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On generalized difference rough ideal convergent of triple sequence defined by Musielak-Orlicz function

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ABSTRACT

We introduce a rough ideal convergent of triple sequence spaces defined by Musielak-Orlicz function, using an four dimensional infinite matrix and a generalized difference Zweier matrix operator $B^{p}_{(abc)}$ of order p. We obtain some topological and algebraic properties of these spaces.

***Keywords*:** triple sequences, Wijsman rough convergence, strongly admissible ideal, cluster points

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1. INTRODUCTION

The idea of statistical convergence was introduced by Steinhaus and also independently by Fast for real or complex sequences. Statistical convergence is a generalization of the usual notion of convergence, which has parallels the theory of ordinary convergence.

Let K be a subset of the set of positive integers $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$, and let us denote the set $\{(m,n,k) \in K: m \leq u, n \leq v, k \leq w\}$ by K_{uvw}. Then the natural density of K is given by $\delta(K) = lim_{u,v,w \to \infty} \frac{|K_{uvw}|}{uvw}$, where $|K_{uvw}|$ denotes the number of elements in K_{uvw}. Clearly, a

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finite subset has natural density zero, and we have $\delta(K^c) = 1 - \delta(K)$ where $K^c = \mathbb{N}\backslash K$ is the complement of K. If $K_1 \subseteq K_2$, then $\delta(K_1) \leq \delta(K_2)$.

Throughout the paper, a triple sequence $x = (x_{mnk})$ such that $x_{mnk} \in \mathbb{R}, m, n, k \in \mathbb{N}$.

A triple sequence $x = (x_{mnk})$ is said to be statistically convergent to $0 \in \mathbb{R}$, written as $st - lim \quad x = 0$, provided that the set

$$\{(m, n, k) \in \mathbb{N}^3 : |x_{mnk}| \geq \varepsilon\}$$

has natural density zero for any $\varepsilon > 0$. In this case, 0 is called the statistical limit of the triple sequence x.

If a triple sequence is statistically convergent, then for every $\varepsilon > 0$, infinitely many terms of the sequence may remain outside the $\varepsilon -$ neighbourhood of the statistical limit, provided that the natural density of the set consisting of the indices of these terms is zero. This is an important property that distinguishes statistical convergence from ordinary convergence. Because the natural density of a finite set is zero, we can say that every ordinary convergent sequence is statistically convergent.

If a triple sequence $x = (x_{mnk})$ satisfies some property P for all m, n, k except a set of natural density zero, then we say that the triple sequence x satisfies P for almost all (m, n, k) and we abbreviate this by a.a. (m, n, k).

Let $\left(x_{m_i n_j k_\ell}\right)$ be a subsequence of $x = (x_{mnk})$. If the natural density of the set $K = \left\{\left(m_i, n_j, k_\ell\right) \in \mathbb{N}^3 : (i, j, \ell) \in \mathbb{N}^3\right\}$ is different from zero, then $\left(x_{m_i n_j k_\ell}\right)$ is called a nonthin subsequence of a triple sequence x.

$c \in \mathbb{R}$ is called a statistical cluster point of a triple sequence $x = (x_{mnk})$ provided that the natural density of the set

$$\{(m, n, k) \in \mathbb{N}^3 : |x_{mnk} - c| < \varepsilon\}$$

is different from zero for every $\varepsilon > 0$. We denote the set of all statistical cluster points of the sequence x by Γ_x.

A triple sequence $x = (x_{mnk})$ is said to be statistically analytic if there exists a positive number M such that

$$\delta\left(\left\{(m, n, k) \in \mathbb{N}^3 : |x_{mnk}|^{1/m+n+k} \geq M\right\}\right) = 0$$

The theory of statistical convergence has been discussed in trigonometric series, summability theory, measure theory, turnpike theory, approximation theory, fuzzy set theory and so on.

The idea of rough convergence was introduced by Phu [15], who also introduced the concepts of rough limit points and roughness degree. The idea of rough convergence occurs very naturally in numerical analysis and has interesting applications. Aytar [1] extended the idea of rough convergence into rough statistical convergence using the notion of natural density just as usual convergence was extended to statistical convergence.

Pal et al. [14] extended the notion of rough convergence using the concept of ideals which automatically extends the earlier notions of rough convergence and rough statistical convergence.

Let (X,ρ) be a metric space. For any nonempty closed subsets $A, A_{mnk} \subset X$ $(m,n,k \in \mathbb{N})$, we say that the triple sequence (A_{mnk}) is Wijsman statistical convergent to A is the triple sequence $\left(d(x, A_{mnk})\right)$ is statistically convergent to $d(x,A)$, i.e., for $\varepsilon > 0$ and for each $x \in X$

$$lim_{r,s,t} \frac{1}{rst} |\{m \leq r, n \leq s, k \leq t : |d(x, A_{mnk}) - d(x,A)| \geq \varepsilon\}| = 0.$$

In this case, we write $st - lim_{mnk} A_{mnk} = A$ or $A_{mnk} \to A(WS)$. The triple sequence (A_{mnk}) is bounded if $sup_{mnk} d(x, A_{mnk}) < \infty$ for each $x \in X$.

A triple sequence (real or complex) can be defined as a function $x: \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by Sahiner et al. [16, 17], Esi et al. [3-6], Dutta et al. [7], Subramanian et al. [18-23], Debnath et al. [8], Aiyub et al. [2], Esi et al.[20-22] and Zweier sequence was introduced and investigated at the initial by Karababa et al. [9], Sharma et al. [19], Khan et al. [11] many others.

A triple sequence $x = (x_{mnk})$ is said to be triple analytic if

$$sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty.$$

The space of all triple analytic sequences are usually denoted by Λ^3. A triple sequence $x = (x_{mnk})$ is called triple gai sequence if

$$\left((m+n+k)! \, |x_{mnk}|\right)^{\frac{1}{m+n+k}} \to 0 \text{ as } m,n,k \to \infty.$$

1. 1. Definition

An Orlicz or Musielak-Orlicz function ([see [10]) is a function $M: [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by $M(x+y) \leq M(x) + M(y)$, then this function is called modulus function.

Lindenstrauss and Tzafriri ([12]) used the idea of Orlicz function to construct Orlicz sequence space.

A sequence $g = (g_{mn})$ defined by

$$g_{mnk}(v) = sup\{|v|u - (f_{mnk})(u) : u \geq 0\}, m,n,k = 1,2,\cdots$$

is called the complementary function of a Musielak-Orlicz function f. For a given Musielak-Orlicz function f, [see [13]] the Musielak-Orlicz sequence space t_f is defined as follows

$$t_f = \left\{x \in w^3 : I_f(|x_{mnk}|)^{1/m+n+k} \to 0 \quad as \quad m,n,k \to \infty\right\},$$

where I_f is a convex modular defined by

$$I_f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mnk}(|x_{mnk}|)^{1/m+n+k}, x = (x_{mnk}) \in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mnk}\left(\frac{|x_{mnk}|^{1/m+n+k}}{mnk}\right)$$

is an extended real number.

1. 2. Definition

A triple sequence $x = (x_{mnk})$ is said to be statistically convergent to $l \in \mathbb{R}$, written as $st - lim x = l$, provided that the set

$$\{(m,n,k) \in \mathbb{N}^3 : |x_{mnk} - l| \geq \varepsilon\},$$

has natural density zero for every $\varepsilon > 0$.

In this case, l is called the statistical limit of the sequence x.

1. 3. Definition

Let $x = (x_{mnk})_{(m,n,k) \in \mathbb{N}^3}$ be a triple sequence in a metric space $(X, |.,.|)$ and r be a nonnegative real number. A triple sequence $x = (x_{mnk})$ is said to be $r -$convergent to $l \in X$, denoted by $x \to^r l$, if for any $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that for all $m, n, k \geq N_\varepsilon$ we have

$$|x_{mnk} - l| < r + \varepsilon$$

In this case l is called an $r -$ limit of x.

1. 4. Remark

We consider $r -$ limit set of x which is denoted by LIM_x^r and is defined by

$$LIM_x^r = \{l \in X : x \to^r l\}.$$

1. 5. Definition

A triple sequence $x = (x_{mnk})$ is said to be $r -$ convergent if $LIM_x^r \neq \phi$ and r is called a rough convergence degree of x. If $r = 0$ then it is ordinary convergence of triple sequence.

1. 6. Definition

Let $x = (x_{mnk})$ be a triple sequence in a metric space $(X, |.,.|)$ and r be a nonnegative real number then x is said to be $r -$ statistically convergent to l, denoted by $x \to^{r-st_3} l$, if for any $\varepsilon > 0$ we have $d\big(A(\varepsilon)\big) = 0$, where

$$A(\varepsilon) = \{(m,n,k) \in \mathbb{N}^3 : |x_{mnk} - l| \geq r + \varepsilon\}.$$

In this case l is called $r-$ statistical limit of x. If $r=0$ then it is ordinary statistical convergent of triple sequence.

1. 7. Definition

A class I of subsets of a nonempty set X is said to be an ideal in X provided

(i) $\phi \in I$

(ii) $A, B \in I$ implies $A \cup B \in I$,

(iii) $A \in I, B \subset A$ implies $B \in I$,

I is called a nontrivial ideal if $X \notin I$.

1. 8. Definition

A nonempty class F of subsets of a nonempty set X is said to be a filter in X provided that

(i) $\phi \in F$.

(ii) $A, B \in F$ implies $A \cap B \in F$.

(iii) $A \in F, A \subset B$ implies $B \in F$.

1. 9. Definition

I is a nontrivial ideal in X, $\quad X \neq \phi$, then the class

$$F(I) = \{M \subset X : M = X \backslash A \ \ for\ some\ A \in I\}$$

is a filter on X, called the filter associated with I.

1. 10. Definition

A nontrivial ideal I in X is called admissible if $\{x\} \in I$ for each $x \in X$.

1. 11. Note

If I is an admissible ideal, then usual convergence in X implies I convergence in X.

1. 12. Definition

Let $x = (x_{mnk})$ be a triple sequence in a metric space $(X, |.,.|)$ and r be a nonnegative real number then x is said to be rough ideal convergent or $rI-$ convergent to l, denoted by $x \rightarrow^{rI} l$, if for any $\varepsilon > 0$ we have

$$\{(m,n,k) \in \mathbb{N}^3 : |x_{mnk} - l| \geq r + \varepsilon\} \in I.$$

In this case l is called $rI-$ limit of x and a triple sequence $x = (x_{mnk})$ is called rough $I-$ convergent to l with r as roughness of degree. If $r = 0$ then it is ordinary $I-$ convergent.

1. 13. Remark

If I is an admissible ideal, then usual rough convergence implies rough $I-$ convergence.

1. 14. Note

Generally, a triple sequence $y=(y_{mnk})$ is not $I-$ convergent in usual sense and $|x_{mnk}-y_{mnk}| \leq r$ for all $(m,n,k) \in \mathbb{N}^3$or

$$\{(m,n,k) \in \mathbb{N}^3 : |x_{mnk}-y_{mnk}| \geq r\} \in I.$$

for some $r>0$. Then the triple sequence $x=(x_{mnk})$ is $rI-$ convergent.

1. 15. Note

It is clear that $rI-$ limit of x is not necessarily unique.

1. 16. Definition

Consider $rI-$ limit set of x, which is denoted by

$$I-LIM_x^r = \{L \in X : x \rightarrow^{rI} l\},$$

then the triple sequence $x=(x_{mnk})$ is said to be $rI-$ convergent if $I-LIM_x^r \neq \phi$ and r is called a rough $I-$ convergence degree of x.

1. 17. Definition

A triple sequence $x=(x_{mnk}) \in X$ is said to be $I-$ analytic if there exists a positive real number M such that

$$\left\{(m,n,k) \in \mathbb{N}^3 : |x_{mnk}|^{1/m+n+k} \geq M\right\} \in I.$$

1. 18. Definition

A point $L \in X$ is said to be an $I-$ accumulation point of a triple sequence $x=(x_{mnk})$ in a metric space (X,d) if and only if for each $\varepsilon>0$ the set

$$\{(m,n,k) \in \mathbb{N}^3 : d(x_{mnk}, l) = |x_{mnk}-l| < \varepsilon\} \notin I.$$

We denote the set of all $I-$ accumulation points of x by $I(\Gamma_x)$.

The difference triple sequence space was introduced by Debnath et al. (see [6]) and is defined as

$$\Delta x_{mnk} = x_{mnk} - x_{m,n+1,k} - x_{m,n,k+1} + x_{m,n+1,k+1} - x_{m+1,n,k} + x_{m+1,n+1,k} + x_{m+1,n,k+1} - x_{m+1,n+1,k+1} \text{ and } \Delta^0 x_{mnk} = \langle x_{mnk} \rangle.$$

The generalized difference triple notion has the following binomial representation

$$B^{p}_{(abc)} = \sum_{r=0}^{m} \sum_{s=0}^{n} \sum_{t=0}^{k} \binom{m}{r} \binom{n}{s} \binom{k}{t} p^{(m-r)+(n-s)+(k-t)} q^{r+s+t} x_{(m-ar)(n-bs)(k-ct)}.$$

Let X and Y be two nonempty subsets of the space w of complex sequences. Let $A = \left(a^{k\ell}_{mn}\right), (m, n, k, \ell = 1,2,3, \cdots.)$ be a four dimensional infinite matrix of complex numbers. We write $Ax = \left(A_\ell(x)\right)$ if

$$A_\ell(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a^{k\ell}_{mn} x_{mnk} \qquad (1)$$

converges for each ℓ. If $x = (x_{mnk}) \in X \Rightarrow Ax = \left(A_\ell(x)\right) \in Y$. We say that A defines a matrix transformation from $X \to Y$ and we denote it by $A: X \to Y$.

The triple sequence $y = (y_{mnk})$ which is frequently used as the Z transformation of the triple sequence $x = (x_{mnk})$ i.e., $y_{mnk} = \alpha x_{mnk} + (1 - \alpha) x_{m-1,n-1,k-1}$, where $x_{m-1,n-1,k-1} = 0$; $m, n, k \neq 0$; $1 < m, n, k < \infty$ and Z denotes the matrix $Z = \left(z_{\ell(mnk)}\right)$ defined by

$$z_{\ell(mnk)} = \begin{cases} \alpha & \text{if } \ell = m = n = k; \\ 1 - \alpha & \text{if } \ell - 1 = m = n = k; \\ 0; \text{otherwise} & \end{cases}$$

The Zweier sequence spaces Z as follows: $Z = \{x = (x_{mnk}) \in w^3 : Z(x) \in X\}$

2. DEFINITIONS AND PRELIMINARIES

Throughout the sections 2-4, Q be the set of all seminorms q.

2. 1. Definition

Let β be a nonnegative real number. A triple sequence $x = (x_{mnk}) \in X$ is said to be rough $I_q -$ convergent $\bar{0}$ if for all $q \in Q$ and all $\varepsilon > 0$,

$$\{(m, n, k) \in \mathbb{N}^3 : q(x_{mnk}, \bar{0}) \geq \beta + \varepsilon\} \in I. \quad (2)$$

In this case we can write $I_q - lim x_{mnk} = \bar{0}$. We denote

$$I_q = \left\{\{(m, n, k) \in \mathbb{N}^3 : q(x_{mnk}, \bar{0}) \geq \beta + \varepsilon\} \in I\right\}. \quad (3)$$

Note: If the space X is Hausdorff, then the limit of rough ideal convergent sequence x is unique.

2. 2. Definition

Let β be a nonnegative real number. A triple sequence $x = (x_{mnk}) \in X$ is said to be rough $B^{p}_{(abc)}\left(I_q\right) -$ convergent to $\bar{0} \in X$ if for all $q_p \in Q$ and all $\varepsilon > 0$,

$$\left\{(m,n,k) \in \mathbb{N}: q\left(B^{p}_{(abc)}x_{mnk}, \bar{0}\right) \geq \beta + \varepsilon\right\} \in I. \qquad (4)$$

In this case we can write $I_q - limB^{p}_{(abc)}(x_{mnk}) = \bar{0}$. We denote

$$B^{p}_{(abc)}(I_q) = \left\{\left\{(m,n,k) \in \mathbb{N}^3: q\left(B^{p}_{(abc)}x_{mnk}, \bar{0}\right) \geq \beta + \varepsilon\right\} \in I\right\}, \quad (5)$$

where

$$B^{p}_{(abc)}x_{mnk} = \sum_{r=0}^{m} \sum_{s=0}^{n} \sum_{t=0}^{k} \binom{m}{r}\binom{n}{s}\binom{k}{t} \alpha^{(m-r)+(n-s)+(k-t)} (1-\alpha)^{r+s+t} x_{(m-ar)(n-bs)(k-ct)}.$$

2. 3. Definition

Let β be a nonnegative real number and f be a Musielak Orlicz function. A triple sequence $x = (x_{mnk}) \in w^I\left(B^{p}_{(abc)}, f\right)$ if and only if there exists $\bar{0} \in X$ such that for $q \in Q$ and for every $\varepsilon > 0$,

$$\left\{(i,j,\ell) \in \mathbb{N}^3: \frac{1}{ij\ell}\sum_{m=1}^{i} \sum_{n=1}^{j} \sum_{k=1}^{\ell} \left[f_{mnk}\left(q\left(B^{p}_{(abc)}x_{mnk}, \bar{0}\right)\right)\right] \geq \beta + \varepsilon\right\} \in I \qquad (6)$$

when (5) holds we write $x_{mnk} \to \bar{0}\left(w^I\left(B^{p}_{(abc)}, f\right)\right)$. The condition (5) procides a definition of triple sequence of rough ideal of locally convex space.

3. MAIN RESULTS

3. 1. Theorem

Let β be nonnegative real number, $A = \left(a^{k\ell}_{mn}\right)$ be a four dimensional regular matrix and f be a Musielak Orlicz function. Then the triple sequence $(x_{mnk}) \to \bar{0}\left(w(f,A)\right) \Rightarrow (x_{mnk}) \to \bar{0}\left(B^{p}_{(abc)}(I_q)(A)\right)$.

Proof: Let $q \in Q$. Assume that $(x_{mnk}) \to \bar{0}\left(w(f,A)\right)$ we have

$$lim_{\ell\to\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a^{k\ell}_{mn} \left[f_{mnk}\left(q\left(B^{p}_{(abc)}x_{mnk}, \bar{0}\right)\right)\right] = 0. \quad (7)$$

Let $\varepsilon > 0$ be given. We define

$$K(\beta+\varepsilon) = \left\{(m,n,k) \in \mathbb{N}: q\left(B^{p}_{(abc)}x_{mnk}, \bar{0}\right) \geq \beta + \varepsilon\right\}, \quad (8)$$

and we write

$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_{mn}^{k\ell}\left[f_{mnk}\left(q\left(B_{(abc)}^{p}x_{mnk},\bar{0}\right)\right)\right] =$
$\sum_{(mnk)\in K(\beta+\varepsilon)} a_{mn}^{k\ell}\left[f_{mnk}\left(q\left(B_{(abc)}^{p}x_{mnk},\bar{0}\right)\right)\right] +$
$\sum_{(mnk)\notin K(\beta+\varepsilon)} a_{mn}^{k\ell}\left[f_{mnk}\left(q\left(B_{(abc)}^{p}x_{mnk},\bar{0}\right)\right)\right] \geq \left(\sum_{(mnk)\in K(\beta+\varepsilon)} a_{mn}^{k\ell}\right)[f_{mnk}(\beta+\varepsilon)].$

Then we have $(x_{mnk}) \to \bar{0}\left(B_{(abc)}^{p}(I_q)(A)\right)$.

3. 2. Theorem

Let $A = \left(a_{mn}^{k\ell}\right)$ be a four dimensional regular matrix and f be a Musielak Orlicz function. If the triple sequence $(x_{mnk}) \in \Lambda^3\left(B_{(abc)}^{p}\right)$ and $(x_{mnk}) \to \bar{0}\left(B_{(abc)}^{p}(I_q)(A)\right)$ then $(x_{mnk}) \to \bar{0}\left(w(f,A)\right)$.

Proof: Suppose that $(x_{mnk}) \to \Lambda^3\left(B_{(abc)}^{p}\right)$ and $(x_{mnk}) \to \bar{0}\left(B_{(abc)}^{p}(I_q)(A)\right)$. Then there is a set $K \in \bar{0}\left(B_{(abc)}^{p}(I_q)\right)$ such that

$$lim_{(m,n,k)\in K}\, q\left(B_{(abc)}^{p}x_{mnk}^{1/m+n+k}\right) = 0 \qquad (9)$$

Now

$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_{mn}^{k\ell}\left[f_{mnk}\left(q\left(B_{(abc)}^{p}x_{mnk}^{1/m+n+k}\right)\right)\right] =$
$\sum_{(mnk)\in K(\beta+\varepsilon)} a_{mn}^{k\ell}\left[f_{mnk}\left(q\left(B_{(abc)}^{p}x_{mnk}^{1/m+n+k}\right)\right)\right] +$
$\sum_{(mnk)\notin K(\beta+\varepsilon)} a_{mn}^{k\ell}\left[f_{mnk}\left(q\left(B_{(abc)}^{p}x_{mnk}^{1/m+n+k}\right)\right)\right]$
$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_{mn}^{k\ell}\chi_K(mnk)\left[f_{mnk}\left(q\left(B_{(abc)}^{p}x_{mnk}^{1/m+n+k}\right)\right)\right] +$
$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_{mn}^{k\ell}\chi_{K^c}(mnk)\left[f_{mnk}\left(q\left(B_{(abc)}^{p}x_{mnk}^{1/m+n+k}\right)\right)\right].$

If we consider the four dimensional regular matrix of $A, K^c B_{(abc)}^{p}(I_q)$ and analyticness of triple sequence spaces of (x_{mnk}) right side tends to zero. Hence $(x_{mnk}) \in \bar{0}\left(w(f,A)\right)$.

4. ZWEIER IDEAL TRIPLE SEQUENCE IN A LOCALLY CONVEX SPACE

Let I be an admissible ideal of $\mathbb{N}$ and $A = \left(a_{mn}^{k\ell}\right)$ be a four dimensional infinite matrix. Let f be a Musielak-Orlicz function and $w(X)$ denotes the space of all $X-$ valued triple rough sequence spaces. For each $\varepsilon > 0$ for all $q \in Q$, we define the following rough triple sequence spaces.

$$\left[Z_0\left(A, B_{(abc)}^{p}, f, q\right)\right]_I = \left\{\begin{matrix} x = (x_{mnk}) \in w(X): \\ \left\{\ell \in \mathbb{N}: \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_{mn}^{k\ell}\left[f_{mnk}\left(q\left(B_{(abc)}^{p}x_{mnk}\right)\right)\right] \geq \beta+\varepsilon\right\} \in I \end{matrix}\right\}$$

$$\left[Z_\infty\left(A, B^p_{(abc)}, f, q\right)\right]_I = \left\{ \begin{array}{c} x = (x_{mnk}) \in w(X): \\ \left\{\ell \in \mathbb{N}: \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_{mn}^{k\ell} \left[f_{mnk}\left(q\left(B^p_{(abc)} x_{mnk}\right)\right)\right] \geq K\right\} \in I \end{array} \right\}.$$

The proofs of the following theorems are easy to verify. Therefore we omit the proofs.

4. 1. Theorem

$\left[Z_0\left(A, B^p_{(abc)}, f, q\right)\right]_I$ and $\left[Z_\infty\left(A, B^p_{(abc)}, f, q\right)\right]_I$ are linear space.

4. 2. Theorem

Let $f = (f_{mnk})$ and $g = (g_{mnk})$ be two Musielak-Orlicz functions of rough triple sequence spaces, then the following holds:

$$\left[Z_0\left(A, B^p_{(abc)}, f, q\right)\right]_I \bigcap \left[Z_0\left(A, B^p_{(abc)}, g, q\right)\right]_I \subseteq \left[Z_0\left(A, B^p_{(abc)}, f+g, q\right)\right]_I.$$

4. 3. Theorem

Let $f = (f_{mnk})$ and $g = (g_{mnk})$ be two Musielak-Orlicz functions of rough triple sequence spaces then the following holds:

$$\left[Z_0\left(A, B^p_{(abc)}, g, q\right)\right]_I \subseteq \left[Z_0\left(A, B^p_{(abc)}, fg, q\right)\right]_I.$$

4. 4. Theorem

The inclusions $\left[Z_0\left(A, B^{p-1}_{(abc)}, f, q\right)\right]_I \subset \left[Z_0\left(A, B^p_{(abc)}, f, q\right)\right]_I$, are strict for $p \geq 1$. In general $\left[Z_0\left(A, B^j_{(abc)}, f, q\right)\right]_I \subset \left[Z_0\left(A, B^p_{(abc)}, f, q\right)\right]_I$, for $j = 0,1,2, \cdots p-1$ and the inclusions are strict.

Example: Let $A = [C, 1,1,1]$ $Cesaro\ matrix, f_{mnk}(x) = x$, for all $x \in [0, \infty), (m, n, k) \in \mathbb{N}$. Consider a rough triple sequence $x = (x_{mnk}) = (m^p n^p k^p)$. Then $x = (x_{mnk}) \in \left[Z_0\left(A, B^p_{(abc)}, f, q\right)\right]_I$ but does not belong to $\left[Z_0\left(A, B^{p-1}_{(abc)}, f, q\right)\right]_I$ because $B^p_{(abc)} x_{mnk} = 0$ and $B^{p-1}_{(abc)} x_{mnk} = (-1)^{p-1}(p-1)!$.

4. 5. Theorem

Let $f = (f_{mnk})$ be a Musielak-Orlicz function of rough triple sequence spaces, then the following statements are equivalent:

(i) $\left[Z_0\left(A, B^p_{(abc)}, f, q\right)\right]_I \subseteq \left[Z_0\left(A, B^p_{(abc)}, q\right)\right]_I$

(ii) $\left[Z_0\left(A, B^p_{(abc)}, f, q\right)\right]_I \subseteq \left[Z_\infty\left(A, B^p_{(abc)}, f, q\right)\right]_I$

(iii) $inf_\ell \sum_{(m,n,k)=1}^{\ell} a_{mn}^{k\ell} [f_{mnk}(t)] > 0 \quad (t > 0)$.

Proof: (i) ⇒ (ii) is obvious.
(ii) ⇒ (iii) Suppose $\left[Z_0\left(A, B^p_{(abc)}, f, q\right)\right]_I \subseteq \left[Z_\infty\left(A, B^p_{(abc)}, f, q\right)\right]_I$. We assume that (iii) does not hold. Then for some $t > 0$, $\ inf_\ell \sum_{(m,n,k)=1}^{\ell} a_{mn}^{k\ell}[f_{mnk}(t)] > 0\ = 0$.

We can choose an index sequence (ℓ_j) of positive integer such that

$$\sum_{m,n,k=1}^{\ell_j} a_{mn}^{k\ell}[f_{mnk}(abc)] > \frac{1}{j}, j = 1,2,3,\cdots. \qquad (10)$$

Define a rough triple sequence $x = (x_{mnk})$ by

$$B^p_{(abc)} x_{mnk} = \begin{cases} j & \text{if } 1 \leq \mathrm{m,n,k} \leq \ell_j \\ 0 & \text{if } \mathrm{m,n,k} > \ell_j. \end{cases}$$

Then by equation (4.1) we have rough triple sequence $x = (x_{mnk}) \in \left[Z_0\left(A, B^p_{(abc)}, f, q\right)\right]_I$ but $x = (x_{mnk}) \notin \left[Z_\infty\left(A, B^p_{(abc)}, f, q\right)\right]_I$ which contradicts (ii). Hence (iii) must hold. (iii) ⇒ (i) Let (iii) hold and $x \in \left[Z_0\left(A, B^p_{(abc)}, f, q\right)\right]_I$. Then for every $\varepsilon > 0$, we have

$$\left\{\ell \in \mathbb{N}: \sum_{m,n,k=1}^{\ell} a_{mn}^{k\ell} \left[f_{mnk}\left(q\left(B^p_{(abc)} x_{mnk}\right)\right)\right] \geq \beta + \varepsilon\right\} \in I. \qquad (11)$$

Suppose that $x \notin \left[Z_0\left(A, B^p_{(abc)}, f, q\right)\right]_I$. Then for some integer $\varepsilon_0 > 0$, we have

$$\left\{\ell \in \mathbb{N}: \sum_{m,n,k=1}^{\ell} a_{mn}^{k\ell} \left[f_{mnk}\left(q\left(B^p_{(abc)} x_{mnk}\right)\right)\right] \geq \beta + \varepsilon_0\right\} \notin I. \qquad (12)$$

Therefore we have

$$[f_{mnk}(\beta + \varepsilon_0)] \leq \left[f_{mnk}\left(q\left(B^p_{(abc)} x_{mnk}\right)\right)\right] \quad (13)$$

and consequently by the relation (4.2) we have

$$inf_\ell \sum_{m,n,k=1}^{\ell} a_{mn}^{k\ell}[f_{mnk}(\beta + \varepsilon_0)] = 0 \qquad (14)$$

which contradicts (iii). Hence $\left[Z_0\left(A, B^p_{(abc)}, f, q\right)\right]_I \subseteq \left[Z_0\left(A, B^p_{(abc)}, q\right)\right]_I$.

5. CONCLUSION

In this paper, we have studied a rough ideal convergent of triple sequence spaces defined by Musielak-Orlicz function, using an four dimensional infinite matrix and a generalized difference Zweier matrix operator $B^p_{(abc)}$ of order p and obtained some topological and algebraic properties of these spaces. The results of this of this paper are more general than earlier results.

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