$$\left\{ \infty + \infty = 2 \cdot \infty = \frac{2}{0} \right\} \Leftrightarrow \left\{ \infty \oplus \frac{1}{0} \Leftrightarrow \frac{1}{0} \oplus \infty \right\} \tag{6}$$

for the operation of algebraic addition is symmetric in $\mathbb{R}$. Now on multiplying the formula (4) algebraically by the imaginary unit, the imaginary counterpart of the formula (4) implies

$$\left\{ 2i \cdot \infty = \frac{2i}{0} \right\} \Rightarrow \left\{ i\infty \oplus \frac{i}{0} \right\} \Rightarrow \left\{ i\infty \oplus \left(-\frac{1}{i0} \right) \right\} \Rightarrow \left\{ i\infty \ominus \frac{1}{i0} \right\} \tag{7}$$

where an algebraic interspatial subtraction is denoted by the symbol $\ominus$. This result is due to the operational fact that complexification respects the inversion i/0 = –1/i0 compare [29]. For complexification involving operational infinity see [30]. For concise discussion of an abstract complexification of vector bundles in the traditional rendition see [31], for instance.

Now I should demonstrate by example that the above complexification is operationally quite legitimate. It is easy to see that from the pattern formula (7) we can obtain the formula

$$\left\{ i\infty \oplus \frac{i}{0} \right\} = \left\{ \frac{i}{\infty} \oplus \frac{i}{0} \right\} = \left\{ -\frac{1}{i\infty} \ominus \frac{1}{i0} \right\} = -\left\{ \frac{1}{i\infty} \oplus \frac{1}{i0} \right\} \tag{8}$$

which shows that stacking twin infinities portrayed in paired dual reciprocal spaces upon each other is indeed multiplicative inversion preserving reciprocity – compare [29].

However, the operational equivalences (6) and (7) of the algebraic operations on zero and infinity in conjunction with the equivalence of the structural representations (1) of the two algebraic entities cannot imply their structural equality, because their homogeneous yet mutually exclusive native algebraic bases are quite different and thus require their distinct allocation in separate though paired dual reciprocal spaces. This – "tangential" to our presentation – topic shall be further discussed elsewhere.

5. ADAPTATION OF THE ESSENTIALLY ADDITIVE BORSUK-ULAM THEOREM TO THE INHERENTLY MULTIPLICATIVE MSR SETTING

More precisely, for two equidimensional paired 3D reciprocal spaces P and Q equipped with homogeneous algebraic basis p and q, respectively, the geometric map can be stated as:

$$0_p \rightarrow \infty_q \, , \infty_q \rightarrow \, 0_p \, , 0_q \rightarrow \infty_p \, , \infty_p \rightarrow \, 0_q \tag{9}$$

and the multiplicative inversion of the neutral unit element

$$\frac{1_p}{1_q} = -\frac{1_q}{1_p} \rightarrow |1| \tag{10}$$

which even though expressed in terms of native algebraic bases is actually quite analogous to inversions shown in [29]. This venue shall be further exemplified elsewhere, because it does require more comprehensive discussion of transitions between the reciprocal bases p and q.
In the pattern formulas (9) one could use also the twodirectional arrow ↔ (i.e. equivalence mapping) instead of the singledirectional arrow →, of course, but for the time being it is safer to stick to the single unidirectional arrow, because each direction should involve conversion

between the respective algebraic bases, which has not been specified yet for this purely operational formula. Note that we are actually dealing with twin infinities and twin zeros residing within paired dual reciprocal spaces. These twins differ only in their representations, which depend on the algebraic bases used to denominate and depict the twin entities. Notice that the pattern (9) is only operational formula, whereas conversion between bases (which has already been specified operationally before in another paper) requires also structural (i.e. geometric or quasigeometric) specification, which shall be done elsewhere (mainly to avoid unnecessary complication), for the sake of simplicity of this paper.

6. COMPARISON OF TRADITIONAL SINGLESPATIAL GEOMETRIC INVERSIONS WITH MULTIPLICATIVE INTERSPATIAL INVERSIONS

When it comes to the structural ramification of the BU theorem in the mentioned above three-dimensional case involving 3D Euclidean sphere, if the center of the sphere is identified with the origin of the local coordinate system (i.e. with zero), then every straight line passing through the origin demarcates two antipodal points on the line segment where the line pierces the sphere. Thus, it complies with the fact that in the traditional SSR framework the geometric inversions map $0 \rightarrow 1$, $1 \rightarrow 0$, $\infty \rightarrow \infty$ compare [16] p.39. Hence in the formerly unspoken SSR framework zero plays the role of the additive neutral element and thus also that of reversive pivot between the additive antipodes x and –x on the given sphere.

But due to the fact that in the MSR framework the ascending (neverending) infinity is naturally reciprocal (i.e. multiplicatively inverse) to zero, the multispatial inversion maps zero to infinity: $0 \rightarrow \infty$, and vice versa: $\infty \rightarrow 0$, while the multiplicative neutral unit element (i.e. real unit) $1 \rightarrow (1/1) = 1$ remains unchanged.

Therefore, it plays the role of an interspatial multiplicative reciprocal pivot, provided that the spaces housing zero and infinity are separate and paired as dual reciprocal with respect to each other. Hence, this multiplicative mapping is slightly different than the one deployed in the traditional single-space geometry which was developed in the traditional SSR setting. Yet because the real zero and its reciprocal infinity must dwell in distinct spaces, each equipped with different homogeneous algebraic bases, the multiplicative mapping acquired interspatial character.

7. MULTIPLICATIVE VERSION OF THE BORSUK-ULAM THEOREM

Hence in the MSR framework the multispatial theorem analogous to the BU theorem shall have multiplicative character. Thus, it could be called multispatial Banach-Ulam (MBU) theorem henceforth. However, at present, the intended applicability of the MBU shall be restricted on purpose to objects depicted only within equidimensional 3D algebraic spaces, i.e. within essentially operational, not really geometric, structures, which shall be further discussed elsewhere.

For $0<n<4$ the following two alternative statements of the MBU are formally equivalent:

1) for every interspatial reciprocal (i.e. multiplicative inverse) mapping $f: S^n \rightarrow \mathbb{R}^n$ there exists a point $x \in S^n$ such that $f(x) = 1/f(1/x)$

2) for every multiplicative antipode-preserving map $f : S^n \rightarrow \mathbb{R}^n$ there is a point $x \in S^n$ satisfying the multispatial inversion $f(x) = 0$ and $u \in U^n$ such that the interspatial dual reciprocal relationship $g(u) = 1/f(x)$ is fulfilled when $u=1/x=\infty$ provided that U is dual reciprocal space paired with the given equidimensional primary space S.

Although the MBU theorem could be offered also for 4D quasispatial structures, that shall be done elsewhere, because such structures are not really unique without some additional qualifications pertaining to their – necessarily heterogeneous – algebraic or perhaps geometric/quasigeometric bases, see [32] and [33], which lack of uniqueness has already been recognized also in the traditional mathematics [34] behind the scenes, so to say, but that was virtually never acknowledged in unambiguous conceptual terms. Under the SSR paradigm it remains a conceptual mystery that is still being worked on, to the best of my knowledge.

The alert reader might have already realized that the multiplicative neutral element (i.e. the real unit 1) that is its own reciprocal in the real domain $\mathbb{R}$, plays the role of a pivoting gateway within the given paired dual reciprocal structure $\{S^n, U^n\}$. In algebraic terms I used to preliminarily denote the dual reciprocal spatial structure as a paired duo {P,Q}. But in order to show its affinity to the BU theorem, and to avoid possible confusion with quaternions, for the time being we can use the alternative multispatial notation $\{S^n, U^n\}$.

Here the multispatial analogs of the additively reversive antipodes (x, –x) entertained in the original BU theorem, are real zero and the neverending infinity, which in the new MBU theorem can be called mutually reciprocal (that is multiplicatively inversive) interspatial antipodes (0, ∞), if you will. Just as the real zero virtually played the role of the additively reversive pivot in the BU theorem, the neutral element/unit 1 plays the role of multiplicatively inversive pivot in the MBU theorem.

At this stage I am not interested in proving that such multispatial structure can exist, for it has already been shown that differential operators actually do act simultaneously on formulas patterned on (11) – compare [35] or see the Appendix A for more concise explanation. After finishing the present stage of discovery of multispatial paired structures, the time will eventually come for their more formalized abstract representations presumably based upon factual axiomatics, which shall be done elsewhere. I am not against abstract axiomatization developed after the phase of discovery supplies the necessary and – experimentally confirmed – facts. Nevertheless, I am against making – often quite arbitrary – *a priori* axioms concocted right up front (that is before facts are discovered), which usually include tacitly hidden unwarranted existential postulates.

8. CONCLUSIONS

By analogy to the essentially additive original Borsuk-Ulam theorem that is applicable to additively reversive operations, a multiplicative version of the theorem has been proposed. The significance of the multiplicative version of the theoreem is that it corresponds to multiplicative inverses which are required for proper handling of dual reciprocal spaces in the multispatial framework. Wile the original/additive Borsuk-Ulam theorem pertains only to additively reversive pairs of antipodes such as (–x,+x) or (–∞,+∞) wherein the real zero (as the additive neutral element, in group-theoretical parlance) plays the role of a revolving reversive pivot, the new, multiplicatively inversive multispatial Borsuk-Ulam theorem pertains to reciprocal pairs of antipodes, such as (0,∞) wherein the multiplicatively inversive neutral element/unit 1 plays the role of a revolving inversive pivot due to the mutually reciprocal nature of zero and infinity: for $0 = \frac{1}{\infty}$ and vice versa: $\infty = \frac{1}{0}$, so that $\infty \cdot 0 = 1$.

In the sense, the multiplicatively inversive multispatial version of the Borsuk-Ulam theorem supplies new, multispatial interpretation of the neutral element/unit 1 as being the pivoting gatewy to equidimensional paired dual reciprocal spaces, at least in the 3D case.

APPENDIX A: DIFFERENTIAL AND INTEGRAL MULTIPLICATIVE INVERSION OPERATIONS IN THE MULTISPATIAL MSR SETTING

As reciprocal to covariant representations of functions, their contravariant representations can be expressed in terms of inverse differential operators [36], [37], [38]. Hence, for the integral kernel k() that compounds the scalar functions f() and K(), where K(t,[x]) denotes an influence function in more constructive then the usual approach to the theory of integral kernels [39], the derivative dk(t)/dt with respect to the (actively varying) variable t that is found in the scalar functions f() and K(), can be expressed as

$$\{f(t) \,▫\, K(t,x)\}'_t \coloneqq t \,▫\, \frac{dK(t,[x])}{dt} \oplus \frac{1}{K([t],[x])} \,▫\, \frac{1}{dt} = t \,▫\, K'(t,[x])dt \oplus \frac{1}{K([t],[x])} \,▫\, \frac{1}{dt} \quad (11)$$

where the symbol ⊕ emphasizes here the formal inappropriateness of adding the illegitimate – even though legitimately obtained – expression $\frac{1}{K([t],[x])} \,▫\, \frac{1}{dt}$ which does not really form operationally proper (i.e. ready to become an integrand) covariant compound derivative (under the unspoken SSR paradigm) but a reciprocal contravariant expression whose operational inappropriateness had not been revealed in the traditional mathematics that did not use the extra symbol ▫. Note that functionals are displayed in square brackets by traditional convention. Recall that functionals are as if frozen functions, which are not actively varying, such as the outcomes of integrations.

Although some authors call the function K() integral kernel, the usage can lead – and all too often did – to conceptual confusion, which allowed to forget that it is unacceptable to integrate a function that had not been obtained by lawful differentiation. The routine omission of the extra compounding symbol ▫ and the relaxed naming convention of the allegedly rigorous traditional mathematics generated numerous operational nonsenses without most scientists realizing that. This frivolous yet common practice permitted some hard to discern conceptual disasters to appear in presentations (i.e. lectures and academic publications) of the various topics entertained in traditional mathematics.

The irresponsible omission of the extra compounding multiplicative symbol ▫ produced also the unnecessary debacle insisting that contravariant representations of vectors or functions are not really differentiable, which was incidentally true in the SSR setting. But as the formula (11) shows that within the multispatial framework of the new MSR paradigm, in dual reciprocal space to the primary space, the pesky contravariant function is also legitimately differentiable via inversion/reciprocity, even though the differentiation must take place in distinct and different space.

For we could turn the contravariant expression into operationally legitimate covariant differential that is suitable for prospective integration:

$$\frac{1}{K([t],[x])} \;\text{▫}\; \frac{1}{dt} \Rightarrow \int\{1 \,/\, \frac{\partial K(t,[x])}{\partial t}\} \;\text{▫}\; \frac{1}{dt} = \int\{\frac{\partial t}{K\prime(t,[x])dt} \;\text{▫}\; \frac{1}{dt}\} = \int\{\frac{1}{K\prime(t,[x])dt}\} \qquad (12)$$

but because the inverse integrand on the right-hand side (RHS) of (12) is multiplicatively reciprocal, it must be placed in a reciprocal space that is dual to the primary space in which the K() and f() are represented in the primary space's native basis and thus appear as natively residing therein. Notice the extra symbol ▫ which is routinely absent in the traditional approach to mathematics. It denotes multiplicative algebraic compounding of functions. I am using it because it can make various deceptive misrepresentation of integral kernels in traditional mathematics visible even for alert undergraduate students.

Just as the contravariant representation of function was previously simply disregarded – under auspices of the formerly unspoken (and thus unquestioned) SSR paradigm – because of its allegedly impossible to perform differentiation, so was also the allegedly impossible to unambiguously operate on infinity slighted. Yet in the framework of the MSR paradigm none of the traditional prejudices is justified. Note that the pattern formula (11) which represents realistic (and operationally legitimate) integral kernel complies with twin representation of infinity as well as with the traditional evaluation of certain apparently descending compounded permutations that seemingly tend to zero – compare [40]. Furthermore, the treatment of operational infinity is not like that commonly accepted approach to infinity entertained in the traditional mathematics as it was exemplified in [41] or [42], for instance, because the latter approach tacitly generated numerous formerly undisclosed yet absolutely inadmissible nonsenses, some of which have been exposed in [43].

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