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Some characterizations of a two-parameter Xgamma distribution

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ABSTRACT

The objective of this paper is establish some new characterization results of a two-parameter Xgamma distribution. We have first established our proposed characterization results by taking a relation between left truncated moment and failure rate function. Then, we have characterized the two-parameter Xgamma distribution by taking a relation between right truncated moment and reversed failure rate function. Finally, we have characterized it by order statistics and record values.

Keywords**:** Characterizations, Order Statistics, Record Values, Truncated Moment, Xgamma distribution

1. INTRODUCTION

Recently, many authors and researchers have studied the problems of characterizations of both discrete and continuous probability distributions. It has been found that, in order to apply a particular probability distribution to some real world data, it is very important to characterize it first subject to certain conditions. The objective of this paper is to characterize a two-parameter Xgamma distribution (TPXG) introduced by Sen et al. [1].

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As pointed out by Nagaraja [2], "A characterization is a certain distributional or statistical property of a statistic or statistics that uniquely determines the associated stochastic model". Since a characterization of a particular probability distribution states that it is the only distribution that satisfies some specified conditions, these characterizations may serve as a basis for parameter estimation, see Glänzel et al. [3] and Glänzel [4, 5]. Moreover, Glänzel [5] points out that the characterizations by truncated moments may also be useful in developing some goodness-of-fit tests of distributions by using data whether they satisfy certain properties given in the characterizations of distributions.

These conditions are used by various authors to test goodness of fit, efficiency of a particular test of hypothesis and the power of a particular estimating, etc. For example, Volkova and Nikitin [6] used a well-known characterization result of Ahsanullah [7] to test exponentiality of a distribution. For an excellent survey of goodness-of-fit and symmetry tests based on the characterization properties of distributions, the interested readers are referred to recent nice papers of Nikitin [8], Miloševic [9] and Akbari [10], and references therein. For various characterization techniques of probability distributions, the interested readers are referred to Nagaraja [2], Galambos and Kotz [11], Kotz and Shanbhag [12], Koudou and Ley [13], Ahsanullah and Shakil [14, 15], Ahsanullah et al. [16-18] and Ahsanullah [19], among others. Thus, motivated by the importance of the characterizations of probability distributions in distribution theory, in this paper, we establish some new characterization results by truncated moments, order statistics and record values of the two-parameter Xgamma distribution (TPXG) of Sen et al. [1].

The organization of this paper is as follows: In Section 2, the two-parameter Xgamma distribution (TPXG) of Sen et al. [1] is provided, along with some of its basic essential distributional properties for the purpose of our proposed characterizations. Based on these distributional properties, we establish our proposed characterization results in Section 3. We provide the conclusions in Section 4.

2. TPXG DISTRIBUTION

For a positive continuous random variable X, Sen et al. [20] introduced a one-parameter lifetime distribution, named as Xgamma distribution, with the probability density function (PDF) as

$$f_X(x) = \begin{cases} \dfrac{\alpha^2}{(\alpha+1)}\left(1+\dfrac{\alpha}{2}x^2\right)e^{-\alpha x}, & (0 < x < \infty, \alpha > 0) \\ 0, & \text{otherwise.} \end{cases},$$

and studied its several interesting structural and survival properties that made it useful in modelling time-to-event data sets. Later, by taking the mixture of exponential and gamma distributions for a positive continuous random variable X, Sen et al. [1] proposed a two-parameter generalization of the above-said Xgamma distribution, known as TPXG distribution, with the probability density function (pdf) given by

$$f_X(x) = \begin{cases} \dfrac{\alpha^2}{(\alpha+\beta)}\left(1+\dfrac{\alpha\beta}{2}x^2\right)e^{-\alpha x}, & (0 < x < \infty, \alpha > 0, \beta > 0) \\ 0, & \text{otherwise.} \end{cases} , \tag{2.1}$$

and studied its applications in modelling time-to-event data sets.

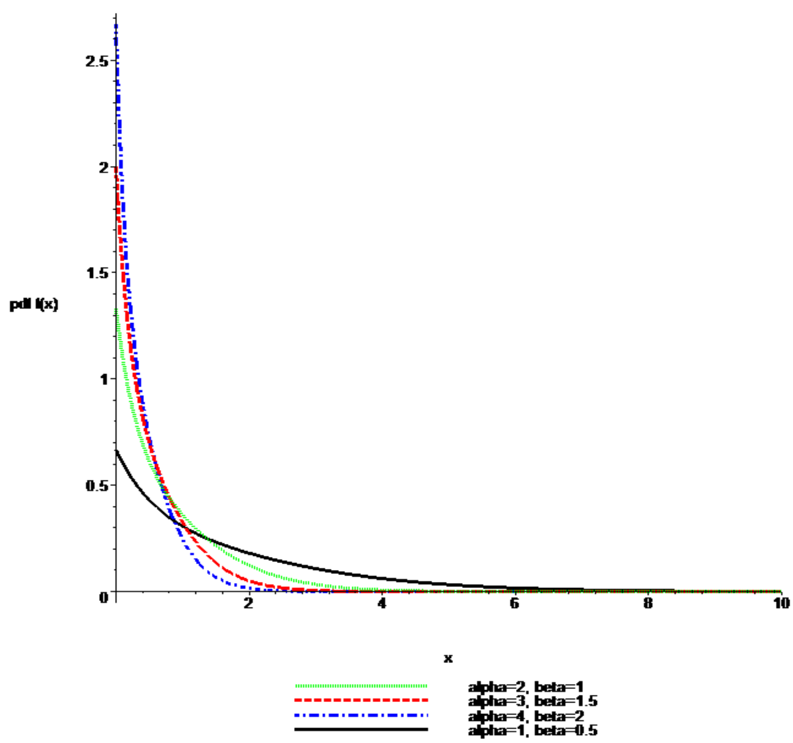
Remark 2.1: For details on TPXG distribution and its applications in modeling real life time-to-event data sets, the interested readers are referred to the paper of Sen et al. [1].

2. 1. Some Basic Distributional Properties

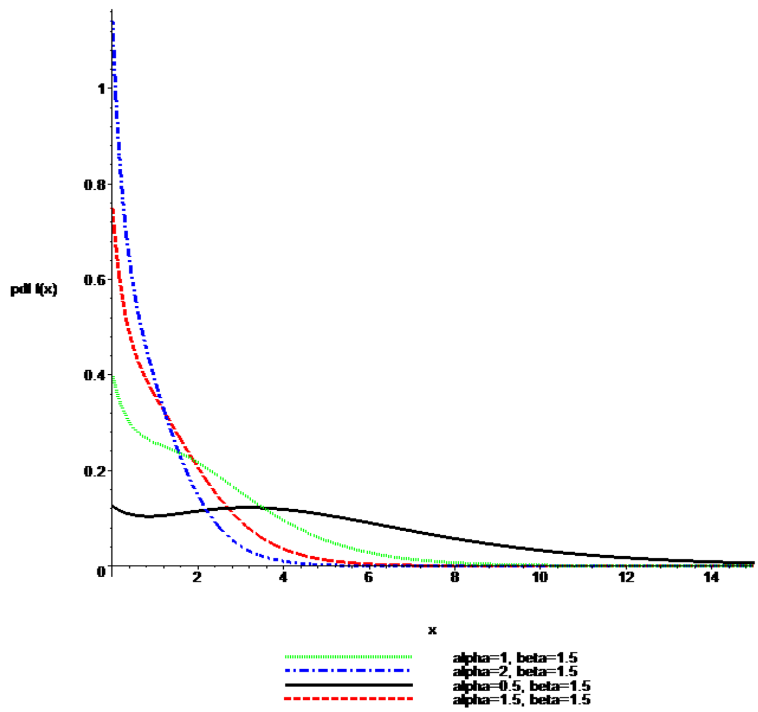
Since the objective of this paper is to characterize the two-parameter Xgamma distribution of Sen et al. [1], in this sub-section, following Sen et al. [1], we will first review and discuss some basic distributional properties of the TPXG distribution, which will be later used in establishing our proposed characterization results.

2. 1. 1. Possible Shapes of the PDF of the TPXG Distribution

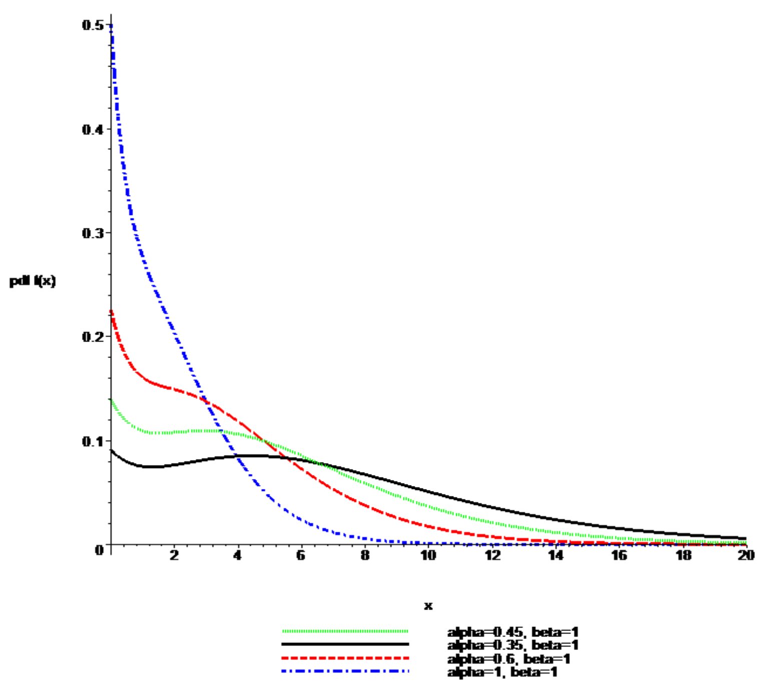
For possible shapes of the pdf (2.1), its graphs are drawn in Figures 2.1 (a – c) for some selected values of the parameters.



(a)



(b)



(c)

Figure 1. Plots of the TPXG pdf (2.1), for:
(a) $\{\alpha = 1, 2, 3, 4;\ \beta = 0.5, 1, 1.5, 2\}$ (left); **(b)** $\{\alpha = 0.5, 1, 1.5, 2;\ \beta = 1.5\}$ (right); and **(c)** $\{\alpha = 0.35, 0.45, 0.6, 1;\ \beta = 1\}$ (bottom).

The effects of the parameters can easily be seen from the above-mentioned graphs. For example, it is clear from these plotted figures that the distributions of the TPXG are positively right skewed with longer and heavier right tails for the selected values of the parameters.

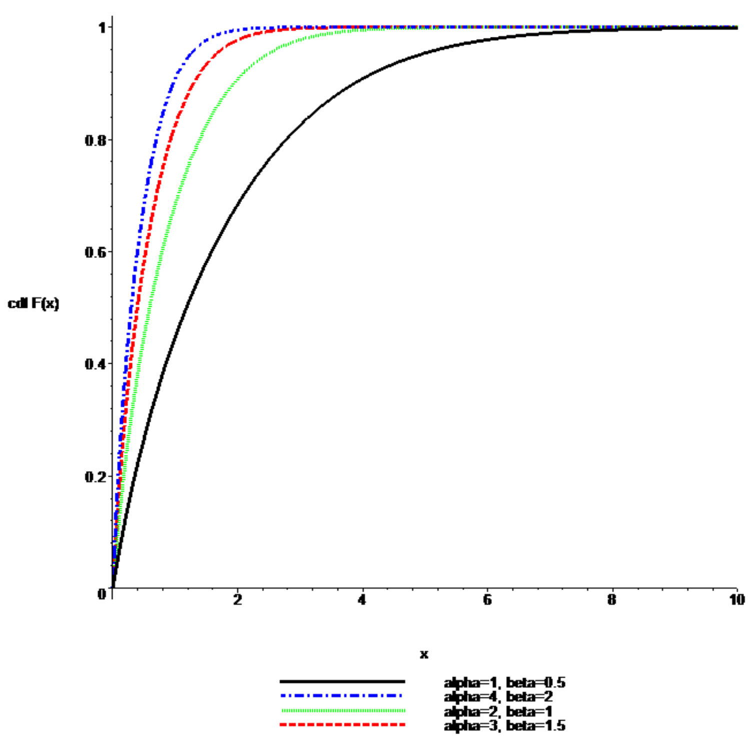
2. 1. 2. Cumulative Distribution Function

The cumulative distribution function (cdf) corresponding to the pdf (2.1) is given by

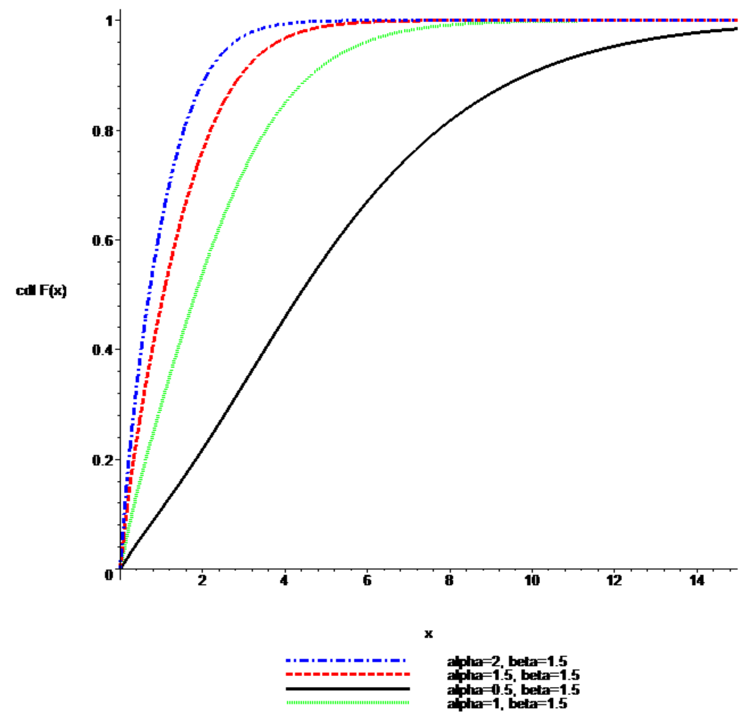
$$F(x) = \int_0^x \left[\frac{\alpha^2}{(\alpha+\beta)} \left(1 + \frac{\alpha\beta}{2} u^2\right) e^{-\alpha u} \right] du \tag{2.2}$$

$$= 1 - \frac{\left(\alpha + \beta + \alpha\beta x + \frac{\alpha^2 \beta}{2} x^2\right)}{(\alpha+\beta)} e^{-\alpha x}, \quad (0 < x < \infty, \alpha > 0, \beta > 0), \tag{2.3}$$

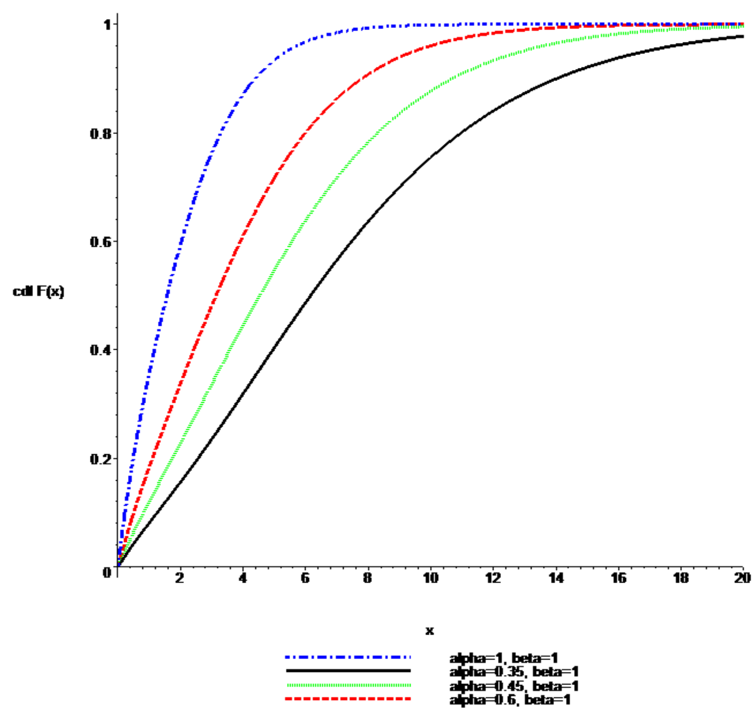
which easily follows by evaluating the above integral on the right side of the Eq. (2.2). It is easily verified, by direct differentiation, that $\frac{dF(x)}{dx} = \frac{\alpha^2}{(\alpha+\beta)}\left(1 + \frac{\alpha\beta}{2} x^2\right) e^{-\alpha x}$, which is the pdf (2.1) under question. For possible shapes of the cdf (2.3), the graphs are drawn in Figures 2 (a – c) for some selected values of the parameters.



(a)



(b)



(c)

Figure 2. Plots of the TPXG cdf (2.3) for:
(a) $\{\alpha = 1, 2, 3, 4;\ \beta = 0.5, 1, 1.5, 2\}$ (left); **(b)** $\{\alpha = 0.5, 1, 1.5, 2;\ \beta = 1.5\}$ (right); and **(c)** $\{\alpha = 0.35, 0.45, 0.6, 1;\ \beta = 1\}$ (bottom).

2. 1. 3. Reliability Analysis

It is well-known that the reliability analysis of lifetime distributions plays important roles in modelling many phenomena in the fields of biological, economics, engineering, physical, and other pure and applied sciences. For a non-repairable population, we define the failure rate as the instantaneous rate of failure for the survivors to time t during the next instant of time. Therefore, in what follows, motivated by the importance of the reliability modelling of real data in the studies of the lifetime distributions, some reliability characteristics of the TPXG distribution are investigated.

Using (2.1) and (2.3), the corresponding survival function, $S(x)$, hazard function (or failure rate function), $h(x)$, and reversed failure rate function, $\eta(x)$, are respectively given as follows:

Survival Function:

$$S(x)=R(x)=1-F_X(x)$$
$$=\frac{\left(\alpha+\beta+\alpha\beta x+\frac{\alpha^2\beta}{2}x^2\right)}{(\alpha+\beta)}e^{-\alpha x},\quad (0<x<\infty, \alpha>0, \beta>0). \qquad (2.4)$$

Hazard Function:

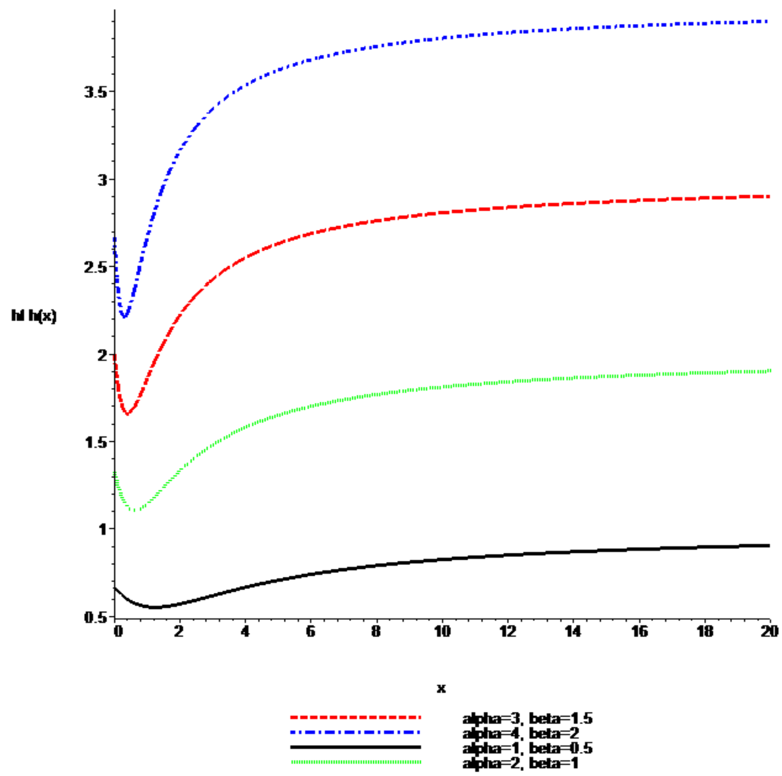
$$h(x)=\frac{f_X(x)}{1-F_X(x)}=\frac{\alpha^2\left(1+\frac{\alpha\beta}{2}x^2\right)}{\left(\alpha+\beta+\alpha\beta x+\frac{\alpha^2\beta}{2}x^2\right)},(0<x<\infty, \alpha>0, \beta>0). \quad (2.5)$$

Reversed Failure Rate Function:

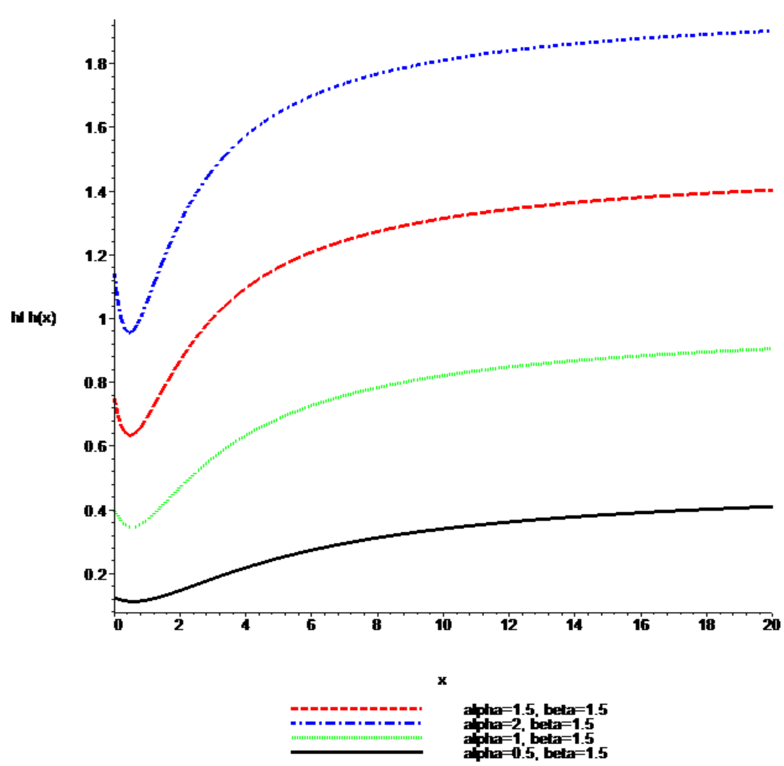
$$\eta(x)=\frac{f_X(x)}{F_X(x)}$$
$$=\frac{\alpha^2\left(1+\frac{\alpha\beta}{2}x^2\right)e^{-\alpha x}}{(\alpha+\beta)-\left(\alpha+\beta+\alpha\beta x+\frac{\alpha^2\beta}{2}x^2\right)e^{-\alpha x}},(0<x<\infty, \alpha>0, \beta>0). \qquad (2.6)$$

Possible Shapes of the Hazard Function of the TPXG Distribution

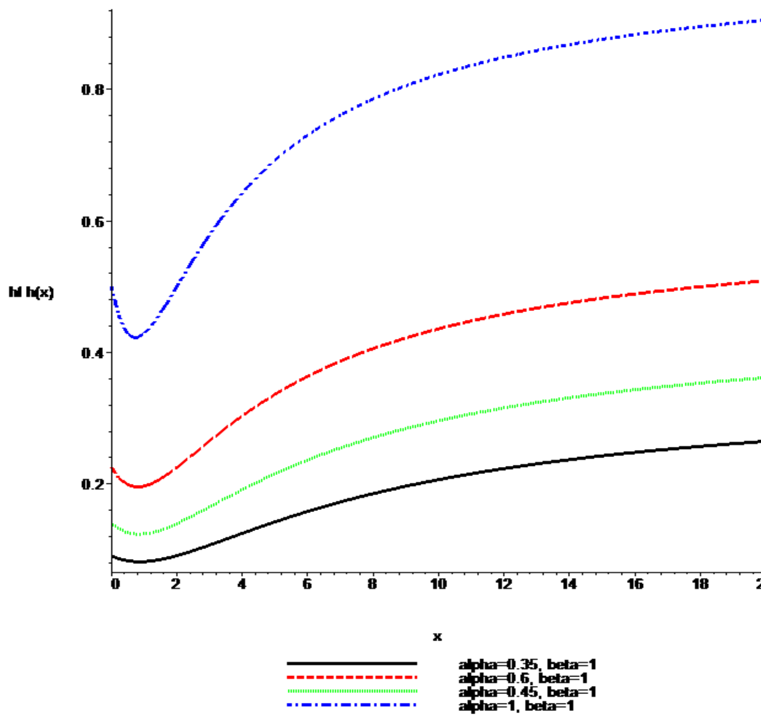
For some special values of the parameters, the graphs of the hazard function (hf) (2.5) are illustrated in Figures 3 (a – c).


hl h(x)
x
alpha=3, beta=1.5
alpha=4, beta=2
alpha=1, beta=0.5
alpha=2, beta=1

(a)


hl h(x)
x
alpha=1.5, beta=1.5
alpha=2, beta=1.5
alpha=1, beta=1.5
alpha=0.5, beta=1.5

(b)



(c)

Figure 3. Plots of the TPXG hf (2.5) for:
(a) $\{\alpha=1,2,3,4;\ \beta=0.5,1,1.5,\ 2\}$ (left); **(b)** $\{\alpha=0.5,1,1.5,\ 2;\ \beta=1.5\}$ (right); and **(c)** $\{\alpha=0.35,0.45,0.6,1;\ \beta=1\}$ (bottom).

The effects of the parameters are obvious from these figures. The decreasing-increasing and bathtub shape (both concave up and concave down) behaviors of the hazard function (hf), $h(x)$, are also evident from these Figures.

Moreover, it is sometimes useful to find the average failure rate function (AFR), over any interval, say, $(0,t)$, that averages the failure rate over the interval, $(0,t)$, see, for example, Barlow and Proschan [21]. Thus, the average failure rate function (AFR) of the TPXG distribution, over the interval $(0,t)$, is given by

$$AFR=\frac{-\ln\left(R(t)\right)}{t}=-\frac{1}{t}\ln\left(\frac{\left(\alpha+\beta+\alpha\beta t+\dfrac{\alpha^2\beta}{2}t^2\right)}{(\alpha+\beta)}e^{-\alpha t}\right),$$

which in view of the expansion of logarithmic function as a power series, is seen to be positive irrespective of the values of the parameters $\{\alpha,\beta\}$. It follows that the TPXG distribution is increasing failure rate on average (IFRA).

Furthermore, a life distribution $F(.)$ is new better than used (NBU) if $R(x+y) \le R(x)R(y), \forall\, x, y \ge 0$, and new worse than used (NWU) if the reversed inequality holds, see, for example, Barlow and Proschan [21]. We note that, for the TPXG distribution from the Eq. (2.4), since

$$R(x+y) = \frac{\left(\alpha+\beta+\alpha\beta(x+y)+\frac{\alpha^2\beta}{2}(x+y)^2\right)}{(\alpha+\beta)} e^{-\alpha(x+y)},$$

and

$$R(x).R(y) = \left[\frac{\left(\alpha+\beta+\alpha\beta x+\frac{\alpha^2\beta}{2}x^2\right)}{(\alpha+\beta)} e^{-\alpha x}\right].\left[\frac{\left(\alpha+\beta+\alpha\beta y+\frac{\alpha^2\beta}{2}y^2\right)}{(\alpha+\beta)} e^{-\alpha y}\right],$$

it is easy see that $R(x+y) \le R(x).R(y)$, which implies that the TPXG distribution has the property of new better than used (NBU).

2. 1. 3. Moments

We derived as given below. For details, we refer to Sen et al. [1].

2. 1. 3. 1. *kth* Moment

For some integer $k > 0$, the kth moment is given by

$$E\left(X^k\right) = \int_0^\infty x^k f(x)\,dx = \int_0^\infty x^k \frac{\alpha^2}{(\alpha+\beta)}\left(1+\frac{\alpha\beta}{2}x^2\right)e^{-\alpha x}\,dx \tag{2.7}$$

$$= \frac{\Gamma(k+1)\left(2\alpha+\beta(k+1)(k+2)\right)}{2\alpha^k(\alpha+\beta)},\ (\alpha>0, \beta>0), \tag{2.8}$$

which easily follows by evaluating the above integral on the right side of the Eq. (2.7).

2. 1. 3. 2. 1st Moment

Now, when $k=1$ in Eq. (2.8), the 1st moment is given by

$$E(X) = \frac{(2\alpha+3\beta)}{\alpha(\alpha+\beta)}.\ (\alpha>0, \beta>0). \tag{2.9}$$

2. 1. 3. 3. kth **Incomplete Moment**

For some integer $k > 0$, the kth incomplete moment is given by

$$I_x = \int_0^x u^k f(u)\,du = \int_0^x u^k \frac{\alpha^2}{(\alpha+\beta)}\left(1+\frac{\alpha\beta}{2}u^2\right)e^{-\alpha u}\,du$$

$$= \frac{\alpha^2}{(\alpha+\beta)}\int_0^x \left(u^k e^{-\alpha u} + \frac{\alpha\beta}{2}u^{k+2}e^{-\alpha u}\right)du \,. \tag{2.10}$$

Now, using $\int_0^z t^{\nu-1}e^{-\mu t}\,dt = \mu^{-\nu}\gamma(\nu,\mu z)$ in Eq. (2.10), where $\gamma(.)$ denotes the incomplete gamma function, see Gradshteyn and Ryzhik [22], Eq. 3.381.1, p. 317, we have

$$I_x = \frac{\alpha^2}{(\alpha+\beta)}\left[\gamma(k+1,\alpha x) + \frac{\alpha\beta}{2}\gamma(k+3,\alpha x)\right], \ (\alpha>0, \beta>0),$$

$$= P_k(x), \text{ say.} \tag{2.11}$$

2. 1. 3. 4. 1st **Incomplete Moment**

When $k = 1$ in Eq. (2.11), the 1st incomplete moment is given by

$$P_1(x) = \frac{\alpha^2}{(\alpha+\beta)}\left[\gamma(2,\alpha x) + \frac{\alpha\beta}{2}\gamma(4,\alpha x)\right], \ (\alpha>0, \beta>0). \tag{2.12}$$

3. CHARACTERIZATION RESULTS

In what follows, in this section, we will establish our proposed characterization results of the TPXG distribution by left and right truncated moments. We first establish our proposed characterization results based on the left truncated moment, that is, by taking a relation between left truncated moment and failure rate function. Then, we characterized the TPXG distribution based on the right truncated moment, that is, by taking a relation between the right truncated moment and reversed failure rate function. Based on these results, we finally characterize the TPXG distribution by order statistics and record values.

3. 1. Characterization by Truncated Moment

In this subsection, we establish our proposed two new characterization results of the TPXG distribution of Sen et al. [1] by truncated moments in Theorems 3.1 and 3.2, respectively. For this, we will need the following assumption and lemmas.

Assumption 3.1. Suppose the random variable X is absolutely continuous with the cumulative distribution function $F(x)$ and the probability density function $f(x)$. We assume that

$\omega = \inf\{x \mid F(x) > 0\}$, and $\delta = \sup\{x \mid F(x) < 1\}$. We also assume that $f(x)$ is a differentiable for all x, and $E(X)$ exists.

Lemma 3.1. If the random variable X satisfies the Assumption 3.1 with $\omega = 0$ and $\delta = \infty$, and if $E(X \mid X \le x) = g(x)\tau(x)$, where $\tau(x) = \dfrac{f(x)}{F(x)}$ and $g(x)$ is a continuous differentiable function of x with the condition that $\int_0^x \dfrac{u - g'(u)}{g(u)} du$ is finite for $x > 0$, then $f(x) = c e^{\int_0^x \frac{u - g'(u)}{g(u)} du}$, where c is a constant determined by the condition $\int_0^\infty f(x)dx = 1$.

Proof. For proof, see Shakil et al. [23].

Lemma 3.2. If the random variable X satisfies the Assumption 3.1 with $\omega = 0$ and $\delta = \infty$, and if $E(X \mid X \ge x) = \tilde{g}(x) r(x)$, where $r(x) = \dfrac{f(x)}{1 - F(x)}$ and $\tilde{g}(x)$ is a continuous differentiable function of x with the condition that $\int_x^\infty \dfrac{u + \left[\tilde{g}(u)\right]'}{\tilde{g}(u)} du$ is finite for $x > 0$, then $f(x) = c e^{-\int_0^x \frac{u + \left[\tilde{g}(u)\right]'}{\tilde{g}(u)} du}$, where c is a constant determined by the condition $\int_0^\infty f(x)dx = 1$.

Proof. For proof, see Shakil et al. [23].

Theorem 3.1. If the random variable X satisfies the Assumption 3.1 with $\omega = 0$ and $\delta = \infty$, then $E(X|X \le x) = g(x)\dfrac{f(x)}{F(x)}$, where

$$g(x) = \frac{P_1(x)}{\left(1 + \frac{\alpha\beta}{2} x^2\right) e^{-\alpha x}}, \tag{3.1}$$

where $P_1(x)$ is given by (2.12), if and only if X has the pdf

$$f_X(x) = \begin{cases} \dfrac{\alpha^2}{(\alpha + \beta)} \left(1 + \dfrac{\alpha\beta}{2} x^2\right) e^{-\alpha x}, & (0 < x < \infty, \alpha > 0, \beta > 0) \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Suppose that $E(X|X \le x) = g(x)\,\frac{f(x)}{F(x)}$. Then, since $E(X|X \le x) = \frac{\int_0^x u\,f(u)du}{F(x)}$, we have $g(x) = \frac{\int_0^x u\,f(u)du}{f(x)}$. Now, if the random variable X satisfies the Assumption 3.1 and has the distribution with the pdf (2.1), then we have

$$g(x) = \frac{\int_0^x u\,f(u)\,du}{f(x)} = \frac{\int_0^x u\left[\left(1+\frac{\alpha\beta}{2}u^2\right)e^{-\alpha u}\right]du}{\left(1+\frac{\alpha\beta}{2}x^2\right)e^{-\alpha x}} = \frac{P_1(x)}{\left(1+\frac{\alpha\beta}{2}x^2\right)e^{-\alpha x}},$$

where $P_1(x)$ is given by (2.12). Consequently, the proof of "if" part of the Theorem 3.1 follows from Lemma 3.1.

Conversely, suppose that

$$g(x) = \frac{P_1(x)}{\left(1+\frac{\alpha\beta}{2}x^2\right)e^{-\alpha x}},$$

where $P_1(x)$ is given by (2.12). Now, from Lemma 3.1, we have

$$g(x) = \frac{\int_0^x u\,f(u)du}{f(x)},$$

or

$$\int_0^x u\,f(u)du = f(x)g(x).$$

Differentiating the above equation with respect to respect to x, we obtain

$$x\,f(x) = f'(x)g(x) + f(x)g'(x),$$

from which, using the definition of the pdf (2.1) and noting that

$$f'(x) = \frac{\alpha^2}{(\alpha+\beta)}\left(\alpha\beta x - \alpha - \frac{\alpha^2\beta}{2}x^2\right)e^{-\alpha x},$$

we easily obtain

$$g^{/}(x)=x-g(x)\frac{\left(\alpha\,\beta\,x-\alpha-\dfrac{\alpha^{2}\,\beta}{2}x^{2}\right)e^{-\alpha x}}{\left(1+\dfrac{\alpha\,\beta}{2}x^{2}\right)e^{-\alpha x}},$$

or

$$\frac{x-g^{/}(x)}{g(x)}=\frac{\left(\alpha\,\beta\,x-\alpha-\dfrac{\alpha^{2}\,\beta}{2}x^{2}\right)e^{-\alpha x}}{\left(1+\dfrac{\alpha\,\beta}{2}x^{2}\right)e^{-\alpha x}}. \tag{3.2}$$

Since, by Lemma 3.1, we have

$$\frac{x-g^{/}(x)}{g(x)}=\frac{f^{/}(x)}{f(x)} \text{ , see Shakil et al. [23],} \tag{3.3}$$

therefore, from (3.2) and (3.3), it follows that

$$\frac{f^{/}(x)}{f(x)}=\frac{\left(\alpha\,\beta\,x-\alpha-\dfrac{\alpha^{2}\,\beta}{2}x^{2}\right)e^{-\alpha x}}{\left(1+\dfrac{\alpha\,\beta}{2}x^{2}\right)e^{-\alpha x}}. \tag{3.4}$$

Now, integrating Eq. (3.4) with respect to x and simplifying, we easily have

$$\ln f(x)=\ln\left(c\left(1+\frac{\alpha\,\beta}{2}x^{2}\right)e^{-\alpha x}\right),$$

or

$$f(x)=c\left(1+\frac{\alpha\,\beta}{2}x^{2}\right)e^{-\alpha x}, \tag{3.5}$$

where c is the normalizing constant to be determined. Thus, on integrating the above equation (3.5) with respect to x from $x=0$ to $x=\infty$, using the condition $\int_0^\infty f(x)dx=1$ and noting that $\int\limits_0^\infty e^{-\alpha x}\,dx=\frac{1}{\alpha}$, and $\int\limits_0^\infty t^{\nu-1}e^{-\mu t}\,dt=\mu^{-\nu}\,\Gamma(\nu)$, where $\Gamma(\nu)$ denotes the gamma function, see

Gradshteyn and Ryzhik [22], Eq. 3.381.4, p. 317, we obtain $c=\frac{\alpha^2}{(\alpha+\beta)}$, and thus $f_X(x)=\frac{\alpha^2}{(\alpha+\beta)}\left(1+\frac{\alpha\beta}{2}x^2\right)e^{-\alpha x}, \quad (0<x<\infty, \alpha>0, \beta>0)$, which is the required pdf (2.1) of the random variable X. This completes the proof of Theorem 3.1.

Theorem 3.2. If the random variable X satisfies the Assumption 3.1 with $\omega=0$ and $\delta=\infty$, then $E(X|X\geq x)=\tilde{g}(x)\frac{f(x)}{1-F(x)}$, where

$$\tilde{g}(x)==\frac{\left(E(X)-g(x)f(x)\right)(\alpha+\beta)}{\alpha^2\left(1+\frac{\alpha\beta}{2}x^2\right)e^{-\alpha x}},$$

where $g(x)$ is given by Eq. (3.1) and $E(X)$ is given by Eq. (2.9), if and only if X has the pdf

$$f_X(x)=\begin{cases}\frac{\alpha^2}{(\alpha+\beta)}\left(1+\frac{\alpha\beta}{2}x^2\right)e^{-\alpha x}, & (0<x<\infty, \alpha>0, \beta>0)\\ 0, & \text{otherwise.}\end{cases}$$

Proof. Suppose that $E(X|X\geq x)=\tilde{g}(x)\frac{f(x)}{1-F(x)}$. Then, since $E(X|X\geq x)=\frac{\int_x^\infty u\,f(u)du}{1-F(x)}$, we have $\tilde{g}(x)=\frac{\int_x^\infty u\,f(u)du}{f(x)}$. Now, if the random variable X satisfies the Assumptions 3.1 and has the distribution with the pdf (1), then we have

$$\tilde{g}(x)=\frac{\int_x^\infty u\,f(u)\,du}{f(x)}=\frac{\int_0^\infty u\,f(u)\,du-\int_0^x u\,f(u)\,du}{f(x)}$$

$$=\frac{\left(E(X)-g(x)f(x)\right)(\alpha+\beta)}{\alpha^2\left(1+\frac{\alpha\beta}{2}x^2\right)e^{-\alpha x}}.$$

Consequently, the proof of "if" part of the Theorem 3.2 follows from Lemma 3.2.

Conversely, suppose that $\tilde{g}(x) = = \frac{(E(X) - g(x) f(x))(\alpha + \beta)}{\alpha^2 \left(1 + \frac{\alpha\beta}{2} x^2\right) e^{-\alpha x}}$. Now, from Lemma 3.2, we have

$$\tilde{g}(x) = \frac{\int_x^\infty u f(u) du}{f(x)},$$

or

$$\int_x^\infty u f(u) du = f(x).\tilde{g}(x).$$

Differentiating the above equation with respect to respect to x, we obtain

$$-x f(x) = f^{/}(x).\tilde{g}(x) + f(x).\left(\tilde{g}(x)\right)^{/},$$

from which, using the definition of the pdf (2.1) and noting that

$$f^{/}(x) = \frac{\alpha^2}{(\alpha+\beta)}\left(\alpha\beta x - \alpha - \frac{\alpha^2 \beta}{2} x^2\right) e^{-\alpha x},$$

we easily obtain

$$\left(\tilde{g}(x)\right)^{/} = -x - \tilde{g}(x) \frac{\left(\alpha\beta x - \alpha - \frac{\alpha^2\beta}{2} x^2\right) e^{-\alpha x}}{\left(1 + \frac{\alpha\beta}{2} x^2\right) e^{-\alpha x}},$$

or,

$$\frac{x + \left(\tilde{g}(x)\right)^{/}}{\tilde{g}(x)} = -\frac{\left(\alpha\beta x - \alpha - \frac{\alpha^2\beta}{2} x^2\right) e^{-\alpha x}}{\left(1 + \frac{\alpha\beta}{2} x^2\right) e^{-\alpha x}}. \quad (3.6)$$

Since, by Lemma 3.2, we have

$$\frac{f^{/}(x)}{f(x)} = -\frac{x + \left[\tilde{g}(x)\right]^{/}}{\tilde{g}(x)}, \quad \text{see Shakil, et al. (2018),} \tag{3.7}$$

therefore, from (3.6) and (3.7), it follows that

$$\frac{f^{/}(x)}{f(x)} = \frac{\left(\alpha\beta x - \alpha - \frac{\alpha^2\beta}{2}x^2\right)e^{-\alpha x}}{\left(1 + \frac{\alpha\beta}{2}x^2\right)e^{-\alpha x}}. \tag{3.8}$$

Now, integrating Eq. (3.8) with respect to x and simplifying, we easily have

$$\ln f(x) = \ln\left(c\left(1 + \frac{\alpha\beta}{2}x^2\right)e^{-\alpha x}\right),$$

or

$$f(x) = c\left(1 + \frac{\alpha\beta}{2}x^2\right)e^{-\alpha x}, \tag{3.9}$$

where c is the normalizing constant to be determined. Thus, on integrating the above equation (3.9) with respect to x from $x = 0$ to $x = \infty$, using the condition $\int_0^\infty f(x)dx = 1$ and noting that $\int_0^\infty e^{-\alpha x}\,dx = \frac{1}{\alpha}$, and $\int_0^\infty t^{\nu-1}e^{-\mu t}\,dt = \mu^{-\nu}\,\Gamma(\nu)$, where $\Gamma(\nu)$ denotes the gamma function, see Gradshteyn and Ryzhik [22], Eq. 3.381.4, p. 317, we obtain $c = \frac{\alpha^2}{(\alpha+\beta)}$, and thus $f_X(x) = \frac{\alpha^2}{(\alpha+\beta)}\left(1 + \frac{\alpha\beta}{2}x^2\right)e^{-\alpha x}, \quad (0 < x < \infty, \alpha > 0, \beta > 0)$, which is the required pdf (2.1) of the random variable X. This completes the proof of Theorem 3.2.

3. 2. Characterizations by Order Statistics

If $X_1, X_2, \ldots, X_n$ be the n independent copies of the random variable X with absolutely continuous distribution function $F(x)$ and pdf $f(x)$, and if $X_{1,n} \le X_{2,n} \le \ldots \le X_{n,n}$ be the corresponding order statistics, then it is known from Ahsanullah et al. [24], chapter 5, or Arnold et al. [25], chapter 2, that $X_{j,n} \mid X_{k,n} = x$, for $1 \le k < j \le n$, is distributed as the $(j-k)th$ order statistics from $(n-k)$ independent observations from the random variable V having the pdf $f_V(v|x)$ where $f_V(v|x)=\dfrac{f(v)}{1-F(x)}$, $0\le v<x$, and $X_{i,n}|X_{k,n}=x, 1\le i<k\le n$, is distributed as *ith* order statistics from k independent observations from the random variable W having the pdf $f_W(w|x)$ where $f_W(w|x)=\dfrac{f(w)}{F(x)}$, $w<x$.

Let $S_{k-1}=\frac{1}{k-1}\left(X_{1,n}+X_{2,n}+\ldots+X_{k-1,n}\right)$, and $T_{k,n}=\frac{1}{n-k}\left(X_{k+1,n}+X_{k+2,n}+\ldots+X_{n.n}\right)$.

Theorem 3.3: Suppose the random variable X satisfies the Assumption 3.1 with $\omega=0$ and $\delta=\infty$, then $E(S_{k-1}|X_{k,n}=x)=g(x)\tau(x)$, where $\tau(x)=\dfrac{f(x)}{F(x)}$ and $g(x)=\dfrac{P_1(x)}{\left(1+\frac{\alpha\beta}{2}x^2\right)e^{-\alpha x}}$

where $P_1(x)$ is given by (2.12), if and only if X has the pdf

$$f_X(x)=\begin{cases}\dfrac{\alpha^2}{(\alpha+\beta)}\left(1+\dfrac{\alpha\beta}{2}x^2\right)e^{-\alpha x}, & (0<x<\infty, \alpha>0, \beta>0)\\ 0, & \text{otherwise.}\end{cases}$$

Proof: It is known that $E(S_{k-1}|X_{k,n}=x)=E(X|X\le x)$; see Ahsanullah et al. [24], and David and Nagaraja [26]. Hence, by Theorem 3.1, the result follows.

Theorem 3.4: Suppose the random variable X satisfies the Assumption 3.1 with $\omega=0$ and $\delta=\infty$, then $E(T_{k,n}|X_{k,n}=x)=\tilde{g}(x)\dfrac{f(x)}{1-F(x)}$, where $\tilde{g}(x)==\dfrac{\left(E(X)-g(x)f(x)\right)(\alpha+\beta)}{\alpha^2\left(1+\frac{\alpha\beta}{2}x^2\right)e^{-\alpha x}}$

, $g(x)$ is given by Eq. (3.1) and $E(X)$ is given by Eq. (2.9), if and only if X has the pdf

$$f_X(x)=\begin{cases}\dfrac{\alpha^2}{(\alpha+\beta)}\left(1+\dfrac{\alpha\beta}{2}x^2\right)e^{-\alpha x}, & (0<x<\infty, \alpha>0, \beta>0)\\ 0, & \text{otherwise.}\end{cases}$$

Proof: Since $E(T_{k,n}|X_{k,n}=x)=E(X|X\ge x)$, see Ahsanullah et al. [24], and David and Nagaraja [26], the result follows from Theorem 3.2.

3. 3. Characterization by Upper Record Values

For details on record values, see Ahsanullah [27]. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed absolutely continuous random variables with distribution function $F(x)$ and pdf $f(x)$. If $Y_n=\max(X_1, X_2, \ldots, X_n)$ for $n\ge 1$ and

$Y_j > Y_{j-1}$, $j>1$, then X_j is called an upper record value of $\{X_n,\ n \geq 1\}$. The indices at which the upper records occur are given by the record times $\{U(n) > \min(j \mid j > U(n+1), X_j > X_{U(n-1)}, n>1)\}$ and $U(1)=1$. Let the *nth* upper record value be denoted by $X(n) = X_{U(n)}$.

Theorem 3.5: Suppose the random variable X satisfies the Assumption 3.1 with $\omega = 0$ and $\delta = \infty$, then $E(X(n+1) \mid X(n) = x) = \tilde{g}(x)\dfrac{f(x)}{1-F(x)}$,

where

$$\tilde{g}(x) == \frac{\left(E(X) - g(x) f(x)\right)(\alpha+\beta)}{\alpha^2\left(1+\frac{\alpha\beta}{2}x^2\right)e^{-\alpha x}},$$

$g(x)$ is given by Eq. (3.1) and $E(X)$ is given by Eq. (2.9), if and only if X has the pdf

$$f_X(x) = \begin{cases} \dfrac{\alpha^2}{(\alpha+\beta)}\left(1+\dfrac{\alpha\beta}{2}x^2\right)e^{-\alpha x}, & (0<x<\infty, \alpha>0, \beta>0) \\ 0, & \text{otherwise.} \end{cases}$$

Proof: It is known from Ahsanullah et al. [24], and Nevzorov [28] that $E(X(n+1) \mid X(n) = x) = E(X \mid X \geq x)$. Then, the result follows from Theorem 3.2.

4. CONCLUSIONS

Since a characterization of a particular probability distribution states that it is the only distribution that satisfies some specified conditions, in this paper we have established some new characterization results of the TPXG distribution of the Sen et al. [1] by the left and right truncated moments, order statistics and record values. The first characterization result is based on a relation between left truncated moment and failure rate function. The second characterization result is based on a relation between right truncated moment and reversed failure rate function. Based on these characterizations, we have also characterized the TPXG distribution by order statistics and record values. We hope the findings of the paper will be quite useful for the practitioners in various fields of sciences.

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