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Concepts Arising from Strong Efficient Domination Number. Part I

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ABSTRACT

Let $G = (V, E)$ be a simple graph. A subset S of $V(G)$ is called a strong (weak) efficient dominating set of G if for every $v \in V(G), |N_s[v] \cap S| = 1. (|N_w[v] \cap S| = 1)$, where $N_s[v] = \{u \in V(G) : uv \in E(G), \deg u \geq \deg v\}$. $(N_w[v] = \{u \in V(G) : uv \in E(G), \deg v \geq \deg u$ The minimum cardinality of a strong (weak) efficient dominating set of G is called the strong (weak) efficient dominating set of G and is denoted by $\gamma_{se}(G)$ $(\gamma_{we}(G))$. A graph G is strong efficient if there exists a strong efficient dominating set of G. The strong efficient bondage number $b_{se}(G)$ of a non empty graph G is the minimum cardinality among all sets of edges $X \subseteq E$ such that $\gamma_{se}(G - X) > \gamma_{se}(G)$. In this paper, the strong efficient bondage number of some path related graphs and some special graphs are studied.

***Keywords*:** domination, strong efficient domination, strong efficient bondage number

AMS Subject Classification (2010): 05C69

1. INTRODUCTION

Throughout this paper finite, undirected graphs without loops or multiple edges are considered. Let G = (V,E) be a simple graph. N. Meena et al. defined strong efficient domination in graph. The bondage number $\mathrm{b}(G)$ of a graph is the minimum cardinality among

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all sets of edges $X \subseteq E$ for which $\gamma(G - X) > \gamma(G)$. N. Meena defined strong efficient bondage number of a graph. In this paper, the strong efficient bondage number of path related graphs and some special graphs are studied.

2. PRELIMINARIES

Definition 1.1: Let $G = (V, E)$ be a simple graph. A subset S of V(G) is called a strong (weak) efficient dominating set of G if for every $v \in V(G)$, $|N_s[v] \cap S| = 1(|N_w[v] \cap S| = 1)$ where $N_s(v) = \{u \in V(G): uv \in E(G), \deg u \geq \deg v\}$ and $N_s[v] = N_s(v) \cup \{v\}(N_w(v) = \{u \in V(G): uv \in E(G), \deg v \geq \deg u\}$ and $N_w[v] = N_w(v) \cup \{v\}$. The minimum cardinality of a strong(weak) efficient dominating set of G is called the strong(weak) efficient domination number of G and is denoted by $\gamma_{se}(G)$ $(\gamma_{we}(G))$. A graph G is strong efficient if there exists a strong efficient dominating set of G. A graph G is strong efficient if there exists a strong efficient dominating set of G. A graph which is not strong efficient is called a non strong efficient graph.

Definition 1.2: The regular spanning sub graph of degree 1 is called 1 – Factor.

Definition 1.3: The twig graph G_n is obtained from the path P_n by attaching exactly two pendent edges to each internal vertex of path.

Definition 1.4: The corona $G_1 \odot G_2$ of two graphs G_1 and G_2 is defined as the graph G by taking one copy of G_1 (which has p_1 points) and p_1 copies of G_2 and then joining the i^{th} point of G_1 to every point in the i^{th} copy of G_2

Theorem 1.5: $K_{n,n} - 1\,F$ is strong efficient and $\gamma_{se}(K_{n,n} - 1\,F) = 2$, for all $n \in N$.

Theorem 1.6:
For any path P_m,

$$\gamma_{se}(P_m) = \begin{Bmatrix} n \text{ if } m = 3n, n \in N \\ n + 1 \text{ if } m = 3n + 1, n \in N \\ n + 2 \text{ if } m = 3n + 2, n \in N \end{Bmatrix}$$

Observation 1.7: γ_{se} (G) = 1 if and only if G has a full degree vertex.

Definition 1.8: The strong efficient bondage number b_{se}(G) of a non empty graph G is the minimum cardinality among all sets of edges X ⊆ E such that γ_{se} (G – X) > γ_{se} (G).

Definition 1.9: Let G = (V, E) be a graph with V = $S_1 \cup S_2 \cup \ldots \cup S_t \cup T$, where each S_i is a set of vertices having atleast two vertices and having the same degree and $T = V - \cup S_i$. The degree splitting graph of G is denoted by DS (G) and is obtained from G by adding the vertices $W_1, W_2, \ldots, W_t$ and joining W_i to each vertex of S_i, $1 \leq i \leq t$.

Definition 1.10: The coconut tree graph $T(n,m)$ is obtained by joining the central vertex of the star $K_{1,m}$ and a pendant vertex of a path P_n by an edge.

3. MAIN RESULTS

Theorem 2.1: $b_{se}(K_{n,n} - 1F) = n - 1, n \geq 2.$

Proof: Let G = $K_{n,n} - 1F$. Let V (G) = $\{v_i, u_i : 1 \leq i \leq n\}$ and E(G) = $\{v_i u_j : 1 \leq i, j \leq n, i \neq j\}$. Then $\{v_i, u_i : 1 \leq i \leq n\}$ are the γ_{se} −sets of G and γ_{se} (G) = 2. Let anyone edge e be removed from G. Let $e = v_i u_k$, $1 \leq i, k \leq n, k \neq i$. Then $\{v_t, u_t : 1 \leq t \leq n, t \neq i$ and $t \neq k\}$ are the γ_{se} −sets of G – e. γ_{se}(G – e) = 2 = γ_{se}(G). Hence b_{se} (G) ≥ 2. Let $S_j = \{v_i u_j : 1 \leq i, j \leq n, i \neq j\}$. $|S_j| = n - 1$. Let H be a new graph obtained by removing the edges of S_j from G. V (H) = V (G). deg$v_i = 0$ and deg$v_j = n - 1$, $1 \leq j \leq n, i \neq j$. deg $u_k = n - 2$, $1 \leq k \leq n, k \neq i, j$. $T_i = \{u_1, u_2, u_3, \ldots, u_n, v_i, 1 \leq i \leq n\}$ is the unique strong efficient dominating set of H. $\gamma_{se}(H) = n + 1 > 2$. $b_{se}(G) \leq n - 1$. Suppose (n – 2) edges are removed then also one of sets $\{v_i u_i\}$ is the γ_{se} −set of G. Hence $b_{se}(G) \geq n - 1$. Therefore $b_{se}(G) = n - 1$.

Example 2.2: Let $G = K_{4,4} - 1F$, where $1F = \{v_1 u_1, v_2 u_2, v_3 u_3, v_4 u_4\}$. γ_{se}(G) = 2. Let $X = \{v_1 u_2, v_1 u_3, v_1 u_4\}$. The graphs G and G – X are given below

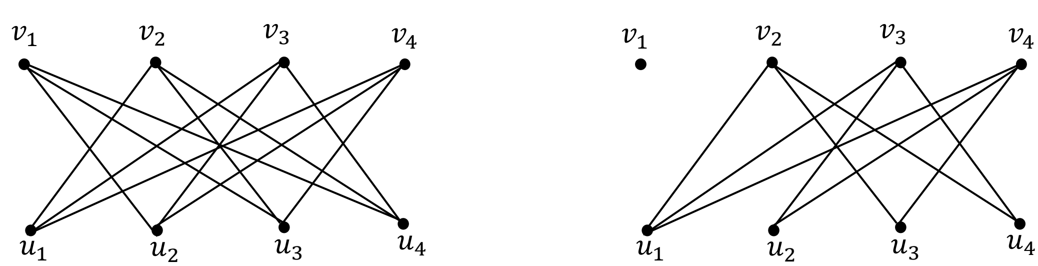


Figure 2.1.

$\{u_1, u_2, u_3, u_4, v_1\}$ is the unique γ_{se} −set of G – X. γ_{se} (G – X) = 5 > γ_{se}(G). Therefore $b_{se}(G) = 3$.

Theorem 2.3: $b_{se}(G_n) = \begin{cases} 1 \; if \; n = 3k, k \geq 1 \\ 2 \; if \; n = 3k+1, k \geq 1 \\ 3 \; if \; n = 3k+2, k \geq 1 \end{cases}$

Proof: $V(G_n) = \{v_i, u_j, w_j : 1 \leq i \leq n, 2 \leq j \leq n - 1\}$ and $E(G_n) = \{v_i v_{i+1}, v_j u_j, v_j w_j : 1 \leq i \leq n - 1, 2 \leq j \leq n - 1\}$.

Case 1: When $n = 3k,\ k \geq 1$.

Subcase1a: Let k = 1. $G_n = k_{1,4}$. $\gamma_{se}(G_n) = \gamma_{se}(k_{1,4}) = 1$. By removing any edge from G_n the resulting graph is $k_{1,3} \cup k_1$. $\gamma_{se}(G_n - e) = \gamma_{se}(k_{1,3}) + \gamma_{se}(k_1) = 1 + 1 = 2 > \gamma_{se}(G_n)$. Hence $b_{se}(G_n) = 1\ if\ n = 3$.

Subcase1b: When $n = 3k, k \geq 2$. $S = \{v_2, v_5, \dots, v_{3k-1}, u_3, u_6, \dots, u_{3k-3}, u_4, u_7, \dots, u_{3k-2}, w_3, w_6, \dots, w_{3k-3}, w_4, w_7, \dots, w_{3k-2}\}$ is the unique strong efficient dominating set of G_n. $|S| = 5k - 4, k \geq 2$. Therefore $\gamma_{se}(G_n) = 5k - 4,\ k \geq 2$. Let $e = v_2v_3$. Then $G_n - e = H_n$,
where

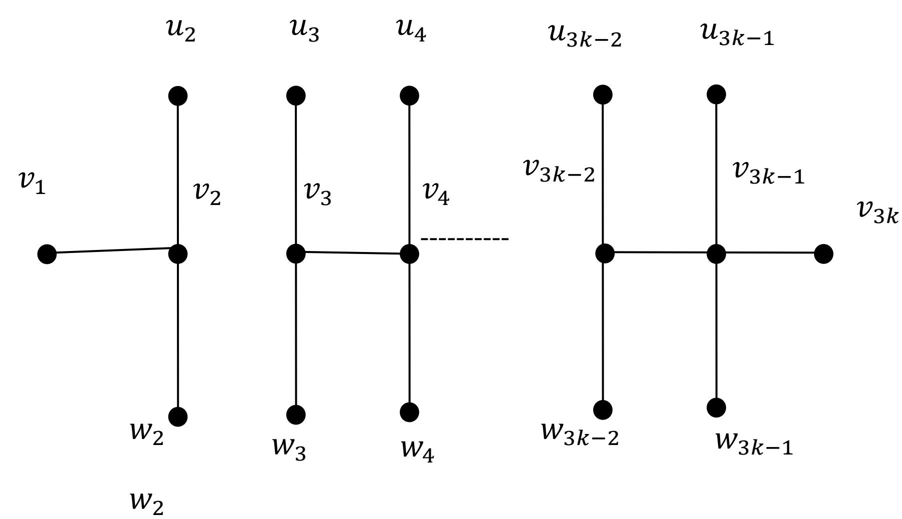
$H_n =$ 

Figure 2.2.

T = $\{v_2, v_4, \dots, v_{3k-2}, v_{3k}, u_3, u_6, \dots, u_{3k-3}, u_5, u_8, \dots, u_{3k-1}, w_3, w_6, \dots, w_{3k-3}, w_5, w_8, \dots, w_{3k-1}\}$ is the unique γ_{se} set of H_n . $|T|$ = 5k – 3, k ≥ 2. Therefore γ_{se} (G_n – e) = 5k – 3 > γ_{se} (G_n). Hence $b_{se}(G_n) = 1\ if\ n = 3k,\ k \geq 2$.

Case 2: When $n = 3k + 1, k \geq 1$. S1 = $\{v_2, v_5, \dots, v_{3k-1}, v_{3k+1}, u_3, u_6, \dots, u_{3k}, u_4, u_7, \dots, u_{3k-2}, w_3, w_6, \dots, w_{3k}, w_4, w_7, \dots, w_{3k-2}\}$ and
S2 = $\{v_1, v_3, v_6 \dots, v_{3k}, u_2, u_5, \dots, u_{3k-1}, u_4, u_7, \dots, u_{3k-2}, w_2, w_5, \dots, w_{3k-1}, w_4, w_7, \dots, w_{3k-2}\}$ are the two γ_{se} sets of G_n .
Therefore $|S_1| = |S_2| = 5k - 1,\ k \geq 1$. Hence γ_{se} (G_n) = 5k – 1, k ≥ 1. Let $e = \{v_1v_2, v_3v_4\}$. $G_n - e = H_n$,
where

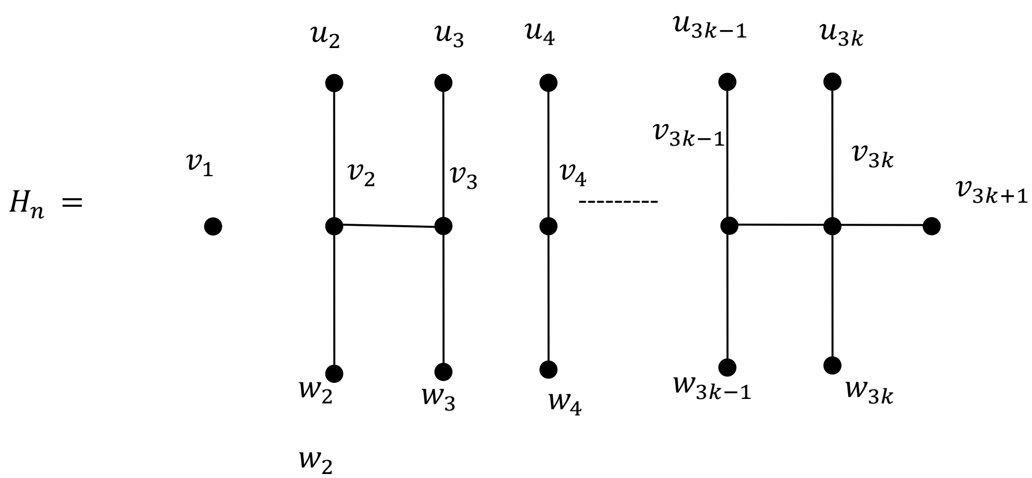


Figure 2.3.

T = { $v_1, v_2, v_5 \dots, v_{3k-1}, v_{3k+1}, u_3, u_6, \dots, u_{3k}, u_4, u_7, \dots, u_{3k-2}, w_3, w_6, \dots, w_{3k}, , w_4, w_7, \dots,$ $w_{3k-2}\}$ is the unique strong efficient dominating set of H_n. $|T| = 5k,\ k \geq 1$. Therefore γ_{se} $(G_n -$ e) = 5k > γ_{se} (G_n). Hence $b_{se}(G_n) \leq 2\ if\ n = 3k+1,\ k \geq 1$. Let $X = \{e_i\}$. Further it is verified that there is no γ_{se} set of $G_n - X$ satisfying γ_{se} $(G_n -$X$)$ > γ_{se} (G_n). Hence $b_{se}(G_n)$ $b_{se}(G_n) \geq 2\ if\ n = 3k+1, k \geq 1$. Therefore $b_{se}(G_n) = 2$.

Case 3: When $n = 3k+2, k \geq 1$. S = $\{v_1, v_3, v_6, \dots, v_{3k},\ v_{3k+2},\ u_2, u_5, \dots, u_{3k-1},\ u_4,\ u_7,$ $\dots, u_{3k+1},\ w_2, w_5, \dots, w_{3k-1}, w_4, w_7, \dots, w_{3k+1}\}$ is the unique strong efficient dominating set of G_n. $|S| = 5k+2,\ k \geq 1$. Therefore $\gamma_{se}(G_n) = 5k+2,\ k \geq 1$.
Let $e = \{v_2u_2, v_3u_3, v_4u_4\}$. $G_n - e = H_n$,where

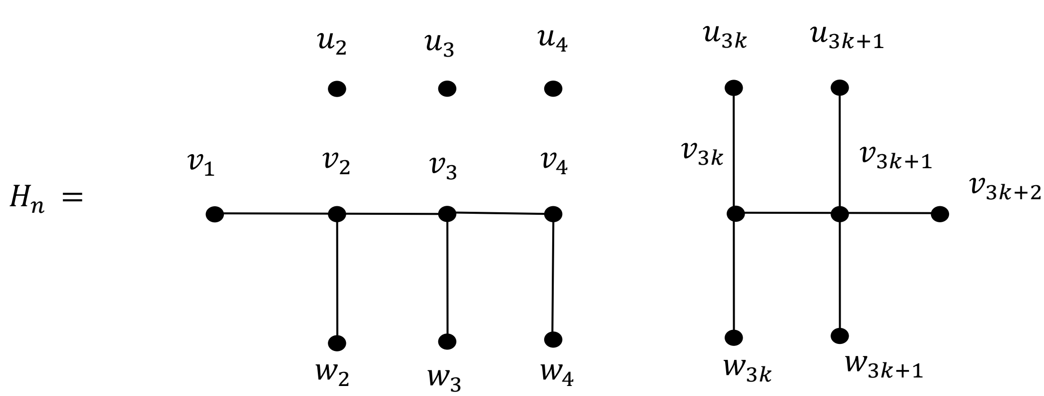


Figure 2.4.

T = $\{v_1, v_3, v_6, \dots, v_{3k},\ v_{3k+2},\ u_3, u_2, u_5, \dots, u_{3k-1}, u_4, u_7, \dots, u_{3k+1}, w_2, w_5, \dots, w_{3k-1}, w_4,$ $w_7, \dots, w_{3k+1}\}$ is the unique γ_{se} set of H_n. $|T|$ = 5k + 3 , k ≥ 1. Therefore γ_{se} $(G_n -$ e) = 5k + 3 > γ_{se} (G_n). Hence $b_{se}(G_n) \leq 3\ if\ n = 3k+2,\ k \geq 1$. Let $X = \{e_i\}\ or\ \{e_i, e_j\}$. Further it is verified that there is no γ_{se} set of $G_n - X$ satisfying $\gamma_{se}(G_n - X) > \gamma_{se}(G_n)$. Hence $b_{se}(G_n) \geq 3$ if $n = 3k + 2,\ k \geq 1$. Therefore $b_{se}(G_n) = 3$.

Theorem 2.4: Let G = $P_n \odot \overline{K_n}$ when $n \geq 1$.

Then $b_{se}(G) = \begin{Bmatrix} 2\ if\ m = 3k, k \geq 2 \\ 2\ if\ m = 3k+1, k \geq 1 \\ 1\ if\ m = 3k+2, k \geq 1 \end{Bmatrix}$

Proof: $V(G) = \{v_i, v_{ij}: 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(G) = \{v_i, v_{ij}, v_i v_{i+1,} 1 \leq i \leq m - 1, v_m v_{mj} 1 \leq j \leq n\}$

Case1: When m = 3k , $k \geq 2$. $S = \{ v_2, v_5, \ldots, v_{3k-1}, v_{1\,1}, v_{1\,2}, \ldots, v_{1\,n}, v_{4\,1}, v_{4\,2}, \ldots, v_{4\,n}, v_{3k-2\,1}, v_{3k-2\,2}, \ldots, v_{3k-2\,n}, v_{3\,1}, v_{3\,2}, \ldots, v_{3\,n}, v_{6\,1}, v_{6\,2}, \ldots, v_{6\,n}, v_{3k\,1}, v_{3k\,2}, \ldots, v_{3k\,n} \}$ is the unique strong efficient dominating set of. $|S| = (2n+1)k,\ k \geq 2, n \geq 1$. Therefore $\gamma_{se}(G) = (2n+1)k, k \geq 2, n \geq 1$. Let $e = \{v_2 v_{21}, v_3 v_{31}\}$. Let $G - e = H$. Then $T = \{ v_2, v_5, \ldots, v_{3k-1}, v_{1\,1,} v_{1\,2}, \ldots, v_{1\,n}, v_{2\,1}, v_{4\,1}, v_{4\,2}, \ldots, v_{4\,n}, v_{3k-2\,1}, v_{3k-2\,2}, \ldots, v_{3k-2\,n}, v_{3\,1}, v_{3\,2}, \ldots, v_{3\,n}, v_{6\,1}, v_{6\,2}, \ldots, v_{6\,n}, v_{3k\,1}, v_{3k\,2}, \ldots, v_{3k\,n}\}$ is the unique strong efficient dominating set of. $|T| = (2n+1)\,k + 1,\ k \geq 2,\ n \geq 1$. Therefore $\gamma_{se}(G) = (2n+1)k + 1,\ k \geq 2,\ n \geq 1$. Therefore $\gamma_{se}(G - e) = (2n+1)\,k + 1 > (2n+1)\ k = \gamma_{se}(G)$. Hence $b_{se}(G) \leq 2$, if $n = 3k,\ k \geq 1$. Let $X = \{e_i\}$. Further it is verified that there is no γ_{se} set of $G - X$ satisfying $\gamma_{se}(G - X) > \gamma_{se}(G)$. Hence $b_{se}(G) \geq 2$ if $n = 3k,\ k \geq 2$. Therefore $b_{se}(G) = 2$.

Case 2: When $m = 3k + 1, k \geq 1$. $S_1 = \{v_2, v_5, \ldots, v_{3k-1}, v_{3k+1}, v_{1\,1}, v_{1\,2}, \ldots, v_{1\,n}, v_{4\,1}, v_{4\,2}, \ldots, v_{4\,n}, v_{3k-2\,1}, v_{3k-2\,2}, \ldots, v_{3k-2\,n}, v_{3\,1}, v_{3\,2}, \ldots, v_{3\,n}, v_{6\,1}, v_{6\,2}, \ldots, v_{6\,n}, v_{3k\,1}, v_{3k\,2}, \ldots, v_{3k\,n}\}$.
$S_2 = \{ v_1, v_3, v_6, \ldots, v_{3k}, v_{2\,1}, v_{2\,2}, \ldots, v_{2\,n}, v_{5\,1}, v_{5\,2}, \ldots, v_{5\,n}, v_{3k-1\,1}, v_{3k-1\,2}, \ldots, v_{3k-1\,n}, v_{4\,1,} v_{4\,2}, \ldots, v_{4\,n}, v_{7\,1}, v_{7\,2}, \ldots, v_{7\,n}, v_{3k+1\,1}, v_{3k+1\,2}, \ldots, v_{3k+1\,n}\}$. Therefore $|s_1| = |s_2| = (2n+1)\ k + 1, k \geq 1, n \geq 1$. Therefore $\gamma_{se}(G) = (2n+1)k + 1,\ k \geq 1, n \geq 1$. Let $e = \{v_1 v_{11}, v_2, v_{21}\}$. Let $G - e = H$. $T = \{v_1, v_3, v_6, \ldots, v_{3k}, v_{1\,1}, v_{2\,1}, v_{2\,2}, \ldots, v_{2\,n}, v_{5\,1}, v_{5\,2}, \ldots, v_{5\,n}, v_{3k-1\,1}, v_{3k-1\,2}, \ldots, v_{3k-1\,n}, v_{4\,1}, v_{4\,2}, \ldots, v_{4\,n}, v_{7\,1}, v_{7\,2}, \ldots, v_{7\,n}, v_{3k+1\,1}, v_{3k+1\,2}, \ldots, v_{3k+1\,n}\}$ is the unique strong efficient dominating set of H . $|T| = (2n+1)\,k + 2,\ k \geq 1,\ n \geq 1$.
Therefore $\gamma_{se}(G - e) = (2n+1)k + 2 >,\ k \geq 1,\ n \geq 1$. Therefore $\gamma_{se}(G - e) = (2n+1)\ k + 2 > (2n+1)\ k + 1 = \gamma_{se}(G)$. Hence $b_{se}(G) \leq 2$ if $n = 3k + 1, k \geq 1$. Let $X = \{e_i\}$. Further it is verified that there is no γ_{se} set of $G - X$ satisfying $\gamma_{se}(G - X) > \gamma_{se}(G)$. Hence $b_{se}(G) \geq 2$ if $n = 3k + 1,\ k \geq 1$. Therefore $b_{se}(G) = 2$.

Case 3: When $m = 3k + 2, k \geq 1$. $S = \{v_1, v_3, v_6, \ldots, v_{3k}, v_{3k+2}, v_{2\,1}, v_{2\,2}, \ldots, v_{2\,n}, v_{5\,1}, v_{5\,2}, \ldots, v_{5\,n}, v_{3k-1\,1}, v_{3k-1\,2}, \ldots, v_{3k-1\,n}, v_{4\,1}, v_{4\,2}, \ldots, v_{4\,n}, v_{7\,1}, v_{7\,2}, \ldots, v_{7\,n}, v_{3k+1\,1}, v_{3k+1\,2}, \ldots, v_{3k+1\,n}\}$ is the unique strong efficient dominating set of G. $|S| = (2n+1)\,k + 2,\ k \geq 1,\ n \geq 1$.
Therefore $\gamma_{se}(G) = (2n+1)\ k + 2,\ k \geq 1,\ n \geq 1$. Let $e = \{v_1 v_{1\,1}\}$.

Let $G-e=H$. T = $\{v_1, v_3, v_6, \dots, v_{3k}, v_{3k+2}, v_{1\,1}, v_{2\,1}, v_{2\,2}, \dots, v_{2\,n}, v_{5\,1}, v_{5\,2}, \dots, v_{5\,n}, v_{3k-1\,1}, v_{3k-1\,2}, \dots, v_{3k-1\,n}, v_{4\,1}, v_{4\,2}, \dots, v_{4\,n}, v_{7\,1}, v_{7\,2}, \dots, v_{7\,n}, v_{3k+1\,1}, v_{3k+1\,2}, \dots, v_{3k+1\,n}\}$ is the unique strong efficient dominating set of G. $|T| = (2n+1)\,k + 3,\ k \geq 1,\ n \geq 1$. Therefore $\gamma_{se}\,(G-e) = (2n+1)\,k+3,\ k \geq 1,\ n \geq 1$. $\gamma_{se}(G-e) = (2n+1)\,k + 3 > (2n+1)k + 2 = \gamma_{se}(G)$.
Hence $b_{se}(G) = 1\ if\ n = 3k+2, k \geq 1$.

Remark 2.5: Let $G = P_2 \odot \overline{K_n}$. $\gamma_{se}(G) = n + 1$. Let e = uu_1

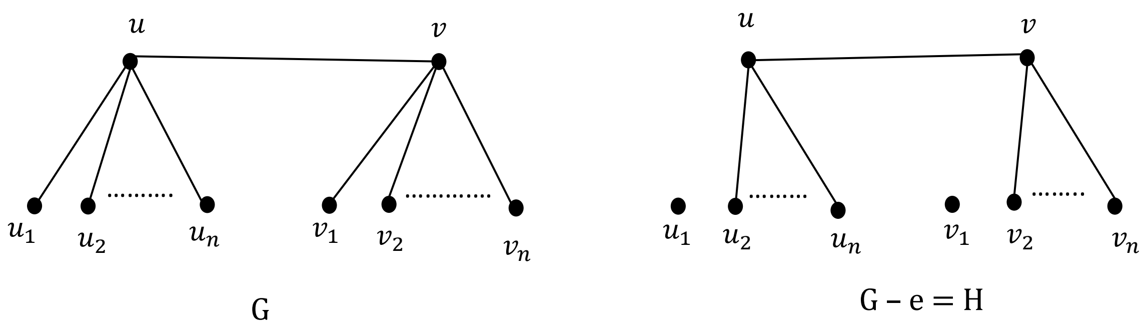


Figure 2.5.

$\{u, u_1, v_1, v_2, \dots, v_n\}$ and $\{v, v_1, u_1, u_2, \dots, u_n\}$ are $\gamma_{se}-$ sets of H. $\gamma_{se}(H) = n+2 > \gamma_{se}(G)$. Therefore $b_{se}(G) = 2$.

Remark 2.6: Let $G = P_3 \odot \overline{K_n}$. $\gamma_{se}(G) = 2n + 1$.

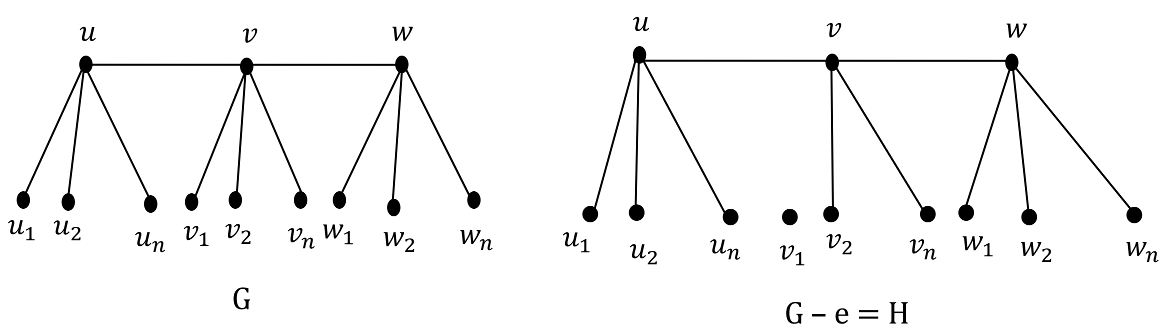


Figure 2.6.

$\{u_1, u_2, \dots, u_n, v, v_1, w_1, w_2, \dots, w_n\}$ is the unique $\gamma_{se}-$ set of H. $\gamma_{se}(H) = 2n + 2 > \gamma_{se}(G)$. Therefore $b_{se}(G) = 1$.

Theorem 2.7: Let $G = DS(P_n), n \geq 2, n \neq 3$. Then $b_{se}(G) = \begin{Bmatrix} 1\ when\ n \geq 6 \\ 2\ when\ n = 2,4,5 \end{Bmatrix}$

Proof: Let $V(P_n) = \{v_i : 1 \leq i \leq n\}$. $\deg v_1 = \deg v_n = 1, \deg v_i = 2\,, 2 \leq i \leq n-1$. Let

$S_1 = \{v_1, v_n\}$ and $S_2 = \{v_2, v_3, \dots, v_{n-1}\}$. Then $S_1 \cup S_2 = V(P_n)$. Let w_1 and w_2 be new vertices.

Case 1: When $n \geq 6$. Let $G = DS(P_n)$. Then $V(G) = V(P_n) \cup \{w_1, w_2\}$. Let $T = \{w_1 v_{i,} w_2 v_j : i = 1, n, 2 \leq j \leq n-1\}$. Let $E(G) = E(P_n) \cup T$. Therefore $\{w_1, w_2\}$ is the unique strong efficient dominating set of G. Therefore $\gamma_{se}(G) = 2$. Let $e = w_1 v_1$. Then $\{w_2, v_n, v_1\}$ is the strong efficient dominating set of G – e. Therefore $\gamma_{se}(G-e) = 3 > \gamma_{se}(G)$. Therefore $b_{se}(G) = 1, n \geq 6$.

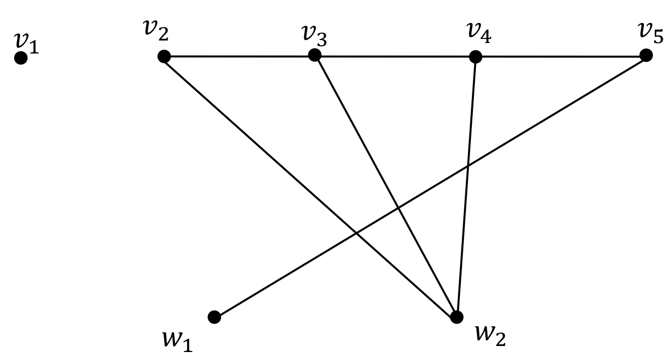
Case 2: When $n = 2$. Then $G = C_3$. Therefore $\gamma_{se}(C_3) = 1$. If one edge is removed, then G is P_3. Again $\gamma_{se}(P_3) = 1$. Hence $b_{se}(G) \geq 2$. If two edges e_1, e_2 are removed then G is $P_1 \cup P_2$. Let $G - \{e_1, e_2\} = H$. Therefore $\gamma_{se}(H) = 2 > \gamma_{se}(G)$. Therefore $b_{se}(G) = 2, n = 2$.

Case 3: When $n = 4$. Let $\{v_2, v_4\}$ and $\{v_1, v_3\}$ are two strong efficient dominating sets of G. Therefore $\gamma_{se}(G) = 2$. Let $e = \{v_1 w_1, v_1 v_2\}$. Let $H = G - e$. $\{v_1, v_3, w_1\}$ is the strong efficient dominating sets of H. Then $\gamma_{se}(H) = 3 > 2 = \gamma_{se}(G)$. Hence $b_{se}(G) = 2,\ n = 4$.

Case 4: When $n = 5$. Let Let $\{w_1, w_2\}$ and $\{v_3, w_1\}$ are two strong efficient dominating sets of G. Therefore $\gamma_{se}(G) = 2$. Let $e = \{v_1 w_1, v_1 v_2\}$. Let $H = G - e$. Then $\{v_1, v_3, v_5\}$ and $\{v_1, v_5, w_2\}$ are the two strong efficient dominating sets of H. Then $\gamma_{se}(H) = 3 > 2 = \gamma_{se}(G)$. Hence $b_{se}(G) = 2,\ n = 5$.

Remark 2.8: When $n = 3$. Then $G = C_4$ which has no strong efficient dominating set.

Example 2.9: Consider the following graph.



$G - e = H$

Figure 2.7.

Let $G = DS(P_5)$, then $\{w_1, w_2,\}$ and $\{v_3, w_1,\}$ are the two strong efficient dominating sets of G. Therefore $\gamma_{se}(G) = 2$. Let $X = \{v_1 w_1, v_1 v_2\}$. Let $H = G - X$ be given in the above figure 2.7. Then $\{v_1, v_3, v_5\}$ and $\{v_1, v_5, w_2\}$ are two strong efficient dominating sets of H. Therefore $\gamma_{se}(H) = 3$. So $\gamma_{se}(H) = 3 > 2 = \gamma_{se}(G)$. Hence $b_{se}(G) = 2$.

Theorem 2.10: Let G = $DS(C_n)$, $n \geq 1$. Then $b_{se}(G) = \begin{Bmatrix} 3 \ if \ n = 3 \\ 1 \ if \ n \geq 4 \end{Bmatrix}$

Proof: Let $V(G) = \{ uv_i: 1 \leq i \leq n\}$ and $E(G) = \{uv_j: 1 \leq j \leq n\} \cup \{v_i \, v_{i+1}, v_n v_1: 1 \leq i \leq n-1\}$.

Case 1: When $n = 3$, $DS(C_3) = K_4$. Then $\gamma_{se}(K_4) = 1$. When one edge e is removed then also $\gamma_{se}(K_4 - e) = 1$. When two edges are removed then for the resulting graph G either $\gamma_{se}(G) = 1$ or $\gamma_{se}(G)$ does not exist. When three edges uv_1, uv_2, uv_3 are removed the resulting graph G is $C_3 \cup K_1$. Therefore $\gamma_{se}(G) = 2 > \gamma_{se}(DS(C_3))$. Hence $b_{se}(DS(C_n)) = 3$, $n = 3$.

Case 2: When $n \geq 4$. u is the unique full degree vertex of $DS(C_n)$. Therefore $\{u\}$ is the strong efficient dominating set of $DS(C_n)$. Therefore $\gamma_{se}(DS(C_n)) = 1$, $n \geq 4$. Let $e_i = uv_i$, $1 \leq i \leq n$. Let $X_i = \{e_i\}$. Let $G = DS(C_n) - X_i$. $\deg u = n-1 = \Delta(G)$. u is the unique maximum degree vertex. $\deg v_i = 2 = \delta(G)$. v_i is the unique minimum degree vertex. $\{v, v_i\}$ is the unique strong efficient dominating set of G. $\gamma_{se}(G) = 2 > \gamma_{se}(DS(C_n))$. Hence $b_{se}(DS(C_n)) = 1$, $n \geq 4$.

Example 2.11: Consider the following graph H = G – e.

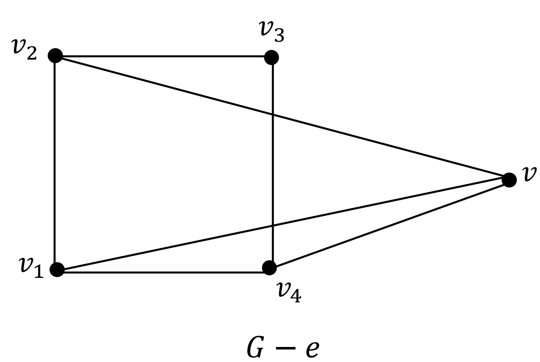


Figure 2.8.

Let G = $DS(C_4)$, $\{v\}$ is the strong efficient dominating set of G. Therefore $\gamma_{se}(G) = 1$. Let e = v_3v . Let $G - e = H$. Then $\{v_3, v\}$ is the strong efficient dominating set of H. Therefore $\gamma_{se}(H) = 2$. Therefore $\gamma_{se}(H) > \gamma_{se}(G)$. Hence $b_{se}(DS(C_4)) = 1, n \geq 4$.

Theorem 2.12: Let G = $DS(K_n), n \geq 1$. Then $b_{se}(G) = n$.

Proof: Let $V(G) = \{v_1, v_2, \ldots, v_{n+1}\}$. $DS(K_n) = K_{n+1}$, $n \geq 1$. Therefore $\gamma_{se}(G) = 1$. $\deg v_i = n, 1 \leq i \leq n$. Each $\{v_i\}$ is a strong efficient dominating set. Atleast n edges should be removed to increase strong efficient dominating number. $b_{se}(G) \geq n$. Let $X_i = \{v_i v_j: 1 \leq j \leq n+1, i \neq j\}$. $H = G - X_i = K_n \cup K_1$. $\gamma_{se}(H) = 2 > \gamma_{se}(G)$. Therefore $b_{se}(G) \leq n$. Hence $b_{se}(G) = n$.

Theorem 2.13: Let $G = T(n,m), n \geq 1, n \neq 3k, k \geq 1$.
Then $b_{se}(G) = \{1 \; if \; n = 3k+1, n = 3k+2, k \geq 1\}$.

Proof: Let $V(G) = \{v, u_i, v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(G) = \{vu_i, vv_1, v_t v_{t+1}, 1 \leq t \leq n-1\}$.

Case 1: Let $n = 3k+1, k \geq 1$. S $= \{v, v_3, v_6, \dots, v_{3k}\}$ is the unique strong efficient dominating set of G. $|S| = k+1$. Therefore$\gamma_{se}(G) = k+1, k \geq 1$. Let $H = G - e_i$ where e_i = vu_i T $= \{u_i, v, v_3, v_6, \dots, v_{3k}\}$ is the unique strong efficient dominating set of H. $|T| = k+2$. Therefore $\gamma_{se}(H) = k+2 > \gamma_{se}(G)$. Hence $b_{se}(G) = 1$.

Case 2: Let $n = 3k+2, k \geq 1$.

Subcase 2a: When k ≥ 2.S= $\{v, v_3, v_6, \dots, v_{3k}, v_{3k+2}\}$ is the unique strong efficient dominating set of G. $|S| = k+2$. Therefore $\gamma_{se}(G) = k+2, k \geq 1$. Let $H = G - e_i$ where e_i = vu_i T = $\{u_i, v, v_3, v_6, \dots, v_{3k}, v_{3k+2}\}$ is the unique strong efficient dominating set of H. $|T| = k+3$. Therefore $\gamma_{se}(H) = k+3 > \gamma_{se}(G)$. Hence $b_{se}(G) = 1$.

Subcase 2b: When k = 1. G = P_{3k+4} = $P_{3(k+1)+1}$. $\gamma_{se}(G) = k+2$. Let e = $v_3 v_4$. Let H = G – e = $P_5 \cup P_{3k-1}$ = $P_5 \cup P_{3(k-1)+2}$ $\gamma_{se}(H) = 3+k+1 = k+4 > \gamma_{se}(G)$. Hence $b_{se}(G) = 1$.

Example 2.14: Let $G = T(7,3)$.The graphs G and G – e are given in the below figure 2.9. $\{v, v_3, v_6\}$ is the unique strong efficient dominating set of G. Therefore $\gamma_{se}(G) = 3$. Let $e = u_1 v$. Let $H = G - e$. $\{u_1, v, v_3, v_6\}$ is the unique strong efficient dominating set of H. Therefore $\gamma_{se}(H) = 4 > \gamma_{se}(G)$. Hence $b_{se}(G) = 1$.

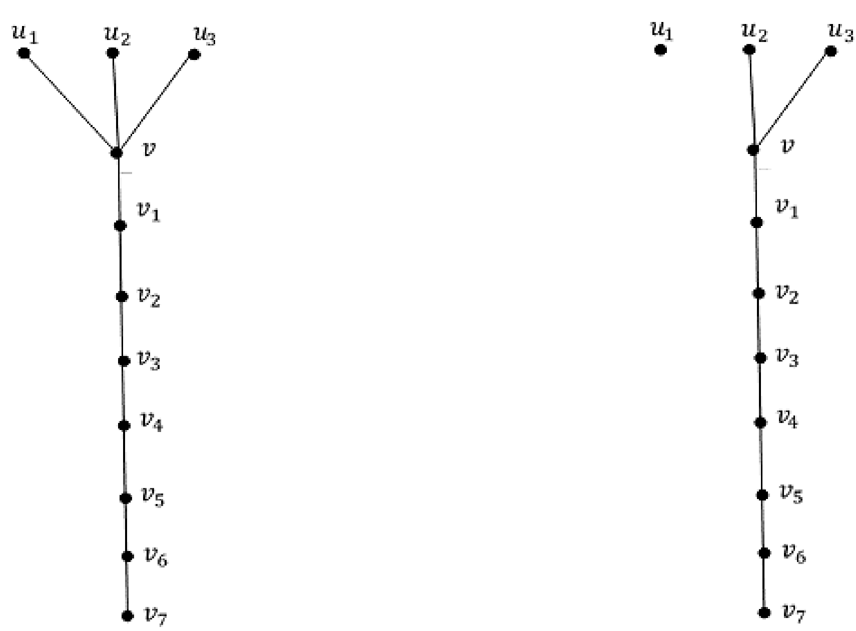


Figure 2.9.

3. CONCLUSION

In this paper, the authors studied the strong efficient bondage number of some path related graphs. Further studies can be made on cycle related graphs. The changing or unchanging the values of strong efficient bondage number when a vertex is removed will be studied. The changing or unchanging the values of strong efficient bondage number when an edge is added or removed from a graph can be found.

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