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R-Countability Axioms

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ABSTRACT

In this article, we use the concept of regular open sets to define a generalization of the countability axioms; namely regular countability axioms, and they are denoted by r-countability axioms. This class of axioms includes r-separable spaces, r-first countable spaces, r-Lindelöf spaces, r-σ-compact spaces and r-second countable spaces. We investigate their fundamental properties, and study the implication of the new axioms among themselves and with the known axioms. Moreover, we study the hereditary properties for r-countability axioms, also we consider some related functions in terms of r-open sets, which preserve these spaces. Finally, we prove that in regular space r-countability axioms and countability axioms are equivalent, while in locally compact T_2 space, the spaces: Lindelöf, r- Lindelöf, σ-compact and r-σ-compact are all equivalent.

***Keywords*:** Countability axioms, σ-compact spaces, Lindelöf spaces, compact spaces, regular open sets

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1. INTRODUCTION

The notion of regular open sets which are stronger form than open sets were introduced by Stone in 1937 [1], where a subset A in a space is called regular open (for short r-open) if A equals to the interior of its closure. More details on r-open sets and their properties can be found

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in [2-5]. The class of r-open sets used to define the semiregularization space of topological spaces, see [1] and [6], also researchers used these sets in a generalization for algebraic openings and closings in a complete lattice [2].

Many studies in the literature have been made on r-open sets, and they used these sets to derive several forms of higher and lower separation axioms and compactness. Levine [7] used r-open sets to define a space which lies between T_0 and T_1; called $T_{\frac{3}{4}}$ (where any singleton is closed or r-open), for more properties on $T_{\frac{3}{4}}$ space see [8]. Then in 2010, Balasubramanian [9] investigated the properties of new spaces called r-T_0, r-T_1 and r-T_2, where he illustrated the relations between these spaces and with some other spaces namely g-T_i spaces (I = 0, 1, 2) using g-open sets. In 1969 Singhal and Mathur [10] used the notion of r-open cover which is a cover by r-open sets to define r-compact (or nearly compact) spaces, they studied their properties, and showed that r-compact space is weaker than compact space, see [11] and [12]. Few years later Balasubramanian [13] introduced and studied the notion of r-Lindelöf (or nearly Lindelöf) spaces, he proved that this space lies between Lindelöf and weakly Lindelöf spaces. More properties on r-Lindelöf spaces were given in [14] and [15]. Jankovic and Konstadilaki in 1996 [16] investigated the covering properties by regular closed sets, and used it to define rc-compact and rc-Lindelöf spaces, also see [17].

The major goal of this paper is to use the concept of r-open sets and the known countability axioms to define r-countability axioms, which include the spaces: r-separable spaces, r-first countable spaces, r-Lindelöf spaces, r-σ-compact spaces and r-second countable spaces. We study the fundamental properties for these spaces, also we investigate some related functions in terms of r-open sets, which preserve these spaces, and then we study the hereditary properties for r-countability axioms. Finally, we consider r-countability axioms in regular spaces and also in locally compact T_2 spaces, and prove some statements.

2. REGULAR OPEN SETS

This section consists definitions and results regarding regular open sets, then we introduce r-closure, r-interior and r-derived of a set, and we consider some of their properties and their relations with the classical closure, interior and derived. Moreover, we recall some related functions as r-irresolute, strongly r-open and strongly r-closed. Throughout this paper (X, τ) or simply X and (Y, τ^*) or simply Y denote topological spaces, and the closure, interior and derived of a set A are respectively denoted by $\overline{A}$, A^o, A'.

Definition 2.1. [1] A subset A of a space X is called regular open (r-open) if A = $\overline{A}^o$, and the complement of r-open set is called regular closed (r-closed). The family of all r-open sets and r-closed sets in X are denoted by RO(X, τ) and RC(X, τ), respectively.

Proposition 2.1. [1] A subset B of a space X is regular closed if B = $\overline{B^o}$.

Remarks 2.1. [10]

1) Every r-open set is open, but not conversely.
2) Every r-closed set is closed, but not conversely.

r-open $\Longrightarrow$ open
r-closed $\Longrightarrow$ closed

Examples 2.1.

1) Let X = {a, b, c} and τ = {X, ∅, {a, b}}, then RO(X, τ) = {X,∅}, and the set A = {a, b} is an open set in X but not r-open, while the set {c} is a closed set in X but not r-closed.
2) In the discrete topological space X, RO(X, τ) = τ.
3) In the usual topological space (ℝ, μ), the set (0, 1)∪(1, 2) is open set but not r-open, since = $\overline{(0,1) \cup (1,2)}^{o}$ = (0, 2).

Proposition 2.2. [3]

1) Finite intersection of r-open sets is r-open.
2) Union of r-open sets is not necessarily r-open.

Definition 2.2. [3] A subset A of a space X is said to be r-clopen if it is both r-open and r-closed in X. The family of all r-clopen sets of a space X is denoted by RCO(X, τ).

Remarke 2.2. [3] In a topological space, a subset A is r-clopen iff A is clopen.

Definition 2.3. [3] Let A be a subset of X then, the r-closure of A defined as the intersection of all r-closed sets containing A, and its denoted by $\overline{A}^{r}$.

Proposition 2.3. [3] Let X be a space and A, B ⊆ X, then:

1- $\overline{A}^{r}$ is r-closed set.
2- $A \subseteq \overline{A}^{r}$.
3- A is r-closed if and only if $A = \overline{A}^{r}$.
4- $x \in \overline{A}^{r}$ if and only if $A \cap U \neq \emptyset$, for any r-open U containing x.
5- $A \subseteq \overline{A} \subseteq \overline{A}^{r}$.
6- $\overline{\overline{A}^{r}}^{r} = \overline{A}^{r}$.
7- If A ⊆ B, then $\overline{A}^{r} \subseteq \overline{B}^{r}$.

Definition 2.4. [3] Let X be a subset of X then, the r-interior of A is defined as the union of all r-open sets of contained, and its denoted by $A^{\circ r}$.

Proposition 2.4. [3] Let X be a space, and A, B ⊆ X, then:

A) $A^{\circ r}$ is r-open set.
B) $A^{\circ r} \subseteq A$.
C) A is r-open if and only if $A^{\circ r} = A$.
D) $A^{\circ r} \subseteq A^{\circ} \subseteq A$.
E) If A ⊆ B, then $A^{\circ r} \subseteq B^{\circ r}$.

Definition 2.5. [3] Let X be a space, and A ⊆ X. An r-neighborhood of A is any subset of X which contains an r-open set containing A.

Definition 2.6. [3] Let A be a subset of a space X. A point x in X is said to be r-limit point of A if for each r-open set U contains x implies that $U \cap A \setminus \{x\} \neq \emptyset$. The set of all r-limit points of A is called r-derived set of A, and its denoted by A'^{r}.

Proposition 2.5. [3] Let X be a topological space, and A, B ⊆ X, then

1) $A' \subseteq A'^{r}$.
2) If A ⊆ B, then $A'^{r} \subseteq B'^{r}$.
3) $\overline{A}^{r} = A \cup A'^{r}$.

Theorem 2.1. [10] Let A be subset of a space X. If A is an open or dense in X then: $RO(A, \tau_A) = \{V \cap A : V \in RO(X, \tau)\}$.

Proposition 2.6. [3] Let A ⊆ Y ⊆ X. Then: If A is a r-open (r-closed) set in Y, and Y is an r-open (r-closed) set in X, then A is an r-open (r-closed) set in X.

Theorem 2.2. [10] Let Y be a subspace of a space X, if Y is an open set in X and U ⊆ Y, then U is an r-open set in Y iff U is an r-open set in X.

Definition 2.7. [3] A function $F: X \longrightarrow Y$ is said to be:

1) An r-irresolute if the inverse image under F of an r-open set is r-open.
2) An strongly r-closed if the image under F of an r-closed set is r-closed.
3) An strongly r-open if the image under F of an r-open set is r-open.

3. COUNTABILITY AXIOMS

The class of spaces satisfing the axioms of countability are called: separeble spaces, first countable spaces, Lindelöf spaces, σ-compact spaces and second countable spaces. In this section we recall the definition and some properties concerning countability axioms which we need in the sequel. See [18], [19] and [20].

3. 1. Separable spaces

Definition 3.1.1. [20] A subset D of a topological space X is called dense if $\overline{D} = X$.

Theorem 3.1.1. [20] The set D is dense in a space X iff every non-empty open set in X contains points of D.

Definition 3.1.2. [20] A topological space X is said to be separable space if there exist a countable dense subset of X.

Theorem 3.1.2. [20]

1) An open subspace of separable space is separable.
2) Image of separable space under continuous map is separable.

3. 2. First countable spaces

Definition 3.2.1. [20] In a topological space X, a collection $\mathfrak{B}_x$ of open sets that contains x is called basis at x if for any open set U such that $x \in$ U there exists B_x in $\mathfrak{B}_x$ such that $x \in B_x \subseteq$ U.

Definition 3.2.2. [20] A topological space X is said to be first countable space if for every $x \in$ X there is a countable local base $\mathfrak{B}_x$ at x.

Theorem 3.2.1. [20]

1) A subspace of first countable space is first countable.
2) Image of first countable space under continuous and open map is first countable.

3. 3. Lindelöf spaces

Definition 3.3.1. [20] A topological space X is said to be Lindelöf space if every open cover of X has a countable subcover.

Corollary 3.3.1. [20] Every compact space is Lindelöf.

Theorem 3.3.1. [20]

1) Closed subspace of Lindelöf space is Lindelöf.
2) Image of Lindelöf space under continuous map is Lindelöf.

3. 4. $\boldsymbol{\sigma}$-Compact spaces

Definition 3.4.1. [20]. A topological space X is said to be a $\boldsymbol{\sigma}$-compact space if it is the union of countable many compact subsets of X.

Corollary 3.4.1. [20]

1) Every $\boldsymbol{\sigma}$-compact space is Lindelöf.
2) Every compact space is $\boldsymbol{\sigma}$-compact

Theorem 3.4.1. [20] Lindelöf locally compact T_2 is $\boldsymbol{\sigma}$-compact.

$$\text{Lindelöf} \xRightarrow{\text{locally compact+ } T_2} \boldsymbol{\sigma}\text{-compact}$$

Theorem 3.4.2. [20]

1) Closed subspace of $\boldsymbol{\sigma}$-compact space is $\boldsymbol{\sigma}$-compact.
2) Image of $\boldsymbol{\sigma}$-compact space under continuous map is $\boldsymbol{\sigma}$-compact.

3. 5. Second countable spaces

Definition 3.5.1. [20] Let X be a topological space, then sub collection $\mathfrak{B}$ of τ is said to be a base for τ if each member of τ can be expressed as a union of members of $\mathfrak{B}$.

Definition 3.5.2. [20] A topological space X is satisfies second countable axioms if X has a countable base $\mathfrak{B}$.

Corollary 3.5.1. [20] Every second countable space is first countablel, separable, and Lindelöf.

Corollary 3.5.2. [20] Second countable locally compact T_2 is $\boldsymbol{\sigma}$-compact.

second countable $\xRightarrow{\text{locally compact}+T_2}$ $\boldsymbol{\sigma}$-compact

Theorem 3.5.1. [20]

1) A subspace of second countable space is second countable.
2) Image of second countable space under continuous and open map is second countable.

We summarize the relations between countability axioms in diagram 1.

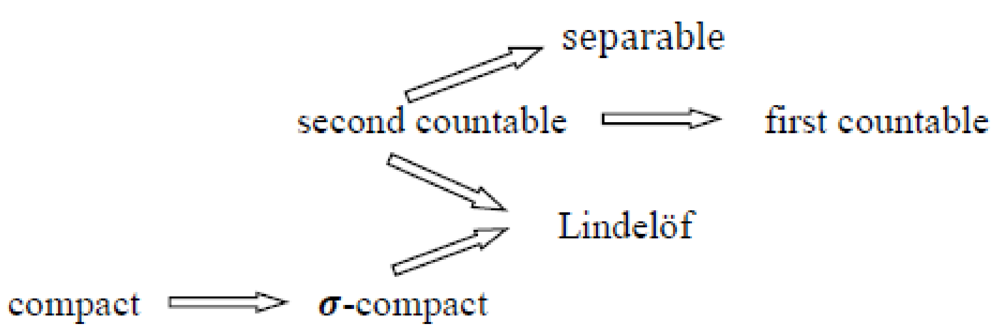

Diagram 1. Relarions between countability axioms.

4. R-COUNTABILITY AXIOMS

In the following section, we use the concept of regular open sets to define a lower class of countability axioms namely, regular countability axioms (for short r-countability axioms), where this class of axioms conclude: r-separable spaces, r-first countable spaces, r-Lindelöf spaces, r-$\boldsymbol{\sigma}$-compact spaces and r-second coumtable spaces. We study the topological properties of r-countability axioms and derive their inter relations.

4. 1. R-separable spaces

Definition 4.1.1. If A is a subset of a topological space X, then A is said to be r-dense if $\overline{A}^r = X$.

Examples 4.1.1.

1) In the cofinite topological space X, any non-empty set is r-dense, so any finite set is r-dence but not dense.
2) Let $X = \mathbb{R}$ with $\tau = \{\emptyset\} \cup \{U \subseteq \mathbb{R} : 0 \in U\}$, then $RO(X, \tau) = RC(X, \tau) = \{\mathbb{R}, \emptyset\}$, so any non-empty proper subset of $\mathbb{R}$ is r-dense but not dense.
3) In the usual topological space $\mathbb{R}$, the sets $\mathbb{Q}$ and K are r-dense, but $\mathbb{Q} \cap K = \emptyset$ is not r-dense.

Corollary 4.1.1. Every dense subset of a space X is r-dense, but not conversly, see example (4.1.1(1)).

Proof: Let A be a dense subset of a space X, i.e. $\overline{A} = X$, since $\overline{A} \subseteq \overline{A}^{r}$, then $X \subseteq \overline{A}^{r}$, and $\overline{A}^{r} \subseteq X$, thus $\overline{A}^{r} = X$.
dense $\Longrightarrow$ r-dense

Corollary 4.1.2. The set D is an r-dense in a space X iff every non empty r-open set in X contains points of D.

Proof: $\Longrightarrow$ Let D be an r-dense in X, and let V be a non-empty r-open set, so there is $x \in V$ and since $\overline{D}^{r} = X$ we have $D \cap V \neq \emptyset$.
$\Longleftarrow$ Let x be an arbitrary element in X, then any r-open set that contains x intersect D, i.e. $x \in \overline{D}^{r}$, so $\overline{D}^{r} = X$.

Remarks 4.1.1.

1) If A is r-dense subset in B, and B is r-dense subset in X, then A is r-dense subset in X.
2) Intersection of r-dense sets not necessarily r-dense, see example (4.1.1(3)).
3) Union of r-dense sets is r-dense.

Theorem 4.1.1. [10] A space X is regular space iff for every $x \in X$ and each open set U in X such that $x \in U$ there exists an open set V such that $x \in V \subseteq \overline{V} \subseteq U$.

Lemma 4.1.1. If V is a subset in a space X, then $\overline{V}^{o}$ is r-open set. i.e $RO(X, \tau) = \{\overline{V}^{o} : \text{where } V \in \tau\}$.

Proof: Since $\overline{V}^{o} \subseteq \overline{V}$, then $\overline{\overline{V}^{o}} \subseteq \overline{\overline{V}}$, i.e. $\overline{\overline{V}^{o}} \subseteq \overline{V}$, we get $\overline{\overline{V}^{o}}^{o} \subseteq \overline{V}^{o} \longrightarrow \boxed{1}$.
Now since $\overline{V}^{o} \subseteq \overline{\overline{V}^{o}}$, then $\overline{V}^{oo} \subseteq \overline{\overline{V}^{o}}^{o}$, we get $\overline{V}^{o} \subseteq \overline{\overline{V}^{o}}^{o} \longrightarrow \boxed{2}$.
From $\boxed{1}$ and $\boxed{2}$ we get $\overline{\overline{V}^{o}}^{o} = \overline{V}^{o}$, so V is r-open.

Lemma 4.1.2. In regular space, any open set can be expressed as a union of r-open sets, i.e. for any open set U, then $U = \bigcup_{\alpha \in I} V_{\alpha}$; where V_{α} is r-open for any α.

Proof: Let U be an open subset in a space X and $x \in$ U, since X is regular space, then there is an open set V_x such that $x \in V_x \subseteq \overline{V_x} \subseteq U$, so $x \in V_x = V_x^{o} \subseteq \overline{V_x}^{o} \subseteq U^{o}$, i.e. $x \in V_x \subseteq \overline{V_x}^{o} \subseteq$ U. From lemma (4.1.1), we have $\overline{V_x}^{o}$ is r-open set, and since x is arbitrary point then:
$U = \bigcup_{x \in X} \overline{V_x}^{o}$.

Theorem 4.1.2. If X is regular space, then D is dense subset if and only if D is r-dense.

Proof:

$\Longrightarrow$ Direct from corollary (4.1.1).
$\Longleftarrow$ Let D be an r-dense subset of X, and U be a non-empty open subset. By lemma (4.1.2), $U = \bigcup_{\alpha \in I} V_\alpha$, where V_α is r-open subset, since $U \neq \emptyset$, then there is $\alpha \in I$ such that $V_\alpha \neq \emptyset$, then $D \cap V_\alpha \neq \emptyset$, from corollary (4.1.2) we get D is r-dense, so $D \cap U \neq \emptyset$, i.e. D is dense.
dense $\overset{\text{regular}}{\Longleftrightarrow}$ r-dense

Definition 4.1.2. A topological space X is said to be r-separable if there exist a countable r-dense subset of X.

Examples 4.1.2.

1) The cofinite topological space X is r-separable, but it is not separable.
2) The discrete topological space on uncountabe is not r-separable.

Corollary 4.1.3. Every separable space is r-separable, but not conversely, see example (4.1.2(1)).

Proof: Let X be a separable space, then X has a countable dense subset, and since every dense subset is r-dense, hence X is a r-separable.
separable $\Longrightarrow$ r-separable

Theorem 4.1.3. In regular, r-separable space is separable.

Proof: Let X be a regular, r-separable space, then X has a countable r-dense subset, so from theorem (4.1.2) we get X is separable.

separable $\overset{\text{regular}}{\Longleftrightarrow}$ r-separable

Theorem 4.1.4. An r-open subspace of r-separable is r-separable.

Proof: Let Y be an r-open subspace of r-separable space X, then X has a countable r-dense subset A, since Y is an r-open subspace, then $Y \cap A$ is r-dense and countable subset in Y, hence Y is r-separable space.

Theorem 4.1.5. If a space X has a r-separable subspace which is dense subset in X, then X is r-separable.

Proof: Let A be an r-separable subspace which is dense subset in X, then A has a countable r-dense subset B, by remark (4.1.(1)), then B is a countable r-dense in X, thus X is r-separable.

Theorem 4.1.6. An r-irresolute image of an r-separable space is r-separable.

Proof: Let F: X ⟶ Y be an r-irresolute function from an r-separable space X , then there exists a countable r-dense subset A of X, so $\overline{A}^{r}$ = X, and F(X) = F($\overline{A}^{r}$) ⊆ $\overline{F(A)}^{r}$, since F is an r-irresolute, then F(A) is a countable r-dense subset of F(X), hence F(X) is r-separable.

4. 2. R-first countable spaces

Definition 4.2.1. Let X be a topological space, and x ∈ X, an r-local basis at x is a colection $\mathfrak{B}_x^*$ of r-open sets containing x such that: for any r-open set V such that x ∈ V there exists B_x^* ∈ $\mathfrak{B}_x^*$, such that $x \in B_x^* \subseteq$ V.

Examples 4.2.1.

1) Let X = {a, b, c} and τ = {∅, X, {a}, {b}, {a, b}, {a, c}}. Then RO(X, τ) = {∅, X, {b}, {a, c}}. Note that $\mathfrak{B}_a$ = {{a}} is a local base at a but not r-local base, while $\mathfrak{B}_a^* = \{\{a, c\}\}$ is an r-local base at a but not local base.
2) In the usual topology on ℝ, the collection $\mathfrak{B}$ = {(-a, a): a ∈ ℕ} is a local base and r-local base at 0.

Definition 4.2.2. A topological space X is said to be r-first countable space if for every x ∈ X there is a countable r-local base $\mathfrak{B}_x^*$ at x.

Examples 4.2.2.

1) The cofinite topological space X is r-first countable, but it is not first countable, since {X} is a countable r-local base for any x ∈ X.
2) The discrete topological space on uncountable X is r-first countable, but it is not r-separable.

Theorem 4.2.1. First countable space is r-first countable, but not conversely, see example (4.1.1(1)).

Proof: Let x ∈ X, and let $\mathfrak{B}_x = \{B_\alpha\}_{\alpha \in I}$ be a countable local base at x. Now consider $\mathfrak{B}_x^* = \{\overline{B_\alpha}^{o}\}_{\alpha \in I}$, clear from lemma (4.1.1) $\mathfrak{B}_x^*$ is a countable collection of r-open sets.
Now we need to prove $B_x^* = \{\overline{B_\alpha}^{o}\}_{\alpha \in I}$ is an r-local base at x. For any r-open set U in X, such that x ∈ U, there is α ∈ I such that $x \in B_\alpha \subseteq$ U (since U is open, and $\mathfrak{B}_x$ is local base at x). Then $x \in B_\alpha \subseteq \overline{B_\alpha} \subseteq \overline{U}$, so $x \in B_\alpha \subseteq \overline{B_\alpha}^{o} \subseteq \overline{U}^{o}$ = U, i.e. for any r-open set U, x ∈ U, there exist $\overline{B_\alpha}^{o} \in \mathfrak{B}_x^*$ such that $x \in \overline{B_\alpha}^{o} \subseteq$ U, we get $\mathfrak{B}_x^*$ is a countable r-local base at x, first countable space ⟹ r-first countable

Theorem 4.2.2. If X is regular space, then any r-local bese is local base.

Proof: Let $\mathfrak{B}_x^*$ be an r-local base at x, and let U be an open set such that $x \in$ U. Since X is regular, and from lemma (4.1.2) then U = $\bigcup_{\alpha \in I} V_\alpha$ where V_α is r-open for any α. Since $x \in$ U then there is α such that $x \in V_\alpha \subseteq$ U, so there is $B_x^* \in \mathfrak{B}_x^*$ such that $x \in B_x^* \subseteq V_\alpha \subseteq$ U; i.e. $\mathfrak{B}_x^*$ is a local base at x.

r-local base $\xrightarrow{\text{regular}}$ local base

Corollary 4.2.1. In regular space, any countable r-local base is countable local base. Moreover, from any countable local base we can construct a countable r-local base.

Proof: Direct, from theorems (4.2.1) and (4.2.2).

Theorem 4.2.3. In regular space, r-first countable space and first countable space are equivalent.

Proof: Direct from corollary (4.2.1).

First countable space $\xleftrightarrow{\text{regular}}$ r-first countable

Theorem 4.2.4. An r-open subspace of r-first countable space is r-first countable.

Proof: Let A be an r-open subspace of an r-first countable space X, then any $x \in$ X has a countable r-local base $\mathfrak{B}_x^*$, hence { $B_x^* \cap$A: $B_x^* \in \mathfrak{B}_x^*$} is a countable r-local base for A, then A is an r-first countable space.

Theorem 4.2.5. Image of r-first countable space under r-irresolute and strongly r-open map is r-first countable.

Proof: Let F be an r-irresolute and strongly r-open map from r-first countable space X, then X has a countable r-local base $\mathfrak{B}_x^*$ at $x \in$ X, we get F($\mathfrak{B}_x^*$) is a countable r-local base at F(x), hence F(X) is an r-first countable space.

4. 3. R-Lindelöf spaces

R-Lindelöf space is a space that satisfy every r-open cover has a countable subcover. This space was introduced by Balasubramanian in 1982 [13] by the name nearly Lindelöf space, when he proved that this space is placed between Lindelöf and weakly Lindelöf spaces. In this section we recall the definition of r-Lindelöf space, also we select some results concerning the charactraization of r-Lindelöf spaces, then we study its relation with the other spaces that we define.

Definition 4.3.1. [13] A topological space X is called r-Lindelöf (nearly Lindelöf) space if every r-open cover (a cover by r-open sets) of X has a countable subcover.

Examples 4.3.1.

1) The space [0, w_1) is r-Lindelöf, but it is not Lindelöf.

2) The discrete topological space on uncountable is r-first countable, but it is not r-Lindelöf.
3) The closed ordinal space $[0, w_1]$ is r-Lindelöf, but it is not r-first countable and r-separable.
4) The sorgenfry plane is r-separable, but it is not r-Lindelöf.

Corollary 4.3.1. [15] Every Lindelöf space is r-Lindelöf, but not conversely, see example (4.3.1(1)).
Lindelöf $\Longrightarrow$ r-Lindelöf

Corollary 4.3.2. [14] Regular r-Lindelöf space is Lindelöf.

Lindelöf $\overset{\text{regular}}{\Longleftrightarrow}$ r-Lindelöf

Theorem 4.3.1. [14] Regular closed subspace of a r-Lindelöf space is r-Lindelöf.

Proposition 4.3.1. [14] Let X be a space and A an open subset of X, then A is r-Lindelöf if and only if it is r-Lindelöf relative to X.

Corollary 4.3.3. [14] A clopen of an r-Lindelöf space is r-Lindelöf.

Theorem 4.3.2. An r-irresolute map of r-Lindelöf is r-Lindelöf.

Proof: Let F: X⟶Y be a r-irresolute map from an r-Lindelöf space X, and let $\{U_\alpha: \alpha \in \Gamma\}$ be an r-open cover for F(X), since F is r-irresolute function and $F(X) \subseteq \bigcup_{\alpha\in\Gamma} U_\alpha$ so $X = \bigcup_{\alpha\in\Gamma} F^{-1}(U_\alpha)$, we have $\{F^{-1}(U_\alpha): \alpha \in \Gamma\}$ is an r-open cover for X, and since X is r-Lindelöf space then there exists countable subcover such that $X = \bigcup_{i=1}^{\infty} F^{-1}(U_{\alpha_i})$ so $F(X) \subseteq \bigcup_{i=1}^{\infty} U_{\alpha_i}$, then $\{U_{\alpha_i}\}_{i=1}^{\infty}$ is a countable subcover for F(X). Hence F(X) is r-Lindelöf space.

4. 4. R-σ-compact spaces

Definition 4.4.1. [3] A space X is called r-compact (nearly compact) space if every r-open cover of X has a finite subcover.

Examples 4.4.1.

1) Let $X = \mathbb{R}$ with $\tau = \{\emptyset\} \cup \{A \subseteq \mathbb{R}: 0 \in A\}$, then (X, τ) is r-compact (since the only r-open sets are $\mathbb{R}$ and $\emptyset$), but it is not compact.
2) Let $X = \mathbb{R}$, with the usual topology, then $\mathbb{R}$ is r-Lindelöf, but it is not r-compact.

Proposition 4.4.1. [3]

1) Every compact space is r-compact, but not conversely, see example (4.4.1(1)).
2) Every r-compact space is r-Lindelöf, but not conversely, see example (4.4.1(2)).
compact $\Longrightarrow$ r-compact $\Longrightarrow$ r-Lindelöf

Theorem 4.4.1. [3]

1) An r-closed subset of compact space is r-compace.

2) Every r-compace subset of T_2 space is r-closed.
3) In any space, the intersection of two r-compact sets is r-compact.

Theorem 4.4.2. Regular r-compact space is compact.

Proof: Let $\mathcal{U}$ be an open cover for regular, r-compact space X, and let U ∈ $\mathcal{U}$, then by lemma (4.1.2) $U = \bigcup_{\alpha \in I} V_\alpha$ where V_α is an r-open set for any α. So $X = \bigcup_{U \in \mathcal{U}} U = \bigcup_{U \in \mathcal{U}} (\bigcup_{\alpha \in I} V_\alpha)$, we obtain $\{V_\alpha\}$ an r-open cover for X, since X is r-compact, there is a countable subcover, say $\{V_{\alpha_i}\}_{i=1}^{n}$ so we can chooes a finite subcover from $\mathcal{U}$, i.e. X is compact.

compact $\overset{\text{regular}}{\Longleftrightarrow}$ r-compact

Definition 4.4.2. A space X is said to be a r-$\boldsymbol{\sigma}$-compact space if it is the union of a countable many r-compact subsets of X.

Examples 4.4.2.

1) The closed ordinal space $[0, w_1]$ is r-$\boldsymbol{\sigma}$-compact, but it is not r-first countableand and r-separable space.
2) The sorgenfery line $\mathbb{R}_s$ is r-separable space, but it is not r-$\boldsymbol{\sigma}$-compact.
3) The discrete topological space on uncountable is r-first countable, but it is not r-$\boldsymbol{\sigma}$-compact.
4) The irrational numbers with usual topology is r-Lindelöf, but it is not r-$\boldsymbol{\sigma}$-compact.

Corollary 4.4.1. Every $\boldsymbol{\sigma}$-compact space is r-$\boldsymbol{\sigma}$-compact, but not conversely, see example (4.4.2(1)).

Proof: Let X be a $\boldsymbol{\sigma}$-compact space, i.e. X is the union of a countable many compact subsets of X, since every compact subset is r-compact, then X is the union of a countable many by r-compact subset of X, thus X is a r-$\boldsymbol{\sigma}$-compact space.
$\boldsymbol{\sigma}$-compact ⟹ r- $\boldsymbol{\sigma}$-compact

Proposition 4.4.2.

1) Every r-$\boldsymbol{\sigma}$-compact space is r-Lindelöf.
2) Every r-compact space is r-$\boldsymbol{\sigma}$-compact.
r-compact ⟹ r-$\boldsymbol{\sigma}$-compact ⟹ r-Lindelöf

Theorem 4.4.3. In locally compact T_2 space, any r-Lindelöf is $\boldsymbol{\sigma}$-compact.

Proof: Let X be an r-Lindelöf locally compact T_2 space, and $x \in X$, since X is locally compact T_2, then there exist an open set U_x, such that $x \in U_x \subseteq \overline{U_x}$ where $\overline{U_x}$ is compact subset in X, then $\{U_x : x \in X\}$ is an open cover for X. Note that: $x \in U_x \subseteq \overline{U_x}^{o} \subseteq \overline{U_x}$, since $\overline{U_x}^{o}$ is r-open by lemma (4.1.1), then $X = \bigcup_{x \in X} U_x = \bigcup_{x \in X} \overline{U_x}^{o} = \bigcup_{x \in X} \overline{U_x}$, since X is r-Lindelöf $\{\overline{U_x}^{o}\}$ is an r-open cover for X, then there exists a countable subcover $\{\overline{U_\alpha}^{o}\}_{\alpha \in I}$ for X, then $X = \bigcup_{\alpha \in I} \overline{U_\alpha}^{o} = \bigcup_{\alpha \in I} \overline{U_\alpha}$ i.e. $X = \bigcup_{\alpha \in I} U_\alpha$, so X is $\boldsymbol{\sigma}$-compact.

r-Lindelöf $\xRightarrow{\text{locally compact+ } T_2}$ $\boldsymbol{\sigma}$-compact

Corollary 4.4.2. In locally compact T_2 space, any r-Lindelöf is r-$\boldsymbol{\sigma}$-compact.

r-Lindelöf $\xRightarrow{\text{locally compact+ } T_2}$ r-$\boldsymbol{\sigma}$-compact

Corollary 4.4.3. In locally compact T_2 space, these stetments are equivalent:

1) Lindelöf space.
2) r-Lindelöf space.
3) $\boldsymbol{\sigma}$-compact space.
4) r-$\boldsymbol{\sigma}$-compact space.

Proof:

1⟹2) Direct, from theorem (4.3.1).
2⟹3) Direct, from theorem (4.4.3).
3⟹4) Direct, from corollary (4.4.1).
4⟹1) Direct, since r-$\boldsymbol{\sigma}$-compact space is r-Lindelöf, from theorem (4.4.3) then r-Lindelöf is $\boldsymbol{\sigma}$-compact, and any $\boldsymbol{\sigma}$-compact is Lindelöf.

Theorem 4.4.4. In regular space, r-$\boldsymbol{\sigma}$-compact space is $\boldsymbol{\sigma}$-compact.

Proof: Let X be a regular, r-$\boldsymbol{\sigma}$-compact space, then $X = \bigcup_{\alpha\in I} \mathcal{U}_\alpha$ where $\mathcal{U}_\alpha$ is r-compact subset for all α, then by theorem (4.4.2) $\mathcal{U}_\alpha$ is compact. Thus X is $\boldsymbol{\sigma}$-compact.

$\boldsymbol{\sigma}$-compact $\xLeftrightarrow{\text{regular}}$ r-$\boldsymbol{\sigma}$-compact

Theorem 4.4.5. An r-closed subspace of r-$\boldsymbol{\sigma}$-compact space is r-$\boldsymbol{\sigma}$-compact.

Proof: Let X be a r-$\boldsymbol{\sigma}$-compact space, and Y be an r-closed subset of X, such that $X = \bigcup_{\alpha=1}^{\infty} U_\alpha$, where U_α are r-compact subsets of X, then $U_\alpha \cap Y$ is r-compact in Y, and $Y = (\bigcup_{\alpha=1}^{\infty} U_\alpha \cap Y)$ is union of a countable subsets of X, thus Y is r-$\boldsymbol{\sigma}$-compact space.

Example 4.4.3. A subspace of an r-$\boldsymbol{\sigma}$-compact space need not be r-$\boldsymbol{\sigma}$-compact, for example the space $[0, w_1]$ is r-$\boldsymbol{\sigma}$-compact, but the subspace $[0, w_1)$ is not r-$\boldsymbol{\sigma}$-compact.

Theorem 4.4.6. An r-irresolute image of r-$\boldsymbol{\sigma}$-compact space is r-$\boldsymbol{\sigma}$-compact.

Proof: Let F: X ⟶ Y be an r-irresolute map from r-$\boldsymbol{\sigma}$-compact space X, then X is union of a countable r-compact subsets as, $X = \bigcup_{i\in I} U_i$, then $F(X) \subseteq F(\bigcup_{i\in I} U_i) = \bigcup_{i\in I} F(U_i)$, so F(X) is r-$\boldsymbol{\sigma}$-compact space
Diagram 2, shows the relations between r-compactness and compactness spaces.

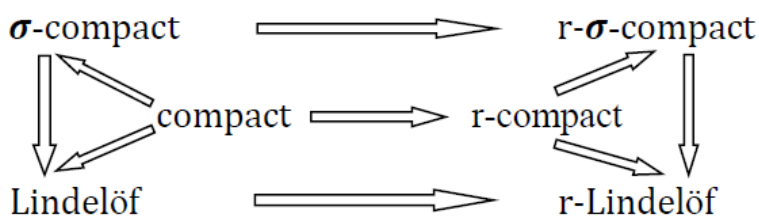

Diagram 2. Relations between r-compactness and compactness spaces.

4. 5. R-second countable spaces

Definition 4.5.1. Let X be a topological space, then the collection $\mathfrak{B}^*$ of r-open sets is called r-base if each r-open set in X can be expressed as a union of members of $\mathfrak{B}^*$.

Examples 4.5.1.

1) Let X = {a, b, c} and τ = {∅, X, {a}, {b}, {a, b}, {a, c}}.then RO(X, τ) = {∅, X, {b}, {a, c}}. Note that $\mathfrak{B}$ = {{a}, {b}, {a, c}} is a base for X but not r-base; while $\mathfrak{B}^*$ = {{b}, {a, c}} is an r-base for X but not base.
2) In the usual topology, the collection {(a, b): a<b, a, b ∈ ℝ} is a base and an r-base for ℝ.

Definition 4.5.2. A topological space X is called r-second countable space if X has a countable r-base.

Examples 4.5.2.

1) The cocountable topological space is r-second countable, but it is not second countable.

2) The irrational numbers with the usual topology is r-second countable, but it is not r-**σ**-compact.
3) The discrete space on uncountable is r-first countable, but it is not r-second countable.
4) The sorgenfry plane is r-separable, but it is not second countable.
5) The closed ordinal space [0, w_1] is r-Lindelöf and r-**σ**-compact, but it is not r-second countable.

Theorem 4.5.1. Every second countable space is r-second countable, but not conversely, see example (4.5.2(1)).

Proof: Let X be a second countable space, then there is a countable base $\mathfrak{B} = \{B_\alpha\}_{\alpha\in I}$. Now we want to prove $\mathfrak{B}^* = \{\overline{B_\alpha}^{o}\}_{\alpha\in I}$ is a countable r-base. From lemma (4.1.1) we have $\overline{B_\alpha}^{o}$ is r-open, now suppose V is r-open set then V is open, i.e. $V = \bigcup_{j\in J} B_{\alpha_j}$ where J ⊆ I, so $B_{\alpha_j} \subseteq \overline{B_{\alpha_j}} \subseteq \overline{V}$ for any α_j then $B_{\alpha_j}{}^{o} \subseteq \overline{B_{\alpha_j}}^{o} \subseteq \overline{V}^{o}$ and since B_{α_j} is open and V is r-open we obtain $B_{\alpha_j} \subseteq \overline{B_{\alpha_j}}^{o} \subseteq V$, then $\bigcup_{j\in J} B_{\alpha_j} \subseteq \bigcup_{j\in J} \overline{B_{\alpha_j}}^{o} \subseteq V$ and since $V = \bigcup_{j\in J} B_{\alpha_j}$, thus $V = \bigcup_{j\in J} \overline{B_{\alpha_j}}^{o}$.
second countable ⟹ r-second countable

Theorem 4.5.2. In regular space, any r-bese is a base.

Proof: Let $\mathfrak{B}^*$ be an r-base for a regular space X, and let U be an open set in X, since X is regular U can be expressed as a union of r-open sets, and since $\mathfrak{B}^*$ is an r-base and each r-open set can be expressed as a union of elements in $\mathfrak{B}^*$, so U is a union of some elements of $\mathfrak{B}^*$ which are open sets, i.e. $\mathfrak{B}^*$ is a base for X.

r-base $\xrightarrow{\text{regular}}$ base

Corollary 4.5.1. In regular space, any countable r-base is countable base. Moreover, from any countable base we can construct a countable r-base.

Proof: Direct, from theorems (4.5.1) and (4.5.2).

Proposition 4.5.1. Every r-second countable space is r-first countable, r-separable, and r-Lindelöf.

Theorem 4.5.3. In locally compact T_2 space: any r-second countable space is $\boldsymbol{\sigma}$-compact.

r-second countable $\xrightarrow{\text{locally compact}+T_2}$ $\boldsymbol{\sigma}$-compact

Remark 4.5.1. In locally compact T_2 space: any r-second countable space is r-$\boldsymbol{\sigma}$-compact.

Theorem 4.5.4. In regular space, the spaces r-second countable and second countable are equivalent.

Proof: Direct froom corollary (4.5.1).

second countable $\overset{\text{regular}}{\Longleftrightarrow}$ r-second countable

Theorem 4.5.5. An r-open subspace of an r-second countable space is r-second countable.

Proof: Let A be an r-open subspace of an r-second countable space X, then X hase a countable r-base $\mathfrak{B}^*$ for r-open subsets, since A is an r-open in a space X, i.e. $\mathfrak{B}_A{}^* = \{ \beta^* \cap A : \beta^* \in \mathfrak{B}^* \}$ is a countable r-base in A, then A is an r-second countable space.

Theorem 4.5.6. An r-irresolute strongly r-open image of r-second countable space is r-second countable.

Proof: Let F: X $\longrightarrow$ Y be an r-irresolute and strongly r-open map from an r-second countable space X, then X has a countable r-base $\mathfrak{B}^*$, since F is an strongly r-open and r-irresolute, hence $\mathfrak{B}^*$is a countable r-base for F(X), hence F(X) is an r-second countable space.
The implication of r-countability axioms among themselves is shown in Diagram 3.

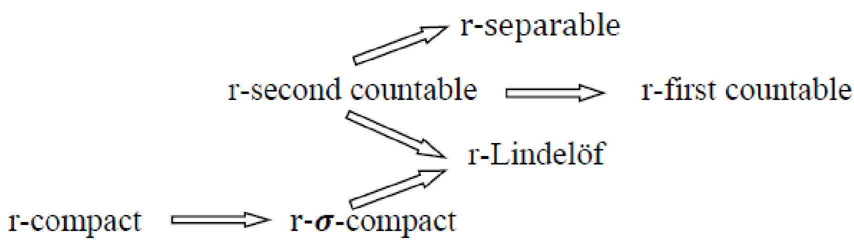

Diagram 3. Relations between r-countability axioms.

5. CONCLUSIONS

In this paper we introduce a generalization of the countability axioms, namely r-countability axioms by using regular open sets. This class of axioms includes; r-separable space, r-first countable space, r-Lindelöf space, r-σ-compact space and r-second countable space. We study the characharization of these spaces and how they relate to the classical countability axioms.

Here we summarize our results:

A) R-Countability axioms are lower than countability axioms.

B) The implication of r-countability axioms among themselves and with the classical countability axioms are investigated, and shown in the following Diagram 4:

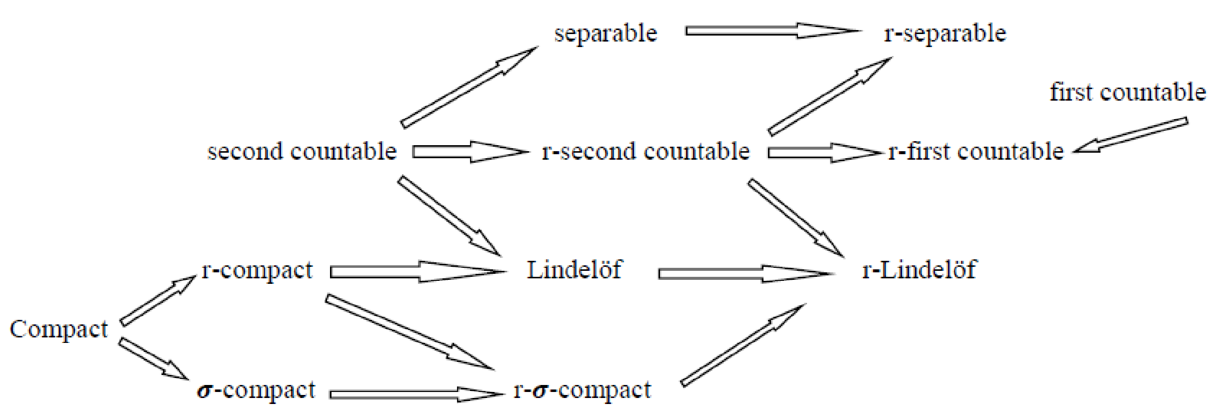

Diagram 4. Relations between r-countability axioms and countability axioms.

C) R-Irresolute map preserves r- separable and r-σ-compact spaces, while r-irresolute and strongly r-open map preserves r-first countable and r-second countable spaces.

D) Regular open subspace of first countable (r-separable, r-second countable) space is r-first countable (r-separable, r-second countable). While regular closed subspace of r-σ-compact space is r-σ-compact.

E) In locally compact T_2 space, these spaces are all equivalent:

Lindelöf space, r-Lindelöf space, $\boldsymbol{\sigma}$-compact space and r-$\boldsymbol{\sigma}$-compact space.

F) In the regular space, these statments are hold:

(1) Every open set can be expressed as a unuin of r-open sets.
(2) Every r-local base at a point in a space is local base.
(3) Every r-base of a space is base.
(4) R-Countability axioms and countability axioms are equivalent. Moreover, r-compact and compact spaces are also equivalent.

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