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Flawed fundamentals of tensor calculus

Jakub Czajko

Science/Mathematics Education Department, Southern University and A&M College, Baton Rouge, LA 70813, USA

E-mail address: sunswing77@netscape.net

ABSTRACT

Mathematically illegitimate operational fundamentals of tensor calculus are exposed and shown what they do. Yet tensorial methods are still helpful for handling purely radial physical phenomena, such as those encountered in general relativity, but only in local setting. Tensorial approach is unsatisfactory when applied to partly radial or other than radial (i.e. nonradial) phenomena. But for large-scale phenomena, both local and cosmic, even some radial predictions based on tensorial methods blatantly disagreed with quite unbiased experimental and observed facts. Tensor calculus is fundamentally flawed and had already generated numerous faulty conclusions because it was conceptually compromised, which was done presumably in order to fit the inadequate, as being quite arbitrarily preconceived and espoused by default, formerly unspoken and thus never really challenged, mathematical paradigm of single space reality. This assertion pertains to the abstract operational mathematical reality and to physical reality too. Due to those (operational, conceptual, and even structural/geometrical) problems, tensor calculus remains operationally incomplete and thus faulty.

***Keywords*:** Flawed tensor fundamentals, operational incompleteness

1. INTRODUCTION

Traditional mathematics was developed under the formerly unspoken – and therefore unquestioned – single space reality (SSR) paradigm, which is essentially a concept rooted in set theory. The multispatial reality (MSR) paradigm was unknown until I have introduced it

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fairly recently, not because I particularly like it, but because some surprising and formerly quite unanticipated experimental results compelled me to espouse it, for I was able to explain and reconcile some of the previously unreconciled experimental results with the help of the mathematical methods that I have invented based upon the new MSR paradigm and its mathematical consequences. Nevertheless, the MSR paradigm faces strong though clearly unreasonable opposition against the ideas upon which it was formed, presumably because the paradigm exposed several embarrassing inconsistencies and, consequently, nonsenses in the traditional ways of doing mathematics. There seem to exist also nonmathematical reasons for that defiant rejection of the abstract mathematical multispatial reality, which also underlies the physical reality we live in. This venue shall be further discussed elsewhere.

In the SSR setting, wherein even spaces of physical significance are often identified with just sets and thus treated as mere selections from the single space that allegedly underlies the mathematical reality, even rigorous derivations can be easily compromised and even some unrealistic mathematical topics presented as if they were legitimately derived. The faulty derivation is then touted as proof of validity of the theorems involved in the scientific topic. Tensor calculus (TC) is just one example of such compromised topics, which are routinely presented as valid also beyond the proper scope of their partial validity. I do not want to be the judge of whether the compromise was just an unintended accidental flop, which might have resulted from overhyping, or perhaps just an unintentional slap in the face of honest and alert students of differential calculus. I let the reader be the judge.

Emergence of tensor calculus marks not only the resuscitation of a few conceptually backward trends in the development of abstract mathematics but also tacit concealment of some inconvenient, yet otherwise unquestioned on merit, mathematical discoveries.

Nevertheless, it is important to keep in mind that despite its incompleteness, or perhaps because of it, tensor calculus still remains partly valid and thus helpful for describing purely radial local phenomena. The results of a test of general theory of relativity (GTR) with the use of a hydrogen-maser frequency standard in a spacecraft launched nearly vertically upward to 10000 km agree with the GTR predication at the 70×10^{-6} level [1]. That was practically perfect agreement with the prediction of Einstein's GTR, which is purely radial theory of gravitation due to its reliance upon the principle of equivalence. But in other than purely radial cases, experimental results either disagreed with predictions based upon the GTR – see [2], [3], for instance, or hinted at the possibility of disagreement with predictions of the GTR if also other relevant data had been recorded [4], [5]. For not exactly radial phenomena, however, the TC produced either incorrect or quite unacceptable predictions [2].

Since many physical phenomena – such as radial attractive gravitational force or radial Coulomb interactions in electrostatics – happen in or along the radial directions, and as TC was successful in handling purely radial physical phenomena, any attempts at its rectification have been disdained as unsubstantiated and scientific papers trying to clarify the TC were routinely rejected despite undeniably unbiased experimental evidence to the contrary.

Although the proposed corrections did not deprecate the validity of TC for purely radial phenomena, the suggested expansion of the TC also onto nonradial phenomena aroused unreasonable resistance for several reasons. First, scientists did not realize that even purely radial/center-bound force fields create also other than radial physical effects (i.e. nonradial: tangential and/or binormal) whose presence cannot be disregarded anymore [6] when it comes to explaining and reconciling of some global experiments involving atomic clocks, which just cannot be retracted at one's whim [4].

Second, the experimentally confirmed presence of potential energy exchange happening along tangential directions in equipotential plane [7], [3], stirred disbelief, even though Einstein himself admitted that nonradial effects cannot be ruled out, while clarifying his statement that they would be too slight when measured on the earth's surface [8]. Well, not anymore – see [3], [4], for instance. Third, our contemporary mathematics is too obsolete and too entrenched in old ideas already condemned to oblivion, that some influential scientists cannot imagine that many proven theorems are wrong just because they believe in absolute validity of proofs. Yet proofs cannot ascertain validity of theorems but only show that the proven theorems can be derived from preconceived axioms and undefinable primitive notions.

Thus, theorems derived from (or proved based upon) discounted axiomatics or faulty primitives can become invalid. This possibility was yet another mishap that the TC suffered from. Notice that axioms can become invalid not only because these were wrongly conceived, but also because some subsequent paradigm shifts can make them both conceptually and/or operationally obsolete.

2. TENSOR CALCULUS SIMPLIFIED ITS METHOD BY DISREGARDING KNOWN FACTS OF DIFFERENTIAL CALCULUS AND GEOMETRY

Ricci, the inventor of tensor calculus [9], which was previously known as the absolute differential calculus among Italian and German scientists, defined the notions of covariance and contravariance via transformations [10]. As transformations involve certain operations, the reliance on transformations looks like safe bet, provided that the abstract operations upon which the transformations are conceived and implemented are in compliance with all the proven laws and operational rules of mathematics that are applicable to the intended scope of the transformations.

If any of the pertinent operational laws and rules are either incomplete or out of the expected/intended operational scope, then the deficient operations underpinning the transformations can become mathematically illegitimate and consequently thus can make some of the adversely affected transformations either inadmissible or faulty or perhaps both. I want to show in this paper that this is exactly what actually happened to the tensor calculus even though neither mathematicians nor physicists realized that.

Ricci's student and successor, Tullio Levi-Civita, continued development of TC without essential conceptual changes but with clarifications and formalizations, yet in the apodictic tradition of the German mathematics that started dominating Europe in late 19th and early 20th century, presumably because it was easier to grasp and quicker to learn than the more nuanced, abstract and highly sophisticated French mathematics of that time. The adherence to German mathematical tradition rather than to the – more advanced at the peak of 19th century – French mathematics may be the reason for the operational illegitimacy of the TC. For had they really understood the conceptual and operational implications of the Frenet-Serret and Darboux's formulas of differential geometry, not to mention their mindboggling structural consequences, then the whole story of their absolute differential calculus might have been quite different.

This my impression is pretty easy to make in hindsight, but even today the Frenet-Serret formulas are often discounted as just strange though undeniably valid curiosity incidentally discovered at the previously unexpected junction of mathematics and physics.

3. TENSOR CALCULUS WAS FOUNDED UPON OPERATIONALLY INCOMPLETE DIFFERENTIAL EVALUATION OF WORK DONE

Levi-Civita touted transformation by invariance as the operational guiding idea for the most abstract algebraic foundation of tensor calculus [11] p.62. Then he went on to devise also the structural/geometric foundation for the TC upon the allegedly uncontestable algebraic foundation. Such a structural and definitely geometric foundation must rely on vectorial differential calculus, of course, but at the center of the algebraic operational guiding idea should be a certain vector-dependent scalarlike entity that could also be transformed as if quite independently of the unbreakable strictly geometric vectorial transformational laws. Although nothing is absolutely unbreakable, purely algebraic transformational laws are easier to circumvent than structural laws of differential geometry, because usually it is fairly easy to see when the transformed structure is broken.

In addition to the operational guiding idea he introduced also the – corresponding to it – quasi-structural guiding idea, for he very confidently wrote: "The most suitable criterion to take as a guide in making our choice [of an invariance-preserving transformation] is found by introducing, alongside the given vector, a scalar quantity with a physical significance which is transformed by invariance. In this case we take two infinitely near points whose co-ordinates differ by dx, dy, dz; then the work of the force whose components are X, Y, Z, in passing from one of these points to the other, will be

$$dW = Xdx + Ydy + Zdz \tag{1}$$

this scalar quantity has a physical significance which is invariant, and it can therefore be concretely determined." – see [11] p.63. Then he proceeds to rewrite the eq. (1) (which is marked as eq. (2) in the 4th chapter of the quoted book [11] p.63) from Cartesian to curvilinear coordinates. The omitted multiplication sign is, of course, the scalar multiplication of vectors, as the text indicates, but his omission to mention that is not the only cause of the mathematical problem that is tacitly neglected and effectively circumvented in the quoted reasoning.

The eq. (1) is just one example of misleadingly worded deception among numerous other – presumably inadvertently – distorted reasonings found in the – previously unchallenged – traditional ways of doing mathematics. The subtlety of coordinate representation is used superbly, presumably to (inadvertently or not) conceal the mathematical trickery of the allegedly rigorous yet actually rather irresponsible reasonings that apparently try to defend the never mentioned nor disclosed SSR paradigm, which is indefensible from the outset. Note that the mathematical trickery is just a noble attempt to circumvent the confines imposed on rational thinking by the treacherously deceptive, formerly unspoken SSR paradigm. It is very commonly used oversimplification, even today. Yet absurd consequences of the mathematical trickery to be shown below, lead to mathematically illegitimate reasonings which constitute either an omission or an intentional yet tacitly obscured intellectual fraud.

Nevertheless, if the clever split of the vector of force **F**(X,Y,Z) and of the pointing radius vector **r**(x,y,z) within the conventional rectangular/Cartesian coordinate system (x,y,z) is set aside just for a moment, then an abstract generalization of the eq. (1) would simply read

$$dW = \mathbf{F}(X,Y,Z) \bullet d\mathbf{r}(x,y,z) \;\Rightarrow\; dW = \mathbf{F} \bullet d\mathbf{r} \tag{2}$$

which does comply with flawed traditional definition of the differential dW of work done W that is rendered here in a more concealed mathematical notation though. Now we can plainly see the conceptual absurd cunningly concealed within the operational fundamentals of tensor calculus. For $dW=\mathbf{F} \bullet d\mathbf{r}$ is not really an operationally complete differential of work done, but only a radial part of it ensuing from the virtually constant force **F** acting on (or carried over) a changing distance d**r**. Notice that the formula (2) is just a conceptual prescription for calculating the differential dW that is local by definition and thus the operational pattern that it embodies must be always respected, no matter in what kind of coordinate system it is going to be implemented, whether in a rectangular system or a curvilinear one.

The generalized formula (2) is still just as deceptive (or masterfully crafted, if you will, to conceal what Levi-Civita apparently did not want us to perceive) as the original formula (1), but its obvious conceptual arrogance is on display here for anyone to see, provided the reader is both awake and alert and has finished attending at least the first semester of the differential calculus course, which ends with the standard product differentiation rule (PDR)

$$\{ (uv)' = uv'+vu' \} \Leftrightarrow \{ d(uv) = udv + vdu \} \tag{3}$$

for derivative and in differential notation, respectively, of two representationswise compatible scalar functions u() and v() see [12-24], or any other college level calculus textbook. In the SSR framework the formulas (3) hold for either scalar or vector functions, provided they are representable in the given space under consideration. Recall that the PDR has been proved even in the context of the formerly unspoken SSR paradigm [21] and validity of the proof was not questioned on merit.

Furthermore, integrating the PDR leads to integration by parts [25], whose applications supplied numerous verifiable examples indicating thus indirectly that the PDR is indeed – in numerous cases involving composition of variationwise compatible scalarvalued functions – the operationally right method of performing their product's differentiation.

If representations of the functions involved would not be quite compatible then the PDR is still valid even though its formal scope must be expanded onto certain multispatial structures (such as mutually dual paired reciprocal spaces) [26], but that is another story. The conceptual validity of the PDR for the operationally compatible scalar functions was never questioned on merit. The PDR is proven in the context of the formerly unspoken SSR paradigm [21], even though the operational validity of the PDR is shown as being expandable under the MSR paradigm [26], which demands (operationally and structurally) different treatment of incompatibly varying scalar functions [27, 28]. The MSR did not really invalidate the PDR but merely revealed some of its previously unknown additional features. The MSR showed that there is yet another way of looking at mathematics responsibly beside the traditional way of doing mathematics, which previously generated faulty yet rarely (if ever) contested physics and even absurdly inapplicable to reality applied mathematics.

The rewritten formula (2) implies that the work is being done by an unchanging, hence constant, force vector **F** over a changing distance pointed to by the differential d**r** of the radius vector **r**. Therefore, the formula (2) cannot be reasonably applied even to a sled. Only a few scientists, most notably Andre Mercier, explicitly contested formulas like (2) in [29] and [30], but he apparently could not find any resolution of the lingering issue.

If the differential dW(), which is the rate of change of the work done function W() of the given radial/center-bound gravitational force field, is defined as scalar product of the vectors involved: the radial attractive gravitational force vector **F** acting over the radial distance vector **r**, then – according to the mathematically legitimate and rigorously proven [21] product differentiation rule (PDR) – the differential dW() of the work done function W() should read

$$dW(\mathbf{F},\mathbf{r}) \coloneqq d(\mathbf{F}\bullet\mathbf{r}) = \mathbf{F}\bullet d\mathbf{r} + \mathbf{r}\bullet d\mathbf{F} \qquad (4)$$

which clearly shows that the actual rendering of the fundamental formula (2) of TC that was cleverly disguised as the conspicuously good looking yet mathematically counterfeit formula (1), is operationally incomplete. Hence, the original fundamental formula (1) of TC is flawed and its uncovered version (2) too is operationally illegitimate from the standpoint of realistic/uncompromised physics. The highlighted in yellow and displayed in red lettering term **r**•d**F** standing on the right-hand side (RHS) of the operationally correct pattern formula (4) is obviously missing in (2) and (1), and thus tacitly disregarded in the TC.

The pattern formula (4) does not require an advanced training in mathematics to be understood. Even undergraduates can comprehend it even if they do not really understand its necessity. Note that understanding is higher conceptual attainment than mere comprehension. Any undergraduate college student should be able to grasp the obvious operational validity of formula (4) and thus recognize the fundamental flaw of TC.

The operational legitimacy of the pattern formula (4) is ensured by the PDR as well as by the conceptual definition of work done: W = **F•r** as the scalar product of the vector of force **F** over (the vector of) radial distance pointing vector **r**, which definition is accepted in both: applied mathematics and physics. Consequently, the differential dW() – as the rate of change – of the work done $dW(\mathbf{F},\mathbf{r}) \coloneqq d(\mathbf{F}\bullet\mathbf{r}) = \mathbf{F}\bullet d\mathbf{r} + \mathbf{r}\bullet d\mathbf{F}$ and could only be reduced to **F•**d**r** if d**F** = 0, which means if the force **F** would remain unchanged (in magnitude and angle) over the radial distance **r**. Yet in radial/center-bound force fields the magnitude of force is unchanged only on equipotential surfaces, not in radial directions. This fact implies that the TC was established upon operationally illegitimate foundation. Although in some cases the formula (4) could be approximated by the formula (2) for practical reasons, conceptually speaking the TC is virtually rigged to disregard other than radial (i.e. nonradial) effects (NE) of radial/center-bound force fields, which NE are inevitable even in purely radial force fields. The nonradial effects depend on density of matter [7] in close vicinity of the massive body that generates the locally dominant gravitational field [3] and can generate repulsive forces [31]. Besides, one may always approximate a complete and legitimately obtained formula, not a tacitly maimed formula like (1) or (2). The latter case and its often reason-defying thoughtless defense imply a deliberate rigging bordering on tacitly veiled intellectual fraud.

Since the creators of the highly acclaimed yet often inconsiderately defended TC were sophisticated mathematicians, the operationally illegitimate foundations of TC might have suggested their (whether inadvertent or not, yet) tacitly perpetrated intellectual fraud, which is still thoughtlessly perpetuated in academic textbooks and lectures.

I am not criticizing Levi-Civita or Ricci for what they did not do. They probably did their best and I can certainly appreciate that and thus admire their acceptable contributions to mathematics. But the fact that we still perpetuate conceptually deceptive – and clearly not quite legitimate mathematically, not to say operationally illegitimate – reasonings, suggests to me that there may exist (heretofore overlooked) systemic problems with some afficionados of the venerable tensor calculus, which can sometimes be helpful, especially in purely radial local cases, but not always so, and may not be entirely useful even in some purely radial cases, as it is commonly misinterpreted. One just cannot establish a method based on obviously flawed foundations and then claim its unrestricted validity. Note that the Levi-Civita's book [11] was contracted by David Hilbert, known for his fondness for unfounded foundations of geometry, and – as it is common for contractors – Levi-Civita may have felt compelled to rush through his book without paying attention to correctness of his derivations.

Tensor calculus virtually restricts reasonings to purely radial ones, but it was thoughtlessly advertised as absolutely (or universally) valid. Nobody should be criticized for being wrong. But the tacit suppression of serious attempts aimed at revealing its flaws and rectification of the conceptual and operational faults that I have found in tensor calculus is bewildering.

Some authors apparently abstracted themselves away from the actual reality and even created separate domain for covariant vectors and another for contravariant vectors [32] as if these were separate entities rather than just different representations of vectors. Others prefer to keep their feet firmly on the earth and call the rule of [differential] transformations the contravariant scheme for vector components and the covariant scheme for covector's components while admitting that the covectors are just 1-forms [33]. But for most authors the words 'covariant' and 'contravariant' are related to the manner in which the components [of vectors] transform under a change of coordinate system [34].

The other problematic issue of the conceptually misconceived tensor calculus has already been announced by Morse & Feshbach. They wrote: "… the new [transformed] components do not necessarily maintain their dimensionality [in the sense of the dimensional units in which they are designated to be compared and measured] when transforming from [one] coordinate to [another] coordinate. For instance, in spherical coordinates, if [vector] **F** has the dimensions of length, [the covariant] components F_r, F_θ, F_φ, still have the dimensions of length after being transformed whereas [the contravariant components] f^θ and f^φ are dimensionless and f_θ and f_φ have the dimensions of area." [35]. This is not just an issue of assigning dimensional units, but that of the kind of variability and, consequently, of representations of the components, which nobody seems to care about because they only handled radial phenomena while pretending to tackle any phenomena within curvilinear coordinate systems. Functional variability is even more important than algebraic representation, both of which were conceptually screwed in the past, even if it was done quite inadvertently. Henceforth I will call this routinely disregarded and nowadays almost entirely forgotten controversial issue: Morse-trap. Nevertheless, the traditional presentations of Morse-trapped tensor calculus are not concerned with such mundane nuances and thus keep on breeding further generations of tacitly concealed mathematical nonsenses which infest abstract reasonings.

Usually such a problem is avoided, and approximations are recommended. R. Creighton Buck wrote: "… the area of the image of a triangle under a linear transformation of 3-space into itself will depend not only upon the area of the original triangle, but also upon its position. Congruent triangles may have images of different area." see [36] p.383. One might have suggested the use of trigonometric functions to circumvent that, but the *ad hoc* suggestion is beside the point here. For Buck also wrote – in reference to the Fresnel integrals that involve $\sin(x^2)$ and $\cos(x^2)$ – that: "… two equally natural choices of the sequence $\{D_n\}$ led to inconsistent results. The key to this strange behavior is the fact that, in this example [involving Fresnel double integral in polar coordinates] the integral of $|f|$ over D is divergent." see [36] p.223. Recall that polar coordinates (r,θ) involve both radius and angle, whose variability is not operationally compatible, which fact is usually ignored. But it could become problematic in nonlinear cases involving multiple integrals. I am not saying that incomplete or defective ideas should not be published. On the contrary. Even defective ideas could be inspirational. It is the tacit suppression of attempts at their rectification that is deplorable.

4. SOME PRESENTATIONS OF WORK DONE (AND POTENTIAL ENERGY) USE CORRECT RULES WHERAS OTHER USE WRONG ONES

When Richard Feynman discussed work done U=PV (where P denotes the pressure and V the volume), for gases he used the correct product differentiation formula in compliance with the PDR rule (3) for he writes it with the correct root expression dU = PdV + VdP see [37] p.39-5. But when he derives work done from kinetic energy [37] pp.13-4f he used only the radial part of the PDR rule for he expressed the work done functional as $W = \int_1^2 \boldsymbol{F} \cdot d\boldsymbol{s}$ along the path **s** as if only the path was varying and the effective force could not; compare similar expression in reference to electric potential [38]. Some other authors pretended that they do a favor to their students by simply stating W = Fs. Nondifferential expressions of products are mathematical farce that is not really conceptually valid, not even for a sled if the snowy surface is not everywhere perfectly flat.

To the best of my knowledge, only Andre Mercier explicitly contested the validity of the mathematically incomplete differential of work done [29], [30], and a few other scientists were also dissatisfied with the nonsensical presentations. Yet they have resolved the issue insofar as I know. The widespread usage of this kind of infantile formulas consequently means that the total energy might not have been reconciled ever, not even in principle. And the allegedly shrunk muonic hydrogen atom gives an informal evidence for that [39].

5. INCLUDING SOME INDIRECT COMPLEMENTARY VARIABLES

It has been shown that differential formulas can be expanded, and their scope of validity extended when some constants or functionals (that is fixed functions/variables, such as those resulting from integration) appearing in accepted formulas are allowed do vary [40]. Hence, we can unravel certain complementary yet previously hidden variables beneath those constants/functionals. The complementary hidden variables (CHV) are complementing the set of variables explicitly present in the expanded formulas. The CHV are not the meant to be the supplementary hidden variables (SHV) that are meant to make quantum mechanics to appear as being deterministic. But of course, the possibility of revealing CHV strongly suggests that in principle, it is not impossible to expose certain set of SHV as well. However, although some of the hidden variables are unavoidable or even indispensable during a more detailed investigation of given physical phenomena, certain CHV are only indirectly involved in the original formulation of the phenomena.

Every earthly year when rays from Taurus A (that is the visible star in the binary star system Taurus) pass close to our Sun on their way to Earth, their frequency is diminished when they are intercepted on Earth, which fact was observed during few days in the earthly Summer near their yearly occultation by our Sun [41]. Since only the potential of the Sun's gravitational field could be blamed for that frequency decrease, the effective potential V that robs the rays – for frequency stands in for energy, which is the common currency exchanged in physical interactions – must depend on several variables and constant/fixed functionals that are immaterial for the gravitational interactions between the Sun and Earth. Yet for practical purposes, the Taurus A is not directly interacting with our solar system and thus beside the Sun's density of matter Q – which should not be confused with the Sun's density of mass – we got yet another variable that was not taken into account before – see [3], [2], [7], and references therein. Moreover, the rays from Taurus A passed our Sun on almost tangential paths, along some equipotential surfaces, on which the usual radial gravitational potential V(r) remains practically unchanged by its definition. Furthermore, an elapsing time functional [t] could be made involved in more general rendition of the interaction. In order to distinguish the actively varying magnitudes from functionals I shall put the latter in brackets as usual.

Although the minuscule effective mass of the – nominally massless – ray that came from Taurus A, which could be derived from the ray's energy, is insignificant in comparison with the mass of our Sun, the very fact that the ray has lost a part of is effective mass via the (observed and recorded) decrease of the ray's frequency that is equated with its energy (and its effective mass as well) means that the ray must have actually interacted with our Sun. The conclusion is inescapable, because there was no other culprit to blame for the frequency decrease beside our Sun, during the ray's passage in vicinity of our Sun. If so, then I must concede that our traditional gravitational theories, all of which were virtually radial by default, are inadequate and must be complemented by a – nonradial by necessity – theory that might have taken into account the little ray's case without defying the radial theories of gravitation. As purely radial theory, Einstein's GTR is unscathed by presence of complementary nonradial effects of gravitational fields. But the underlying it mathematics, the conceptual core of which was adapted from tensor calculus, needs remediation. It does not mean that the TC should be abandoned, but only that its scope of validity must be restricted to purely radial interactions, where its validity has been indisputably confirmed in numerous local, macro scale yet purely radial experiments, see [1], for instance. Recall that Einstein himself admitted the possibility of existence of some other than radial (i.e. tangential and/or binormal) effects of gravity [8]. The theory of nonradial effects complements and props up the GTR.

If we would substitute certain realistic functions F() and r() into the pattern formula (4)

$$W(\mathbf{F},\mathbf{r},[\mathrm{GM}],[\mathrm{Q}],[\mathrm{t}]\ldots)=\int_{r_0}^{r}\mathbf{F}(\mathrm{r},[\mathrm{GM}])\bullet\mathrm{d}\mathbf{r}+\int_{F(r)_0}^{F(r,Q,\ldots)}\mathbf{r}\bullet\mathrm{d}\mathbf{F}(\mathrm{r},[\mathrm{GM}],[\mathrm{Q}],[\mathrm{t}]\ldots)\quad(5)$$

then the filled in pattern could be integrated. M is the fixed mass of our Sun and G is the gravitational constant. The depth of inquiry is up to you to specify – hence the triple dot.

Nevertheless, if the – formerly viewed as scalar – potential V(r) is also a function of time V(r,[t]) just as the work done function, with which the potential function can be identified, the integration would be more complicated than in traditional approach. For if the potential function V depends on time, the Lagrange equation cannot be derived as usual – see [42] for concise explanation. On integration, the result can be very complicated indeed, depending on to what extent – or how deeply – one wants to go. I used to call it the depth of inquiry.

The fact is, however that I was able to explain an reconcile the formerly unexplained and unreconciled experiments with the rays from Taurus A [41] and with the radio waves that (propagated along the Earth's surface) and controlled by atomic clocks [43] – see simplified account in [3]. Szekeres deemed these experiments unreconcilable by the relevant (and universally accepted) contemporary theories of gravitation or electromagnetism [44] in AD 1968. He was quite correct that the experiments could not be handled by those theories of physics, but this does not mean that the theories are wrong. The actual culprit is the flawed mathematical foundation of TC that precluded thinking in the categories outlined above. The TC virtually strangled theoretical analyses of physical phenomena and especially conceptual thinking in both mathematics and physics by tacitly placing invisible restrictions on thinking about nonradial phenomena. It virtually discarded any possible nonradial effects of physical phenomena. Looking at the operationally ready pattern formula (5), one can see that the ill-founded TC is not equipped to handle it. In fact, the requirements for its integration shred to pieces not only the traditional approach to classical physics but former mathematics as well.

6. HOW TO RECTIFY THE ILLEGITIMATE DIFFERENTIAL OPERATIONS UPON WHICH THE FAULTY TENSOR CALCULUS WAS FOUNDED

On further evaluation of the formula (4) that includes also an additional variable that represents the varying angle α of visibility of the trajectory curve of a satellite (or that of an electron in hydrogen atom in Bohr's classical picture) or an airplane, one gets more comprehensive even though still approximate differential $dW(\mathbf{F},\mathbf{r},\alpha)$ of work done function

$$dW() = -\,dW(\mathbf{r}) + 2dW(\alpha) - dW(\mathbf{F}) = -\,F\cos(2\alpha)\,dr + 2Fr\cos(2\alpha)\,d\alpha - r\cos(2\alpha)\,dF \qquad 6)$$

which is valid for small visibility angles, as the differential dW() is seen from the local gravity center when sweeping its perihelion radius – see [45] and references therein. The evaluation is based on ideas that have already been introduced in [6]. As usual, 3D vectors are displayed in bold font and scalar magnitudes/values of the vectors $F=|\mathbf{F}|$ and $r=|\mathbf{r}|$ are real numbers. The highlighted in yellow and displayed in red letters terms that stand on the right-hand side (RHS) of the equations (4) and (6) are missing in most traditional evaluations of the rate of change dW() of the work done function $W(\mathbf{F},\mathbf{r},\alpha)$, which is conventionally equated with the scalar potential energy V(r) of the given field. If the work done function would depend on two separate angles $W(\mathbf{F},\mathbf{r},\alpha_r,\alpha_F)$, one of which is associated with the radius and the other with the force, then the rate of change of work done dW() would be even more complicated than (6), and so on, if other extra variables would have to be included in the function W(). If some variables could not be represented directly within the given space then the new product differentiation rule for MSR, which was proposed in [26], would have to be applied, of course.

The potential energy is spent on the work done by forces of the local, radial gravitational field of force. It is important to realize that the formula (6) is by no means final. It too could be further expanded and then reevaluated with more pertinent variables involved and taken into account, and thus it can yield even more comprehensive expression.

The routinely neglected term dW(α) shows dependence of the potential energy change on the changing planar angle of visibility (which can actually be a solid angle in more detailed theoretical approach) whereas the other usually omitted term $-dW(\mathbf{F})$ shows dependence of the effective potential energy change on the changing (in general) effective force of the local radial gravitational force field. It is easy to see that while the negative energy terms generate (or correspond to) attractive forces, the positive angle-dependent energy term generates nonradial effects corresponding to repulsive forces. Although the static scalar function $V(r)=1/r$ of the purely radial component of potential energy was traditionally treated as depending only on its position within the radial force field (i.e. on the inverse radius 1/r alone), the effective potential energy should – and actually is, according to formerly unanticipated curious results of some unbiased experiments – virtually depend on both dynamical as well as structural magnitudes characterizing the experimental situation. Therefore, rectifying the aforesaid (and other) blunders of the traditional axiomatic mathematics is extremely important for realistic reconciliation of formerly unreconciled data supplied by the previously curious results of some quite unanticipated yet unbiased experiments [45].

It is also easy to see that the formerly neglected additional radial term –dW(**F**) could be perhaps used (after including some pertinent astrophysical data, of course) to imply that the – allegedly missing – mass as well as the – nowhere to be found – dark energy are impromptu declared medications that were supposed to cure essentially mathematical malady with the use of nonexistent physical concepts [45]. Although I am sure that more matter will be discovered, the more of it is found the more mass will appear as missing, because it is not only physics but mathematics that went amiss.

Fritz Zwicky has apparently assumed that since scientists must have already known everything, including calculus, then their inability to grasp mathematical (that is operational and structural) intricacies should be blamed on something physical, and so he invented the "dark" physical concepts. It may be less obvious that the formerly neglected clearly attractive radial term –dW(**F**) in conjunction with the nonradial, essentially repulsive angular term dW(α) could indicate the need for expanding universe, if pertinent astrophysical data is plugged into it. Furthermore, the deeper variability of the angular term should imply accelerated expansion of the physical universe, as both these previously neglected terms show dependence on the density of clustered together matter. I have already demonstrated the latter dependence [2] by reconciling the formerly unreconciled experiments conducted by Sadeh and his teams [41], [43], which I have also explained – see [3] for simple explanation. They have been deemed as unreconcilable by established theories of physics [44] and rightly so, because the formerly developed physics relied on faulty mathematics.

For example, the conjecture that even the usual radial gravitational force field should exhibit also some nonradial effects, which emerged from an experimental hint (namely the alleged impossibility of reconciliation of the East-West discrepancy that was found in Hafele-Keating and Alley experiments with atomic clocks flown around-the-world [4], [46]) was predicted upon metamathematical reasoning [47] and then verified via new formula derived from conceptual reanalysis of two curious Sadeh experiments [2], [3], and then have been verified also theoretically via abstract differential-geometric reasoning [6], and eventually also derived quite analytically within the conceptual framework of newly enhanced potential theory [31]. Einstein acknowledged the possibility of presence of some nonradial effects even though he considered them as undetectable on the Earth back then [8].

Hans Stephani wrote: "Over large spatial regions when the gravitational field is properly included there is no energy balance equation. It is incorrect to regard this as a violation of energy conservation; there exists in general no local covariant quantity 'energy' to which the property of conservation or nonconservation can be ascribed. None of the foundations of physics are thereby destroyed; energy is only a (very important) auxiliary for describing interactions; but the interaction of all parts of the universe is quite essential for the theory of gravitation." – see [48]. Although what he wrote is true, and it is the energy-momentum that should actually be conserved, it is clear to me that since energy is the currency exchanged in all physical interactions, the lack of covariant representation of energy in the field implies the presence of an extrinsic incompleteness of the traditional description of the gravitational field that was devised under auspices of the SSR paradigm. But since energy of the field is an attribute of the space that underlies the field, then the contravariant representation of the missing term 'energy' could be made covariant in a reciprocal space that is dual to the primary space [26]. We should not try to defend and rationalize the indefensible faulty mathematics.

Recall that differentials pertain to local features of space, and thus defending the lack of unambiguously defined local differentials (which issue is also known as the problem of nonlocalization of energy in purely radial theories of gravitation) via global (i.e. integral) features is a lame duck excuse. The issue at stake here is not about global conservation of energy on the large cosmic scale but rather about the tacitly covered up inability of the contemporary theories of physics that disregard other than radial (that is nonradial) interactions generated within purely radial/center-bound force fields to account for potential energy (or to localize it, which means to represent it locally, of course).

I am not blaming tensor calculus for all the operational misconceptions lingering within the conceptual framework of the (unspoken and therefore unchallenged) SSR paradigm, but am aware of the tensor calculus' uncanny "ability" to conceal questionable mathematical operations under its undeserved aura of formerly unchallenged completeness and alleged generality. My work on finegrained differential operators revealed several previously suppressed mathematical (that is both: operational as well as structural) inconsistencies. There is no reason for suppression of ideas that proved able to reconcile some previously unexplained and unreconciled (and even deemed by some scientists as unreconcilable, by well-known and established theories of physics [44]) physical phenomena – see [2], [3], [45] to mention just a few. It is time to speak the truth rather than tacitly conceal wrongdoings.

7. INVARIANTS OF TENSOR CALCULUS RELY ON TRANSFORMATIONS WHICH TOO CAN BE TRANSFORMED AND CEASE TO BE INVARIANT

I have already shown that rules of operations performed on differential operators can change depending on the choice of paradigm and the framework in which the rules are applied [26]. Hence transformations themselves can be transformed away and thus could cease to be invariant even though they may have been invariant under the prior paradigm.

Levi-Civita diverged from the high-flying abstract mathematical approach that was most visible in the then French mathematics, which had culminated back then in the development of the conceptually realistic – though theoretically still not quite complete yet – Frenet-Serret and Darboux formulas of differential geometry.

As I see it, the profound conceptual difference between the apodictic and the realistic approaches to mathematics is clearly perceptible in their choice of invariants, which are driving the operational methods used in all abstract approaches to mathematical analyses of models describing physical phenomena. While the apodictic approach prefers invariance of transformations, which is destined to stifle the methods that rely on the transformations by making them less flexible for the future, the more realistic approaches tend to rely on fixed/functional invariants that leave room for operational enhancements, which are essential for pursuing differentials of yet unknown anticipated prospective functions.

The Frenet-Serret formulas rely upon length-invariance of unit vectors, which can be used to define isometric algebraic or geometric basis, for functionals whose unit length remains invariant by definition are unaffected by change, of necessity, by the length-invariant unit vectors. But the functionals depend on the moving reference frame (i.e. the comoving 3D trihedron whose internal components (i.e. the units) remain invariant/unchanged by the given abstract motion). That method was generalized by E. Cartan. Thus, the functionals of motion depend mainly on the external parameters of motion that characterize also the comoving trihedron, without being functionally dependent also on transformations of the internal parameters, which cannot be changed due to their constant definition. This easy to control and very realistic invariance leaves their various prospective functions fixed. Yet their possible procedural or operational transformations remain fairly flexible. Nevertheless, E. Cartan apparently did not realize that the tensor calculus is operationally flawed, presumably because he worked within the framework of the unspoken SSR paradigm that precluded even questioning the distinction between external and internal functional dependencies back then.

Although E. Cartan distinguished the exterior product, such as $x \wedge y$ and hyped its actual significance, neither he nor his successors realized that it must have also equally significant counterpart. For the counterpart is actually the conceptually "stubborn" reciprocal interior product $x \vee y$, which was already envisaged by Grassmann. It is often called 'meet' product of multivectors, dual to their outer product [49]. Since some authors frequently identified the term 'interior' – when used in refence to products – with the inner scalar product of vectors $\mathbf{x} \bullet \mathbf{y}$ whose real value evaluates to $xy\cos(\mathbf{x},\mathbf{y})$ in analytic geometry, where x,y are the values/magnitudes of the vectors **x**,**y**, respectively, the names can be confusing. Note that giving a name to an object or an operation does not mean gaining complete understanding of its character, even though naming it can make it somewhat more comprehensible than it was before the object has been defined through an operation. Recall that covariant components of a vector can be defined as scalar products with the basis vectors [50] p.7.

Those following Grassmannian tradition usually call the expression $x \vee y$ the regressive product [51] p.13ff, and interpret it as pertaining to intersection; hence the term 'meet' used by such authors. An important clue to the algebraic and geometric dilemma is given by John Browne who remarked that bivectors are located nowhere [51] pp.7,15. He is right on target saying that there is no place within the single space reality that is assumed as the abstract mathematical universe under auspices of the formerly unmentioned and therefore never questioned SSR paradigm. He virtually acknowledged the conceptual dilemma clearly seen in the incompleteness of geometric representation of multivectors even though he recognized the importance of geometric algebra as their operational representation.

Note that it is the realistic unambiguous geometric construction that actually counts, not a postulative existential declaration that "exists" only on paper or in someone's mind. My new synthetic approach to mathematics demands that to every lawful, legitimate operational procedure, there should exist a constructible spatial or quasispatial structure fitting the procedure, and vice versa. For there is no point in postulating existence of structures that cannot be constructed or operated on, nor in proposing operational procedures for which no unambiguously constructible structure can be conceived to operate on objects immersed in the structure. In other words, we must avoid postulating existence of fictions regardless of whether they are impossible to actually construct mathematical or physical objects.

Other authors rationalized the aforesaid geometric dilemma by calling the topic that deals with the exterior product $x \wedge y$ either geometric algebra, or spacetime algebra, or algebra of space, but carefully avoided calling the topic geometry. For in addition to "space reflection", which means reversing the direction of all vectors in space, they recognized also space conjugation [52] p.17ff. Hestenes remarked that although the imaginary unit i commutes with elements of Pauli algebra, it does not commute with the inner and outer products of vectors [52] p.19. Thus, he defined the [generalized] geometric product **ab** of the two vectors **a,b** as

$$\mathbf{ab} = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \wedge \mathbf{b} \tag{7}$$

containing symmetric and asymmetric parts [52] which can be alternatively designated as the coplanar part **a•b** that represents the scalar product of the two vectors and the orthogonal part **a**∧**b** [53] p.98, that is imaginary dual of the cross product of these vectors, which equals to

$$\mathbf{a} \times \mathbf{b} = \mathrm{i}\, \mathbf{a} \wedge \mathbf{b} \tag{8}$$

see [52] p.18; compare also slightly different approach in [53] p.100. The latter two formulas obviously constitute geometric algebra, not really geometry. These are algebraic operational formulas relating geometrical objects, not geometric structures *per se*.

The geometric algebra – also known as Clifford algebra – and tensor calculus also share the implicit assumption that the formerly unspoken SSR paradigm is so selfevident that it was not even mentioned, not to mention questioned on merit.

This aforesaid dilemma is thus restated in less drastic terms and usually just replaced with the rationalized "rounded" statement that we distinguish axial vectors **a**×**b** also representing geometric areas, from the regular polar vectors. There is nothing wrong with representations of geometrical objects in algebraic terms within the setting of the SSR paradigm, of course, but this does not mean that certain other, alternative interpretations should be demeaned and tacitly suppressed. Any meaningful further discussion of axial vectors would have to involve spinors as well, which shall be done elsewhere.

I think that the unnecessary complexity of the problems of space is a consequence of the formerly unspoken SSR paradigm. E. Cartan already realized that there is no place for spinors in Riemannian geometry [50] p.151. He also remarked, in reference to GTR, that classical Riemannian space is without torsion [54] – just as projective space is without torsion [55] – and the space of absolute parallelism is without curvature [54], which to me is a polite reminder that the Riemannian geometry cannot be ever helpful (in its present renditions within the old SRR setting), for realistic mathematical depiction of physically meaningful 3D phenomena. Recall that while curvature is the rate of change of direction of the curve per unit length, torsion is the rate of twisting of the curve per unit length [56].

Note that [abstract] torsion as an asymmetric part of an affine connection was introduced by Cartan [57]. Duality between curvature and torsion is analogous to the duality between electricity and magnetism [58]. But the Riemannian geometry is [defined as] local (differential) whereas Euclidean geometry is global [59]. Nevertheless, one can introduce the notion of torsion into a Riemannian space with nonabsolute parallelism, but it would be difficult to do so in a general case [60]. The space with torsion and curvature is then equivalent to an elastic continuum which has undergone plastic deformations and, following Sakharov idea of the spacetime as an elastic continuum, we are lead to a gravitational constant, which occurs in the Einstein action, as the metrical elasticity of spacetime with the exact value, without introducing any arbitrary cutoff, when also torsion is considered [61].

The classical notions of curvature and torsion, which can be traced to Frenet and their generalization to Darboux and E. Cartan, did not create the problems when they were still treated in 3D. But their attempted more abstract generalizations in the SSR setting did backfire.

The Frenet trihedron (**t,n,b**) moving along the given curve is defined in terms of the tangential, normal and binormal vectors, respectively, and the instantaneous axis of rotation of the trihedron is defined by the Darboux vector: $\mathbf{d} = \tau\,\mathbf{t} + \kappa\,\mathbf{b}$ where κ and τ are the curvature and torsion of the curve, respectively, compare [62]. In the Kobayashi-Nomizu notation a vector field X along a geodesic is called Jacobi field for it satisfies the Jacobi equation with torsion tensor T and curvature tensor K see [63]. When higher order Lagrangian is written in terms of geometric algebra then it leads to spherically symmetric metrics of Schwarzschild-like form but with slightly different term involving curvature and torsion in place of the Schwarzschild metric [64]. Obviously neither geometric algebra by itself, nor classical differential geometry create unsurmountable inconsistencies. It seems to me that once we assume that the framework of the SSR paradigm is quite correct and thus could be further generalized, then we run into various conceptual problems and other pesky theoretical inconsistencies with apparently overwhelming problems. Although extremely important, the issue of torsion is only tangential to our topics and therefore shall be discussed elsewhere.

I am not criticizing scientists for what they did not do, for they certainly did their best. But some authors tacitly suppressed or openly dismissed the ideas that I have codified under the MSR paradigm, even though their dismissal was not really "open" for all to see, because they were hiding as anonymous referees. I do not know John Browne, but I admire people who have the guts to spit the truth, even if it is an inconvenient truth. For he has virtually acknowledged that the operationally unavoidable multivectors are homeless – or spaceless, if you will – under the formerly unspoken arbitrary SSR paradigm, for there was no viable (i.e. unambiguously constructible) geometric or quasigeometric structure for them to dwell in. Note that realistic constructible spaces must not be identified with sets, which are mere selections from set-theoretical assemblages.

I have exposed heretofore tacitly concealed facts and thus challenge the readers to judge the scope of validity of the inherent – though traditionally unspoken and thus operationally inadmissible – restrictions implicitly imposed on tensor calculus, based on the operational flaws unveiled above.

8. TENSORS RELY ON WRONG SPATIAL REPRESENTATIONS

Levi-Civita identified coefficients – yes, multiplicatively inverse reciprocal coefficients: not the quite independently varying variables denoted therein by x, but the coefficients used to evaluate [the representations of] the variables – as being [represented] in either covariant or contravariant format [11] p.67f. The set of contravariant variables, which he identified with point variables, can be transformed into the set of dual variables, which he called covariant variables [11] p,68. He realized that the two sets of variables are not only separate but also quite different [11] p.67f.

Although it is rather irrelevant how he called the variables, the fact that he accurately pinpointed their conceptually diverse nature and quite distinct character is mathematically important indeed. For by making this distinction, they – Ricci and Levi-Civita – proved, at least to me, that their mathematical instincts were quite correct. But because they – just as everyone else back then – worked under the foggy auspices (or the conceptual cloud, if you will) of the (unwarranted yet never challenged) SSR paradigm, they probably felt that they had to somehow compromise their theoretical exposition so that it would still fit the theoretical confines (or the conceptual straitjacket, if you will) of the SSR paradigm, which they inherited and thus likely espoused by default.

Although Levi-Civita did not mention representations explicitly (in the passages quoted or just paraphrased above) it is important to realize that – in reference to the term 'covariance' and 'contravariance' – he should have been talking about representations of objects. For vectors or vector and/or scalar functions are not being inherently covariant or contravariant, but when depicted as abstract geometrical objects they can be represented either in covariant or in a contravariant form. Furthermore, Einstein has once remarked that the terms designating 'covariant' and 'contravariant' representations should be actually reversed, if logical consistency of the concepts would be desired. Nevertheless, this particular traditional naming convention would be quite immaterial, were it not for the fact that it may also be used – presumably quite inadvertently – to tacitly conceal the routinely unmentioned possibility that the abstract mathematical reality (regardless of how it may be eventually defined) could be actually of multispatial nature and thus possess highly stratified character.

It should also be noted that just because the transition from one of the representations into the other is usually defined via a transformation tied to a particular coordinate system for vectors, it is not really about the vectors themselves but rather about their representations. Vectors and tensors are clearly multifaceted operational entities and thus possess multiple covariant derivatives [65].

Attempts at an abstract generalization of tensors, such as to holors, many of whose applications do not need a consideration of coordinate transformation [66], created more abstract language. Yet holors are just repacked tensors in an allegedly more general but still postulative language apparently created without addressing any of the awkward operating issues plaguing the TC. Postulating existence does not solve any existential or operational or structural issues. It only makes the topic murky without even making an effort to address the most debilitating dilemma of conceptual and operational incompleteness of the TC.

Moreover, the common omission of operationally viable algebraic bases, without which truly realistic representations of scalar functions are pointless, indicates that many traditional ways of doing mathematics are deficient. I am not criticizing Ricci or Levi-Civita for what they could not do; they surely did their best and I admire that. But the fact that their countless successors and aficionados of the (often misrepresented or misinterpreted) tensor calculus did not permit to rectify its deficiencies is disturbing, to say the least. They may have a hidden agenda, because they suppressed any critique of their faults, despite of generating numerous nonsenses, which the indefensible SSR paradigm facilitated. The mathematical nonsenses are destructive not only for theoretical mathematics, but especially if the faulty mathematics is applied to physical sciences, wherein some unbiased experimental results are virtually disregarded [3] or their conclusions tacitly falsified [4] both explicitly and implicitly.

Some physical theories have been openly introduced by obvious, conceptually arrogant mathematical fraud, see for instance theories entertaining the so-called missing mass and the – curiously unobserved or perhaps unobservable even in principle – dark matter/energy [45].

In Galilean coordinates Christoffel symbol is zero and [consequently] the covariant differentiation reduces to ordinary differentiation [67] p.240. It is always possible to find a coordinate system in which all the Christoffel symbols become zero at a previously assigned point and such a system is said to be locally inertial or locally-geodesic [67] p.241.

The fact that in 3D cases one can use locally inertial/geodesic coordinate system does not mean defiance of relativistic treatments, as some authors claim. The option to use the classical coordinate system for 3D cases is meant just to indicate that the tensor calculus can be used to conceal certain nontraditional interpretations of other than radial physical phenomena. The fact that nonradial effects of radial force fields have been revealed in formerly unanticipated results of unbiased experiments [2] can be – and still is – suppressed by some authors under the pretense that tensor calculus did not predict them. Nevertheless, Einstein himself remarked in his original paper on general relativity that although other than radial (or nonradial, if you will) effects of the essentially radial gravitational field cannot be dismissed, they would be too slight when measured on the Earth [8]. But much more precise unbiased experiments revealed presence of measurable definitely nonradial effects of the purely radial attractive gravitational force field [2-7, 46].

9. EQUATING WORK DONE WITH EXPENSE OF POTENTIAL ENERGY

Since nothing physical seems to come for free in the physical universe we live in, where the abstract universal currency in mathematics and physics alike is energy, let us consider the dynamically emerging (DE) forces that are generated by changes to the potential energy that is embedded within definitely radial/center-bound force fields (such as the gravitational or electromagnetic fields), which energy is commonly used to pay for the work being done by the forces of the given force field. I have discovered mathematically presence of certain nonradial DE forces [2], during my investigation of experiments with the atomic clocks flown around-the-world by Hafele-Keating [68], [69], and similar experiments conducted by Alley [70], as well as the experiment with atomic clocks carried along the Earth's surface [43] and the observed frequency decrease in rays coming from the Taurus A star [41]. For simple explanation of the latter two see [3] and references therein.

The regular forces act between pointlike masses or pointlike charges and thus can also be viewed as interactions between them. The nonradial DE forces should not be classified directly in terms of radial/normal interactions, because although they are consequences of radial interactions their instantaneous impact is caused by either tangential or binormal interactions. Therefore, the effects of the nonradial DE forces were unknown in the past. Their impact is often easier to perceive as these forces act on third party bodies, such as radio waves or rays coming from distant stars whose frequency is diminished in close vicinity of our Sun (and planets) when the path of the rays traverses the

Sun's gravitational force field tangentially and/or binormally.

Here I am interested only in purely mathematical, mainly operational description of the local gravitational force field, the presumed locality of which necessarily restricts its abstract rendition to only differential terms. The phrase "dynamically emerging" forces is supposed to mean that the effective DE force fields are being created as result of varying differentials (i.e. rates of change) of the given field's potential energy, which is assumed as the primary physically observable yet essentially abstract mathematical entity that actually resides within the given force field. We usually think of regular radial forces (such as the local, radial/center-bound attractive gravitational forces or the electrostatic Coulomb forces) as permanently residing/existing within their respective force fields and being unleashed by the differences in their potentials. But actually, forces emerge when potential energy is changing. Yet the changes can happen also in other than radial directions which suggests that potential energy is actually a vector representable in a distinct spatial structure.

10. CURIOUS DISCREPANCY BETWEEN COSMOLOGICAL MODELS AND ACTUAL OBSERVATIONS

The exposed above systematic fundamental flaw of the TC, namely the operational incompleteness of estimation of potential energy (understood as the price paid for the work done by the given gravitational force field) is common to almost all presently accepted theories of traditional physics. And the illogical consequences resulting from that previously unrecognized flaw show up in many areas of contemporary physical sciences over and over again. Recent comparison of cosmological models with the observations of gravitational lensing supplies few theoretically disturbing conclusions. Meneghetti and collaborators summarized their findings in their words: "The observed cluster [of galaxies] substructures are more efficient lenses than predicted by CDM [cold dark matter] simulations, by more than an order of magnitude. We suggest that systematic issues with simulations or incorrect assumptions about the properties of dark matter could explain the results." - see the Abstract in [71]. How about both, I would say.

They further wrote "The maximum circular velocity is given by $V_{circ} = max\sqrt{\frac{GM(r)}{r}}$ (1) where G is the gravitational constant, M(r) is the galaxy mass profile, and r is the distance from the galaxy center. Figure 4 shows that, in our lens models, observed galaxies have larger circular velocities than their simulated analogs at a fixed mass. This implies that dark-matter subhalos associated with observed galaxies are more compact than theoretically expected. […] Explaining this difference requires the existence of a larger number of compact substructures in the inner regions of simulated clusters. […] Alternatively, the difference could arise from incorrect assumptions about the nature of dark matter." - compare [71] p.3=p.1349. Their paper is extremely important as it points out quite unanticipated discord between observed results and simulated/theoretical outcomes. Notice formwise resemblance of the circular velocity formula to the escape speed from the galaxy cluster, which would be $\sqrt{2}$ bigger [72].

As you can see their formula for circular velocity is purely radial, which is not incorrect, but it shows prejudice. For nonradial effects of purely radial/center bound gravitational force fields generate also some repulsive components [31], which can add an expansion term to the model [of the given galaxy, cluster, or the whole universe, in general], and would thus require creation of slightly different models. Moreover, the nonradial effects imply the need to take into account also density of matter [45], [7], [3], that was formerly called specific gravity, in addition to the density of mass.

Although mass is commonly associated with measure of material substance, this view is not really acceptable. For mass – as it is equated to energy – should be properly viewed as just an inertial attribute of massive bodies and thus related primarily to their motion in space, not to the composition of massive bodies – see [3] for purposely simplified explanations of the routinely ignored yet experimentally confirmed fact.

Newton, and many others, including Galilei before him and most notably Eötvös, and even some modern experimenters repeating the Eötvös experiments concluded that the ratio of gravitational to inertial mass does not differ from one substance to another [73]; compare also [74, 75]. Nevertheless, presence of certain nonradial (i.e. tangential and/or binormal) effects of gravity theorized in [47] and employed in [6], was confirmed in several experiments [2, 3]. Although the nonradial effects are very small indeed, just as Einstein suspected [8], their recorded existence makes the conclusion drawn from Eötvös-like experiments only valid if we admit that the alleged lack of dependence of gravity on material substance is justified only for purely radial phenomena. These are null experiments only when gravity is assumed as being capable of acting only in purely radial directions. For the nonradial or mixed phenomena, however, the density of matter – as a measure of material composition of gravitating bodies – has to be taken into account [3, 7]. Although the nonradial effects of gravity are expressed in terms of energy (or work done, if you will), the spatial differential $\nabla V(r)$ of the local potential $V(r)$ can be interpreted as equal also to a (tangential or binormal) force $F = -\nabla V(r)$, which would be too small to change perceptibly the magnitude of the radial force even though it might tilt it.

When our Sun robbed the ray from Taurus A of its energy that is equated to effective mass via decreasing its frequency, there was no way for the Sun to accomplish the robbery without applying a mix of radial, tangential and binormal forces, which are responsible also for deflection of light. Since frequency shift is seen near collapsing black hole [76] then less dense material bodies should also experience similar effect, even though its impact might be considerably smaller [2]. There are no separate, distinct laws of physics for black holes pertaining to their gravitational collapse even though some outcomes may be slightly different. Nevertheless, gravitational collapse and rotation of stars, galaxies, etc. cannot be realistically depicted by operationally incomplete equations like (1) and (2), which disregard angles in principle as well as ignore proven operational rules of mathematics, but it should be modelled according to pattern formulas like (4) or (6) that are operationally complete and mathematically legitimate, because they have been derived according to proven mathematical rules including the proven product differentiation rule (3) for composition of compatibly varying functions.

Moreover, as spatial differential (i.e. increase or decrease) of the local potential energy, the magnitude F of the effective force $F = -\nabla V(r) = -\sum \partial V(r)$ which either caused or just contributed to the observed (and unambiguously recorded) decrease of frequency/energy of the rays coming from Taurus A, is just an artefact of the changing potential energy. In other words: the primary concept is energy whereas the notion of force is secondary notion as it actually pertains to change of energy. Yet because the anticipated presence of force is easier to observe (by examples of the rearrangement of the objects in space on which the force acts) than the changing shape of the field's potential energy, then the notion of force was assumed as primary by default. But the overriding principle is that only abstract mathematical laws are truly fundamental, and the physical laws are merely comments on the underlying them mathematical laws.

Although some researchers realized that there should exist a fifth force that messes up the allegedly impeccable experimental results, the extra force can be the nonradial gravitational force spreading in other than radial directions, not entirely distinct fifth interaction nor a force that is not included in the (uncompromised) mathematical laws of energy exchanges. The numerous Eötvös-like experiments did not show that there is no other than radial force, but only that no perceptible difference could be found by scientists who apparently had no idea that the extra nonradial complementary contributions to the radial gravitational force should be superposed with (and in terms of magnitudes just added to) the usual radial forces rather than superimposed onto the latter [77].

More precisely: mathematically speaking, force is just an attribute of the local potential energy which the energy acquires when the magnitude of the energy is changing. If so, then potential energy (and the work done paid by it) must actually be a vector that spreads in three quasispatial directions [2], but in a separate space that is associated with the usual space, yet is distinct from the usual length-based space of motion.

That is why I have invented and developed certain abstract multispatial structures of paired dual reciprocal spaces [78] under auspices of the new multispatial reality (MSR) paradigm, where more advanced and finegrained differential operators [27] appear to act simultaneously on both such 3D structures [28]. Since the pairing of dual reciprocal spaces requires absolutely unrestricted division by zero, I have implemented the previously forbidden operation of division by zero via multiplication by the naturally reciprocal (i.e. multiplicative inverse) to zero infinity [79], where the infinity can be operationalized in algebraic (that is real and complex) domain [80] and then complexified in 2D [81], as well as operationalized in 4D quaternionic domain [82], which then leads to quaternionic twin infinity concept [83]. Needless to say, the unwarranted traditional prohibition of division by zero, which was one of the biggest nonsenses ever decreed in mathematics, must be thrown into the trashcan of pathological mathematical reasonings. For if physics virtually demands removal of mathematical nonsenses then mathematics should better do that or just get lost as uncooperative obstacle (on the road to the prospective development of realistic physical theories), while pretending to be scientific.

It is the inconsiderate traditional mathematics that virtually allows us to make thoughtless associations of concepts even quite inadvertently. Due to omission of the qualifier ‘radial’ and on the basis of that misformed conclusion of Galilei (and others who still repeat his unwarranted generalization of his experiments), some referees and editors of scientific journals kept on rejecting efforts to show that that unqualified conclusion is questionable, to say the least. Due to its operational flaw, the TC should be qualified as “radial” if drawing of conclusions is intended. If not qualified as radial, TC should not be permitted to be used for making abstract generalizations or predictions.

11. TROUBLESOME DIFFERENTIATION OF TENSORS AND DIFFERENTIAL FORMS COMPEL SHIFT TO MULTISPATIAL REALITY PARADIGM

Recall that covariant derivatives of covariant and contravariant vectors are tensors [67] p.239. I am quoting this statement in order for the reader to grasp the sentence that follows.

A certain problem discussed by Theodore Frankel “indicated that it does not seem to be possible to differentiate a contravariant vector field to obtain a tensor field.” see [84] p.73. Within the traditional SSR setting an unambiguous differentiation of contravariant vectors is indeed impossible, because the contravariant representation of vector function cannot be represented directly therein. But in the multispatial framework of the MSR paradigm the contravariant vector function too can be differentiated yet in the dual reciprocal space, not in the primary space. This is because the contravariantly represented vectorial function actually is an inverted covariant function provided that it is depicted in the dual reciprocal space [26].

More traditional way of exemplifying interrelations between covariant and contravariant representations of space and surface vectors has been offered in [85] p.212f; and for concise discussion pertaining to kinematic derivatives of vectors see [85] p.140ff.

Elie Cartan has proved remarkable theorem saying that “<< With the geometric sense we have given to the word “spinor” it is impossible to introduce fields of spinors into the classical Riemannian technique; that is, having chosen an arbitrary system of [contravariant] coordinates x^i for the space, it is impossible to represent a spinor by any finite number N whatsoever, of [covariant] components u_α such that the u_α have covariant derivatives … >>”. The theorem was discussed in [50] p.151; see also the preceding page for other views on the topic.

He realized that spinors, from which the notion of vector can be defined, can also be based on vectors [86], and the importance of preserving fundamental form [87]. These issues shall be discussed elsewhere. Einstein pointed out that although are concepts are creations of humans, their justification and value comes from their ability to explain our experiences [88]. That is why the faulty tensor calculus should be rectified and enhanced. Moreover, it should be complemented by a theory of nonradial effects of radial/center-bound force fields.

The above quotations are not intended as a critique of the reported inconsistencies but only to illustrate how many other unnecessary conceptual problems are tied to the faulty operations, which are nonetheless presented as correct. They just show few conceptual and operational limitations of the fundamentally flawed and operationally faulty tensor calculus.

To say the truth, we do not really have unsurmountable problems with the contravariant differentiation. The alleged problems originated from obviously thoughtless acceptance of the unwarranted yet previously unspoken paradigms which traditional mathematics inherited and left uncontested despite their logical inconsistencies and rationalized conceptual nonsenses.

12. CONCLUSIONS

Tensor calculus is a tool well suited for modeling of purely radial physical phenomena happening within radial/center-bound force fields. Nevertheless, the fundamentally flawed assumptions that resulted in its operational incompleteness and made tensor calculus too oversimplified to be useful for predictions when nonradial effects are involved, created also many nonsenses and inadvertently derailed some reasonable reasonings and conclusions.

Both differential geometry and tensor calculus are conceptually incomplete. Yet while the Frenet-Serret formulas of differential geometry are not yet complete due to theoretical obstacles caused by still lacking mathematical evidence for their prospective enhancements, the incompleteness of tensorial approach is due to tacit coverup of operationally incomplete differentiation, even if it was made quite inadvertently. Since all developing theories are incomplete as being work still in progress, such incompleteness is not unusual. Unlike the incompleteness of differntial geometry, however, the incompleteness of tensor calculus is due to mathematically illegitimate fundamentals if not outright intellectually fraudulent.

What is really unusual in the man-made operational incompleteness of tensor calculus is its tacitly veiled misrepresentation of mthematical fundamentals relies on flawed product differentiation operation upon which thr calculus was founded. There is nothing wrong with admitting that a theory is still incomplete. But to pretend that all students are too stupid to see through the concocted deception is not only an insult to their innate intelligence but perhaps also miscalculation of how far – or maybe for how long – the deception could be perpetuated. Yet the ongoing suppresssion of scientific papers that tried to rectify the tensorial approach suggests that afficionados of the faulty tensor calculus have either a hidden agenda or just perpetrate an intellectual fraud, whether unwittingly or not.

The misconceived and very cleverly formulated foundations of tensor calculus provided theoretical smoke screen behind which even outstanding scientists with world-class knowledge of mathematics may have hard time to see its obviously faulty fundamental assumptions, upon which it had been founded and developed, while declaring clearly illegitimate mathematical operations as correct and thus admissible. The fact that the flaws have not been rectified before is – to the best of my knowledge – due to undue influence of some scientists who tacitly suppressed publications of papers which might have exposed the still ongoing perpetuation of intellectual fraud in mathematics and in physical sciences.

Paraphrasing Richard P. Feynman, who once said that even standing clock is right twice per [24 hours long] day, one may say that despite its reliance on faulty operational formula the tensor calculus is still right twice per 360º, but only in the purely radial planar directions i.e. at 0º and 180º. This means, however, that for the other 358 planar angular degrees, the tensor calculus might quite inadvertently generate inaccurate reasonings, for it disregards the descriptions of physical phenomena that happen – either fully or just partly – in some other than purely radial directions within the given radial/center-bound force fields.

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