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Quaternionic operational infinity

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ABSTRACT

By analogy to the complex analytic representation of infinity that I have already proposed in my prior paper, it is demonstrated here by examples that a hypercomplex representation of infinity can also be conjectured and formulated in a formwise similar way. Unlike the two-dimensional (2D) complex infinity, which holds in 2D spaces equipped with an orthogonal homogeneous algebraic basis, the 4D quaternionic infinity requires an algebraic or quasigeometric basis that would be either heterogenous if established within a single space or, if it would be established upon a quasigeometric multispatial structure then at least two homogeneous bases are needed for the latter structure. In the latter case orthogonality and isometry are preserved. The multispatiality not only conforms to the experimentally confirmed reality of the quantum Cheshire Cat whose grin was located in a separate beam path from the beam path of the cat but also implies the feasibility of prospective mathematical reformulation of the basically complex (and thus essentially 2D) quantum mechanics in terms of certain hypercomplex 4D quasispatial structures under auspices of the multispatial reality paradigm.

***Keywords*:** Quaternionic descending infinity, division by zero, multiplication by infinity

1. INTRODUCTION

My previous investigations of some unbiased physical experiments, whose results have been not only undeniably surprising but formerly quite unanticipated, led me to question and then also challenge the propriety of the traditionally unspoken single space reality (SSR) paradigm. It is presumably because the SSR paradigm was so selfevident that it was unspoken

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and thus uncontested. Inappropriateness aside, the fact that I have explained and reconciled those surprising experiments (some partly and few others fully) led me to believe that the SSR paradigm is conceptually unacceptable and thus must be replaced with a certain multispatial reality (MSR) paradigm. The MSR paradigm originated from my new synthetic approach to mathematical and mathematized sciences, which demands that to each operational procedure there must exist a geometric or quasigeometric structure that fits the procedure and vice versa.

The synthetic approach ensures that the structure that fits the operational procedure is constructible and thus realistic. It precludes postulating existence of nonexistent structures.

However, this particular mathematical paradigm shift requires modifications to several formerly unquestioned mathematical concepts – such as the operational notion of infinity and the unrestricted division be zero that is implemented as multiplication by infinity – as well as certain conceptual, operational and structural enhancements to many physical theories, some of which I have already offered in many of my previously published papers. The operation of quaternionic division by zero has already been discussed and exemplified in rudimentary terms in [1].

This paper addresses primarily the purely operational aspects of the idea of quaternionic descending infinity based upon totally unrestricted division by real zero. Any references to geometric issues are made solely for the sake of explanation of the operational topics and thus can be viewed as constituting only a conceptual background.

2. THE RADITIONAL AND THE SO-CALLED NOVEL APPROACH TO QUATERNIONIC DIFFERENTIALS IN THE SSR FRAMEWORK

Although the domain/set $\mathbb{H}$ of 4D quaternions emerged as an algebraic extension of the 2D complex numbers domain $\mathbb{C}$, and therefore some traditional authors tried to define and impose the analyticity/holomorphicity that is theoretically essential in the complex domain $\mathbb{C}$ also for functions defined in hypercomplex and, in particular, in the quaternionic domain $\mathbb{H}$. But the desired analyticity could not really (or perhaps not always) be effectively enforced for compound functions whose components are defined in both $\mathbb{R}$ and $\mathbb{H}$, in general.

The practical need to use differentials of combinations of different types of functions: one that belongs to the real domain $\mathbb{R}$ and another that is defined in the 4D quaternionic domain $\mathbb{H}$, spurred the creation of the so-called HR calculus with some new differential operators [2] and then the novel or generalized HR (GHR) calculus – see [3] or [4]. As Dennis Morris remarked, each type of space has its own differential operators [5]. The same is true for certain spatial or quasispatial and multispatial structures [6]. Even some mathematical laws and operational rules need enhancements for such structures. That is the main reason why I have proposed new product differentiation rule for such multispatial structures as comprise paired dual reciprocal spaces [7].

Traditional quaternionic differential operators have been briefly reviewed in [8]. Other concise summary of traditional quaternion algebra and calculus was offered by David Eberly [9]. However, in the traditional SSR setting quaternion is simply viewed as an [algebraic] operator which changes one vector into another [10]. Note that Clifford constructed his algebra as a direct product of quaternion algebras [11]. In the MSR framework, however, the quaternion is envisioned as an entity that supports transition from 3D spaces equipped with homogeneous bases to 4D spatial structures equipped with heterogeneous bases.

Application of complex quaternions (biquaternions) and biquats (i.e. biquaternions with zero imaginary part) to new formulation of classical electromagnetic fields has been briefly discussed by Imaeda in [12]. Quaternions and antiquaternions are discussed in the context of electromagnetic fields in [13]. Recall that he quaternion basis (1,j,k,l) is not unique [14] and the quaternion group is of order 8 [15]. Furthermore, quaternion algebra is an associative corpus, but not commutative and the neutral element 1 is foreign for it does not belong to the 3D space with basis (j,k,l) [16].

The facts prompted me to split the 4D algebraic basis and consider it as heterogeneous basis spanned over dual reciprocal structure whose 3D spaces are equipped with their own homogeneous algebraic bases. This approach conforms to the quaternionic principle of duality of abstract points and planes by using the same vector symbol to denote either point or plane [17]. Thus, the vectorial 3D slice Rj + Rk + Rl = Im H of the 4D quaternion space constitutes imaginary vector subspace [18].

Since Frobenius' theorem states that the only division algebras over the real field are $\mathbb{R}$, $\mathbb{C}$ and the Hamilton quaternions $\mathbb{H}$ [19], I shall refrain for the time being from survey of octonions. But it may be of interest to some readers to compare the 4D quaternionic structure in $\mathbb{R}^4$ to the 8D octonionic structure in $\mathbb{R}^8$ in [20]. Since the MSR paradigm is relying on differential aspects of operational, not algebraic issues of composition, I am not going into that topic for it assumes prohibition of division by zero as binding, which I do not agree with. A short review of some aspects of composition algebras in the traditional SSR setting is presented in [21]. Recall that the MSR paradigm demands unrestricted division by zero as prerogative for doing no-nonsense investigation of both differential and purely algebraic operations. Note that prohibition of division by zero is just unwarranted impediment to truly realistic mathematical reasonings. It virtually imposes the SSR paradigm and thus effectively precluded the MSR paradigm from even being mentioned as feasible an option.

Nevertheless, traditional mathematics assumed the SSR paradigm as indisputable by default. It seemed so selfevident that usually it was not even mentioned and thus the former set-theoretical approach reigned supreme and quite uncontested. Therefore, no matter how sophisticated or detailed formulas for differential operators have been devised, they could never capture other than purely radial effects of the basically radial force fields. However, I have discovered mathematically that purely radial/center-bound force fields generate also some other than radial (i.e. nonradial) effects that could be designated as either tangential or binormal or both [22, 23], in the parlance of differential geometry.

This fact should have been known for over 170 years now since the works of Frenet and Serret, yet it was not only ignored but routinely suppressed by some unduly influential scientists. Even nonradial nuances found and recorded in unbiased global experiments [24-27] – such as East-West asymmetries found in time accumulated by flown atomic clocks, which I have investigated – could not be completely formulated with the use of traditional mathematical methods, not to mention reconciled, because no accepted mathematical formula existed into which the recorded inconvenient data might be plugged in.

Yet I have reconciled the formerly unexplained data (some fully and other only partly) with the use of special mathematical methods devised by analogy to methods successfully used in other nonstandard applications [24, 23, 27, 28]. Those experimentally justified new methods hinted at the necessity of paradigm shift, from the SSR to the MSR paradigm, which involves operational infinity too.

3. COMPLEX FORMULA FOR UNRESTRICTED DIVISION BY REAL ZERO FULFILLED VIA MULTIPLICATION BY INFINITY IS INCOMPLETE

Having already proposed complex formula for unrestricted division by real zero as

$$N_{n\uparrow}^{n\cdot\infty} \coloneqq n \cdot \infty = \frac{\mathrm{n}}{0} \eqqcolon N_{n\uparrow}^{\frac{n}{0}} \Rightarrow \sum_0^n \{[[\int_1^\infty tv'([x],t)dt] \,|\, ȸ \oplus \hat{\imath} \left[\int_0^1 \left[\frac{1}{v'([x],t)dt}\right]\right] \,|\, \frac{\mathrm{ȹ}}{\mathrm{i}} \} \qquad (1)$$

that is implemented as multiplication by the neverending real infinity in [29] I can proceed now to extend the 2D complex formula (1) to the conjectured 4D quaternionic form by analogy. The term on the right-hand side (RHS) of (1) is the prospective imaginary yield which attains to the infinitesimal part that corresponds to the descending infinity – compare also [1, 30, 31], and references therein.

The compound nondenominated algebraic operator $\hat{\imath}|\frac{ȹ}{\mathrm{i}}$ was chosen on purpose for the sake of simplicity, as it preserves the inverse algebraic basis ȹ, so that the integral that stands by the – nondescript, at this stage of development of the present topic – algebraic operator $\hat{\imath}|\frac{ȹ}{\mathrm{i}}$ which yields the integral's magnitude taken in the direction of the imaginary unit vector **i** that corresponds to the imaginary algebraic unit operator î as well as to the imaginary geometric unit operator $\vec{\imath}$, of course, so that in terms of values we have $|\hat{\imath}| = |\vec{\imath}| = |\mathrm{i}| = \sqrt{(-1)}$. If I would have chosen a denominated operator instead, as scalar product of the imaginary units, for instance, then our discussion here would require also some introduction to the spaces in which the operators operate on the functions involved therein, which is definitely structural feat (i.e. geometric or perhaps quasigeometric, not just an operational/procedural or algebraic feature); the latter shall be further elaborated elsewhere.

Since 2D complex domain is neither algebraically nor geometrically complete, its abstract features would not quite satisfactorily reflect on real life situations encountered during most realistic macrophysical experiments. Thus, it is imperative to move into 4D quaternions which are more suitable for depicting physical reality even though only algebraically at present. However, this expansion calls for at least preliminary delineation of persistent mishandling of dimensionality within the postulative realm of traditional mathematics. For although abstract mathematics seems happy with its various ideas of dimensionality formed in the SSR setting, formerly unanticipated results of certain physical experiments seemingly put in question both the SSR paradigm as well as some of its consequences.

Most disturbing to me is the tacitly concealed fact that although behind the scenes mathematicians are aware of theoretical controversies of the 4D spacetime, which seemingly require two of such abstract 4D structures [32], they kept on suppressing any attempts to show how this could be done, presumably because the paper [33] was causing embarrassment for scientists adhering to the traditional ways of doing mathematics with under sacrosanct SSR paradigm, whose hidden presence they also denied. Since the quaternion group is of order 8 [15] it is not surprising that two 4D structures might fit into it.

I have already demonstrated that the notion of 4D structure of spacetime requires also the presence of a certain 4D timespace structure of timespace [33, 34]. These two structures are necessary not only for the sake of geometry but also for acknowledgement of some slightly inconvenient – and thus tacitly (but persistently) suppressed in the past – yet mathematically unquestionable theoretical achievements of Abel, Galois, and Lagrange [33]. J.-A. Serret, of the Frenet-Serret fame, also demonstrated the impossibility of general algebraic solutions of equations of degree greater than four [35], which fact also imposes limit on dimensionality of spatial structures capped at 4^{th} [33, 34]. Although I can imagine that not all mathematicians have mastered every mathematical topic they were supposed to learn in college, I do not understand how anyone with Ph.D. degree (or its equivalent) in a mathematical discipline can tacitly stifle publications revealing the disregarded as inconvenient though undeniably correct achievements of such world-class mathematicians as Abel, Galois, Lagrange or Serret, for instance, unless they have some other than mathematical hidden agendas. We just cannot afford to dismiss consequences of their inconvenient yet proven theorems.

Since the infamous prohibition of division by zero seemingly decreed in order to prevent clear operational thinking in mathematics did not stop the division to be eventually implemented, so also the still ongoing though veiled suppression of some previously unanticipated structural consequences of those previously disregarded achievements of Abel, Galois, Lagrange, Frenet and Serret, to name just a few outstanding mathematicians, is unlikely to succeed in demeaning of the realistic mathematics that is corroborated by unbiased physical experiments. Although nobody in their right mind dares to contest their valid mathematical achievements, the most straightforward and direct consequences of their unassailable achievements are routinely silenced or tacitly suppressed. Yet due to the tacit suppression of mathematical truths some most inconvenient physical experiments remained unreconciled, because these openly contradicted the nonsenses that are taught to unsuspected students by the unduly influential but visibly embarrassed scientists.

4. SOME CONCERNS IN REGARD TO EXTENDING THE COMPLEXIFIED 2D INFINITY TO ITS 4D QUATERNIONIC REPRESENTATION

Since the finegrained geometric differential operator GDiff operates simultaneously on both sides of the formula (1) and its equivalent forms – see [36], the procedure of operational pairing seems formally independent of dimensionality of the spaces to be paired. In other words, although the pairing involves abstract spaces, each of which is equipped with an orthogonal homogeneous basis native to the given space, the spatial representations of objects remain unrestricted insofar as the algebraic pairing procedure is concerned.

If the latter conclusion is true, then perhaps the multispatiality of representations that is enshrined in the MSR paradigm transcends the common notion of geometry that Felix Klein related to particular group of transformations in his Erlangener program [37] p.7, [38]. He too realized that the number of dimensions appears as something secondary [37] p.16. In that context the abstract concept of structure – as it relates to realistic mathematical existence – became of paramount theoretical importance [39]. Nevertheless, the bigger theoretical problem is the impossibility of existence that is primarily due to impossibility of meaningful construction of the spaces or geometric structures or geometrical objects immersed within the spaces. By 'realistic' I mean here truly nonpostulative construction. Everything that relies on group-theoretical specifications is a kind of local or internal thing specific to the given single space and thus fits the SSR paradigm. The MSR paradigm transcends the group-theoretical classification of geometries treated as different approaches to geometric issues. In other words, while set-theoretical methods and mappings are indeed helpful for dealing with permutations of elements of a single set considered in the SSR framework, in the MSR setting – at the present stage of its development – some multispatial structures are not always quite unambiguously comparable, and therefore mappings between some structural elements may not be easy to determine. This state of affairs could change as the present understanding of the multispatiality in terms of the MSR paradigm matures.

This issue of structural nonexistence is two pronged: either an object cannot exist because the given spatial structure cannot accommodate its unambiguous actual representation or the prospective spatial structure whose presence the object virtually requires for its true existence, does not exist. For not everything that can be written down (and whose possibility of existence is then usually postulated) can actually exist in the given particular mathematical reality. That is why I proposed the new synthetic approach to mathematics that demands matching abstract structures to operational procedures and vice versa, because there is no point in postulating existence of structures on which it is impossible to meaningfully operate without breaking necessary operational rules or without circumventing some quite legitimately imposed structural laws. Nevertheless, sometimes one can see that if a certain – heretofore unknown – structure (or unrecognized yet but not impossible to imagine operation) would be present, then the alleged impossibility of existence could be overcome. This kind of issues transcends topics handled in conventional geometries. Therefore, this structural transcendence would require the attendance of a sort of quite realistic universal pangeometry, the idea of which was already entertained by Lobachevsky.

He recognized the fact that not everything we know could be derived from then-accepted axiomatics of usual Euclidean geometry when he remarked, for instance, that pangeometry demonstrates that the claim about sum of three angles of a straightlinear triangle in the usual geometry does not necessarily follow from our concepts of space [40]. It is known that manifolds of zero and constant positive curvature admit embeddings in the Euclidean space under which any intrinsic motion of the ambient space (embeddings with maximal mobility) is admissible. However, it is impossible to embed the Lobachevsky plane with maximal mobility in a finite-dimensional Euclidean space even in class C^0 [41]. Furthermore, there are some global issues too. Norden remarked that logical equivalence of the Lobachevsky and the Euclidean geometry does not answer the question of the relation of them to the real world [42]. Thus, I see the pangeometry that Lobachevsky proposed as an umbrella for various particular geometries. Thus, pangeometry should be designed to handle all the global issues which none of the particular geometries is prepared to tackle. To me, such a pangeometry is just a tool, not a goal.

Structurally, the formula (1) can be viewed as a certain pangeometric extension of the imaginary geometry [43]. Otherwise, it could happen just as Lobachevsky suspected, namely that most probably the [presumed as being primitive] Euclid's assumptions might remain unproven forever [44]. In Lobachevsky's view, what attested to the consistency of the geometry that he discovered was that its trigonometric formulas are obtained from the corresponding formulas of spherical geometry by multiplying the sides of a triangle by an imaginary unit [45]. While noneuclidean geometries complement the Euclidean geometry, the former do not only fill out the conceptual hole of the Euclidean geometry but exhibit also the necessity to generalize some geometrical notions of the Euclidean one.

According to Lobachevsky, the properties of objectively existing real space are expressed in sciences by physical and geometric notions, which stand in unbreakable mutual relation with each other [46]. He too challenged the old Kantian doctrine of space as a subjective intuition [47]. Although the Lobachevsky's hyperbolic geometry is conceptually different than the Euclidean one, it can be applied directly to numerous physical theories. The proper geometry of rotating disk is a special Lobachevsky geometry [48] and the heated-plane geometry is exactly that of Lobatschevsky [49]. If "velocity space" is considered as Lobachevsky space, then the relativistic velocity addition theorem for velocities coincides with the vector addition theorem in Lobachevsky geometry [50]. On the 3D pseudoeuclidean space with metric $ds^2 = dt^2 - dx^2 - dy^2$ the Lobachevsky plane is realized in the large as an analytic surface $t^2 - x^2 - y^2$ with t>0, which is the superimposed 3D Euclidean space, that is the same affine space but with different metric, namely $ds^2 = dt^2 + dx^2 + dy^2$ which is one half of an ordinary hyperboloid of revolution of two sheets [51].

Elie Cartan pointed out similarity of points of Riemannian space with those of complex noneuclidean space and ordered pairs in 3D real Lobachevsky space [52]. Furthermore, double numbers (a,b) = a + be where $e^2 = 1$, are applied in Lobachevsky's geometry [53].

Another peculiarity is that in hyperbolic Lobachevsky's geometry similar figures do not exist. A triangle is completely determined by its three angles [54] p.18. In the Euclidean geometry, to determine a segment, it is necessary to give some other segment and indicate the geometric construction with the aid of which the first segment can be obtained from the second. In the Lobachevsky geometry it suffices to indicate only the geometric construction [54] p.18f. If the Lobachevsky geometry holds in real space then the unit of length can be given by means of some geometric construction – in this case the space itself determines some unit of length by means of its geometric properties [54] p.19. Since through each point not on a line b there pass at least two lines not meeting b [55], it was a hint as to distinctive – and perhaps fundamental – characteristics of noneuclidean geometries.

Introducing the concept of quadratic geometry, Poincaré has proved in 1887 that if the fundamental surface is a two-sheeted hyperboloid, then the quadratic geometry does not differ from the Lobachevsky's geometry [56]. Distance as bilinear form was discussed in the context of Lobachevsky's geometry in [57]. According to Cayley-Klein, the hyperbolic geometry is identical – or perhaps can be identified, I would say – with a metric, which for infinitely far elements is given by a surface of second order, which could be considered as spherical – compare [58]. So it should not perhaps come as a surprise that each line of a hyperbolic plane has two points at infinity [59]. Since quaternions represent the operation of compounding rotations in ordinary Euclidean space [60], therefore it is clear that to every point of elliptic space one can associate infinitely many quaternions [61], because there is no compelling limit on compounding rotations about the points. The latter conclusion hints indirectly also at the feasibility of building up multispatial algebraic operational structures under the auspices of the MSR paradigm.

5. A WAY OF OVERCOMING THE AFOREMENTIONED CONCERNS VIA PAIRING OF DUAL RECIPROCAL SPACES

Among geometric characterization of complete surfaces Willmore asserts that every geodesic can be prolonged indefinitely in either direction, or else it forms a closed curve [62] p.133. It is the inddefinite extension, of course, that could pose a problem for the pangeometry as I envisaged it. Moreover, surfaces of negative curvature are the signifier of Lobachevsky geometry. Furthermore, by the proven Hilbert theorem we know that a complete analytic surface, free from singularities, with constant negative Gaussian curvature, cannot exist in 3D Euclidean space [62] p.137. On the other hand, extending dimensionality of a single space would require us to equip the 4D spatial structure with heterogenous geometric basis (such as that of 4D spacetime) and thus negate the chief benefit of preserving orthogonality through maintaining homogeneous basis. The other option that would avoid the latter dilemma is to deploy a quasigeometric multispatial strucure comprising two 3D Euclidean spaces or perhaps some quasieuclidean spaces, each of which should be equipped with their native distinct 3D homogeneous orthogonal basis.

Since in this paper I am talking mainly about purely operational (though not yet structural at this time) notion of infinity, the algebraic (though not yet geometric) bases are of chief importance. Given the fact that both the primary algebraic basis ȸ and the inverse/reciprocal basis ȹ are assumed as homogeneous, it seems to me that one of simplest ways to overcome the concerns listed above would be applying the method of pairing of dual reciprocal spaces or spatial structures, in general.

6. QUATERNIONIC EXTENSION OF THE UNRESTRICTED DIVISION BY REAL ZERO IMPLEMENTED VIA MULTIPLICATION BY INFINITY

The complex 2D formula (1) with the imaginary algebraic operator $\hat{\imath}$ seems to imply also the possibility of its operational (and perhaps structural) extension onto 4D quaternions so that the division by real zero $N_{n\uparrow}^{n\cdot\infty} \coloneqq n\cdot\infty = \frac{\mathrm{n}}{0} \eqqcolon N_{n\uparrow}^{\frac{n}{0}}$ that is implemented as multiplication by real infinity would evaluate to the following expression in the 4D domain of quaternions

$$N_{n\uparrow}^{n\cdot\infty} \equiv N_{n\uparrow}^{\frac{n}{0}} \Longrightarrow \sum_0^n \{[\int_1^\infty tv'([x],t)dt] \mid ȸ \oplus \{\hat{\jmath}+\hat{k}+\hat{l}\} \int_0^1 [\frac{1}{v'([x],t)dt}] \mid \frac{ȹ}{\mathrm{i}}\} \tag{2}$$

where the resulting quaternionic formula (2) is only conjectured here. The compounding interspatial symbol $\oplus$ signifies paring of dual reciprocal spaces.

Since the formula (2) is – at this stage – just a conjecture inferred from the essentially algebraic, complex formula (1), it suffices to say that the inverted integral on the RHS of (2) is projected onto each of the intraspatial imaginary unit operators $\hat{\jmath}, \hat{k}, \hat{l}$, that form the resulting quaternion whose heterogeneous quasigeometric basis is $(1, \hat{\jmath}, \hat{k}, \hat{l})$.

Hence the nondenominated operational components $\hat{\jmath}|\frac{ȹ}{\mathrm{i}}$, $\hat{k}|\frac{ȹ}{\mathrm{i}}$, $\hat{l}|\frac{ȹ}{\mathrm{i}}$, cast the integral's value in the directions of the corresponding to them unit vectors **j**,**k**,**l**, whose value equals to that of $|\vec{\imath}|$, for instance. Further discussion of this topic requires introduction of rules of transition from the inverse 3D homogeneous basis ȹ to the heterogenous 4D quaternionic basis, which shall be discussed elsewhere. The nondenominated status of the quaternionic operators is also needed at this stage because the inverted RHS of the formula (2) belongs in the dual reciprocal space whose primary native algebraic basis is denoted by ȹ.

The main reason for denoting the intraspatial imaginary unit operators $\hat{\jmath}, \hat{k}, \hat{l}$, and the corresponding to them unit vectors **j**,**k**,**l**, is that the regular imaginary unit operator $\hat{\imath}$ shall be reserved for signifying the interspatial imaginary unit operator, which is just a matter of convenience at this time, for the latter is algebraic by convention. In the MSR framework the interspatial imaginary unit operator $\hat{\imath}$ must not be mixed with the intraspatial imaginary unit operators $\hat{\jmath}, \hat{k}, \hat{l}$, in order to avoid confusion.

Moreover, the interspatial imaginary unit operator $\hat{\imath}$ will be handy for physical applications, especially in relativistic physics. Although at first glance this assignment of $\hat{\imath}$ may seem to contradict the statement saying that the scalar part of quaternion can no longer be identified with the imaginary unit [63], the latter assertion is certainly true in the SSR setting but is not necessarily contradictory within the MSR framework if the scalar part belongs in a separate space distinct from the space of quaternions.

Since the interspatial imaginary unit operator $\hat{\imath}$ evaluates to the regular imaginary unit i, the quaternion $q = \alpha + i\mathbf{b}$ is thus a component of the generic element of the geometric algebra called multivector $M = \alpha + \mathbf{a} + i\mathbf{b} + i\beta$ compare [64]. Hence pursuit of geometric aspects of the formula (2) demands first some discussion of vectors and multivectors in terms of the MSR paradigm, which shall be done elsewhere.

7. SIGNIFICANCE OF THE COMPLEX AND QUATERNIONIC INFINITIES

The formula (2) indicates that the neverending ascending infinity on the LHS and the infinitesimal descending infinity on the RHS are acted simultaneously by the sum here as well as by the finegrained geometric differential operator GDiff as it was shown in [36]. By now it should be clear that the twin character of the operational infinity is depicting not only a certain dual reciprocal structure but also prospective multispatial nature of the setvalued infinity and of the physical reality we live in, which shall be further discussed elsewhere.

The significance of multispatial representations and of the MSR paradigm in general is that both support quantum mechanics. It has been demonstrated experimentally that quantum object and its property can travel on different beam paths [65]. Objects and their properties are distinct entities and thus can have different representations. Paraphrasing the quantum Cheshire Cat analogy one can say that while the cat in a Mach-Zehnder-type interferometer is located in one beam path, its grin is located in the other path – see the picture in [65] on p.2. Then, after the two paths are recombined, the cat gets its grin back on its face.

This means that quantum objects and their properties can be split and then simultaneously transported on distinct beam paths – or be represented in different spaces of a multispatial structure – and then recombined. The multispatial objects in complex and quaternionic representations are also acted upon simultaneously by the geometric differential operator GDiff in two distinct paired dual reciprocal spaces each of which is equipped with different basis. The conceptual analogy suggests that the 4D hypercomplex mathematics built upon the MSR paradigm can be applied to issues of quantum entanglement in support of quantum nonlocality because it quite naturally clings to the experimentally confirmed tenets of that quantum reality.

8. CONCLUSIONS

It has been shown by examples in analogy to 2D complex infinity that the algebraic concept of an abstract operational infinity can be interpreted also as a 4D quaternionic or 4D quasigeometric entity combining the real neverending ascending infinity with its imaginary reciprocal counterparts spreading in three imaginary quaternionic dimensions, which correspond to the infinitesimal descending infinity. The twin character of operational infinity was not postulated but has been imposed by proven operational rules of differential calculus.

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