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Unrestricted division by zero as multiplication by the – reciprocal to zero – infinity

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ABSTRACT

It is demonstrated that quite unrestricted operation of conventional division by the real number zero can be implemented via multiplication by the – reciprocal to zero – ascending infinity in paired dual reciprocal spaces, provided that the "real" zero and the infinity are mutually reciprocal (i.e. multiplicatively inverse). Since the infinity is not an absolute concept independent of the particular circumstances in which it is being determined, the value of the setvalued infinity is not fixed but depends on an influence function that is usually applied for evaluation of integral kernels, the operations proposed here are always defined relative to a certain abstract influence function. The conceptual and operational validity of the proposed unrestricted division and multiplication by zero, is authenticated operationally by fairly simple example, using an evaluation of Frullani's integral.

***Keywords*:** Division by zero, multiplication by infinity, paired dual reciprocal spaces

1. INTRODUCTION

The unwarranted presumption that theoretical mathematics should deal with single-space reality (SSR), has become tacitly the implicitly accepted yet usually unmentioned and therefore never challenged paradigm of traditional mathematics. The SSR paradigm envisaged mathematical reality as if built upon an abstract set-theoretical universe of number points

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resembling thus single set-theoretical space. Nonetheless, my analyses of some formerly unanticipated and previously irreconciled curious experimental results (i.e. till I have reconciled some of them with the use of my new synthetic mathematical approach) strongly suggested that perhaps both the abstract mathematical, and the – corresponding to it – physical reality demand an entirely new approach, which insists on espousing a certain multispatial reality (MSR) paradigm instead of the previous unwarranted SSR paradigm. This paradigm shift is also supported by some rigorously proven – and thus unquestionable within their proper scope, which was nevertheless flamboyantly ignored – operational laws and rules (both algebraic and differential) of the undeniably valid (even if sometimes slightly misinterpreted) differential calculus [1].

There are two basic operationally different and conceptually distinct types of singularity: the nice behaving complex infinitesimal (descending) singularities and the more troublesome neverending infinite (ascending) singularities [2]. The ascending infinity that frequently pops up in the domain of real numbers can lead to obvious nonsenses, such as $1 = 0$ [3], when operations on the infinity and zero are performed the traditional way [4], [5] p.65, i.e. as if the ascending infinity were a kind of bottomless trash bin into which anything could be thrown and just disappear without leaving a trace. It is because some mathematical operations demand the presence of quite unambiguously defined (and respected both operationally and structurally) ascending infinity, which was heretofore homeless or perhaps better: spaceless under the previously unspoken SSR paradigm of the traditional mathematics. Simple classification of singularities is given in [6].

Yet meaningful operations involving both zero and the ascending infinity are possible if they will be performed within paired dual reciprocal spaces [7], [8]. Since unambiguously defined finegrained differential operators [9] virtually demand the reciprocity [10] in compliance with prospective extensions of differential operations [11] and their possible incorporation into quasispatial four-dimensional (4D) structures of spacetime and timespace [12], [13], the implementation of unrestricted division by zero through multiplication by infinity is highly desirable.

I shall show that the reciprocal operations are straightforward to be defined theoretically under conceptual auspices of the MSR paradigm and indeed possible to be implemented operationally within the framework of paired dual reciprocal spatial structures.

Unlike the traditional mathematics that tries to derive theorems from preconceived axioms and allegedly undefinable primitive notions, I shall use the new synthetic approach, which I have successfully deployed for reconciling and explaining some formerly quite unanticipated and deemed (by other experts) as unexplainable, results of unbiased experiments. Due to the fact that the previously unanticipated – yet experimentally confirmed – results cannot be built upon axioms of existing theories, the new synthetic approach to mathematics and physical sciences relies primarily on matching abstract operational procedures to the – corresponding to them – spatial or quasispatial structures. The rationale for the new synthetic approach is that there is no point in creation of arbitrary structures that cannot be actually constructed or in devising of operational procedures that cannot really operate on the structures whose existence is virtually unreal (i.e. merely postulated).

The myth that operational proofs do validate the proven (by the proofs) formulas has been shown to be just that: myth [1]. When a new paradigm removes or replaces some axioms or primitive notions, the proofs that rely on the old axioms or primitives might be deprecated.

2. DOUBLE CHARACTER OF THE ASCENDING SINGULARITY

From conventional definition of infinity as multiplicative inverse or reciprocal of zero

$$\frac{1}{0} = \infty \Leftrightarrow 1 = 0 \cdot \infty \tag{1}$$

in conjunction with the – fairly easy to grasp – intuitive implication that, in a sense, the ascending infinity reveals a sort of split "personality" for besides being itself (i.e. as a multivalued and/or setvalued algebraic operational entity) it must also be representable as the singlevalued inverse of the naturally singlevalued zero

$$\left\{ \infty + \infty = 2 \cdot \infty = \frac{2}{0} \right\} \quad \Rightarrow \quad \{ 2 \cdot \infty = \frac{2}{0} = \frac{1}{0} + \frac{1}{0} = \infty \oplus \frac{1}{0} \} \tag{2}$$

and therefore, it should be representable (not to say exist) in two distinct and different spaces at the same time, which – at the present stage of the theory's development – seems only possible if the two spaces are somehow tied together, forming a dual pair of mutually reciprocal spaces, which spatial structure has been devised with the intent to provide algebraically two different operational views of essentially the same structural entity. The ascending infinity is treated here as setvalued operational algebraic entity that is reciprocal (i.e. multiplicative inverse) of zero – see [7], [8] for details.

The interspatial addition sign $\oplus$ indicates the necessity to place the infinity representing inverted zero: $\infty \oplus \frac{1}{0}$ in a separate spatial structure from that of space in which the infinity natively dwells, and thus requires conversion from its native algebraic basis to a foreign basis to be performed. Although the inverse zero that stands for the ascending infinity cannot be represented directly in the very same space in which the zero dwells (i.e. wherein the zero is natively housed), the 1/0 cannot dwell in the same space wherein the infinity dwells (i.e. wherein the infinity is directly representable and thus natively housed) either.

This conclusion means that there must always exist at least two pairs of paired reciprocal dual spaces. Besides, the interspatial sum implicitly designates also the need to convert the term 1/0 to the native algebraic basis of the space into which the term is being temporarily moved.

The rightmost term on the RHS of the implication (0x) also suggests that the ascending infinity is not a singlevalued algebraic entity that fits all cases but its effective value should depend on the specific circumstances of the given case under investigation and thus its particular case-dependent representation should be cast upon the influence function that – for most practical purposes – best describes the particular case.

For example: the ascending infinity that could pertain to the energy spectrum at the Planck scale foreseen for use in string theories may be different than that envisaged for probing of subatomic particles.

Since the neverending ascending infinity is naturally characterized as a certain setvalued algebraic entity, the effective value of the entity called infinity cannot be fixed to any particular single value, which would be valid for all phenomena.

But the algebraic operations devised to be performed on objects represented within the algebraic structures that correspond to the operations should be identical. The same operations should be suitable also for handling the ascending infinity as the natural reciprocal of zero, I surmise.

3. NEW PRODUCT DIFFERENTIATION AND INTEGRATION RULES

The well-known from traditional introductory differential calculus regular scalar product differentiation rules (RSPDR1) for compounded scalarvalued functions u(x) and v(x) that depend on the very same formally independent variable x, read:

$$(uv)' = uv' + vu' \quad \Leftrightarrow \quad d(uv) = udv + vdu \tag{3}$$

where the primed terms u' and v' denote derivatives of the functions u(x) and v(x), respectively, and – as usual – the letter d signifies differential of the varying function that follows it [14] p.134. The number 1 in the moniker RSPDR1 means that the two equivalent rules pertain only to first order derivatives and/or differentials. The RSPDR1 on the left-hand side (LHS) of the equivalence formula (3), which is expressed in terms of derivatives, is conceptually equivalent to the RSPDR1 formula on the RHS of (3) that is shown in terms of differentials. The relation between the differential du(x) = u'(x)dx and the derivative u'(x), which is commonly understood as just limit (of the ratio of differentials of dependent and independent variables: u'(x) = du(x)/dx), is simply explained in [14] on p.228. The RSPDR1 formulas are based upon binomial evaluation of compounded functions [15] p.87 usually attributed to Leibniz; see also [16] p.275. The meaning of the RSPDR1 formulas is that the differential – i.e. the infinitesimal rate of change – of one function is compounded with the other function unchanged and the compounded terms are then added together.

Although the Leibniz's rule that underlies the RSPDR1 formulas was rigorously proved [17], and thus is neither inherently faulty nor incorrectly stated, and yet under the SSR paradigm it has produced also an operationally unacceptable (contravariant) differential expression when it was used to produce/compound two scalar functions [2], [1]. Therefore, I have investigated this issue further in terms of integral kernels. For it is unusual that proven operational rule yields in addition to an admissible (covariant) differential expression also an operationally disallowed (contravariant) differential term. We tend to assume that if a proven rule is applied then it must yield correct results, but this is not always the case [1]. Euclid, the father of rigorous derivations and the king of proofs, if you will, is dead and so are some formerly proven theorems. Yet most mathematicians deny the fact and prefer to tacitly cover up their ambiguous flops. In order not to keep the reader in suspense, I should say that it is not mathematics that allows the flops to happen, but the traditional way of doing mathematics under the unwarranted SSR paradigm is the real – though notoriously undisclosed – culprit.

Using K(t,[x]) as an influence function [18] of independent variable t (and of a functional set [x]) is used to form the integral kernel k(x) = f(t)▫K(t,[x]) according to the RSPDR1 pattern

$$k(x) = f(t)▫K(t,[x]) \tag{4}$$

of a certain auxiliary scalar function f(t)=t combined with the influence function K(t,[x]) of an abstract (often quite arbitrary) integral transformation K(t,x), I have generalized some consequences of the resulting integral kernel k() applied to the pair of scalar functions represented in paired mutually dual reciprocal spaces [1]. The multiplicative compounding symbol ▫is just an extra separator preventing the pretty common occurrence of grave mistakes that have previously been committed in former mathematics, whether inadvertently or not. Lack of the notoriously omitted extra separator permitted tacit cover-up of nonsenses. For brief introduction to kernels of integral operators see [19], [20], [21] p.604f; for more comprehensive discussion see [22]; compare also summary of the integral kernels' relevance to incomplete elliptic integrals [23] also when cast in paired dual reciprocal spaces in [2] and [1]. Functionals in formulas and equations shall be shown in brackets, as usual.

Disambiguation: when I use the term 'functional' as noun (rather than as adjective) and denote the variable x or function x() as [x], for example, it means that the variable/function x enclosed in brackets [x] is not being treated as an actively varying variable/function, at least temporarily during the given operation of differentiation or integration.

While the auxiliary function f(t) is usually required only to be measurable [24], some authors impose on the influence function K(t,x) the demand that it should be complete [25]. Note that some authors consider the influence function K() as the integral kernel itself, because the character of the abstract auxiliary function f(t) is not quite clear under the SSR paradigm, but its formal correspondence to Green's formulas suggests that it should be regarded as being reciprocal, as I do for the sake of its compliance with the MSR paradigm. If the influence function K() is called the integral kernel then the whole topic of integral kernels of operators is tacitly obfuscated, even if it is done quite inadvertently. To the best of my knowledge – or perhaps to my recollection, I should say – it was rather unusual to insist on having the total compounded kernel k() properly differentiated if it has to serve as an integrand and to be eventually integrated, as I have explicitly demanded in [1]. In order to enforce the latter demand, the traditionally neglected use of the separating symbol ▫that I had introduced is thus imperative for the sake of ensuring correctness of the derivations. The notoriously disregarded traditional ambiguity is absolutely inadmissible to me.

To be properly understood let me recapitulate: The RSPDR1 formulas (3) is quite correct if all the variables involved in the kernel function k() are directly representable within the very same spatial structure. If even just one variable used in k() is reciprocal to some other variable(s) then the SSR paradigm is inadmissible (because the reciprocal variable cannot be properly represented directly within the primary space but should be depicted in the paired reciprocal space that is dual to the given primary space) and – consequently – the RSPDR1 formulas must not be applied but the new MSR paradigm should be invoked and pertinent MSR-based formulas should be applied. Now comes very brief restatement of some formerly unrecognized wrongs tacitly buried in the topic of integral kernels, which I have already tackled extensively in [1].

By analogy to the RSPDR1 formulas (3) I have derived in [1] multispatial scalar differentiation rule (MSPDR1) based on multispatial evaluation of the integral kernel derived from the compounded derivative with respect to the actively varying independent variable t:

$$\{u(t) \,\text{▫}\, v(t,b)\}'_t = u(t) \,\text{▫}\, v'_t(t,[b]) \oplus \frac{1}{u'_t(t) \,\text{▫}\, v'_t(t,[b])} \tag{5}$$

and then also the corresponding to it multispatial scalar product integration rule (MSPIR1)

$$\int_0^\infty d\{u(t) \,\text{▫}\, v(t,[b])\} = \int_1^\infty u(t) \,\text{▫}\, v'(t,[b])dt \oplus \int_0^1 \frac{1}{u'(t) \,\text{▫}\, v'(t,[b])dt} \tag{6}$$

which are intended for operations performed on paired dual reciprocal spatial or quasispatial structures. The far RHS parts of the above formulas contradict the usual formulas used in traditional mathematics, which relies on the – formerly unmentioned and thus unchallenged –

unwarranted SSR paradigm. The extra separating symbol▫that was not used in traditional mathematics makes it possible to see the deception of the latter. I do not think that omission of the extra symbol indicates conspiracy to conceal the multispatial reality, but it was certainly convenient to neglect it and permit anyone to make evasive derivations without giving explanations why former mathematics was occasionally either inconsistent or nonsensical. The readers are free to believe whatever they choose, of course. But I am not disposed to cover up such inherited stupendous mistakes, regardless of whether these had been made inadvertently or not.

Notice that only the running variable t is actively varying therein whereas the set of variables codetermining the integral transform v(t,[b]) denoted by the functional [b] remains unchanged during this differentiation process. I am using now the set of variables [b] instead of the usual set of variables [x] that is commonly used in the theory of integral kernels, just in order to avoid possible confusion [1]. For the same reason, the influence function v(t,[b]) of an integral transform v() shall be used henceforth instead of the influence function K(t,[x]) of an integral transform K() that are conventionally used in the traditional theory of integral kernels which was developed under auspices of the unwarranted SSR paradigm. For although the functions u() and v() resemble the functions f(t) and K(t,[x]) of an integral transform K(), they may not necessarily be directly related to integral kernels, but henceforth shall be cast within the conceptual and operational framework of the MSR paradigm.

The evaluation of the rule MSPIR1 (6) was founded upon the derivative (5) because if the differential d{u(t) ▫v(t,[b])} is intended to be used as the integrand in the leftmost integral on the left-hand side (LHS) of (6) then it should be obtained from the differentiation (5), which requirement was not always explicitly enforced in the traditional mathematics.

The latter neglect is the main reason why we often encounter so many conceptually deficient and even operationally faulty evaluations of differentials and the integrals – that are based upon the differentials – as well as tacitly veiled inconsistencies in the allegedly rigorous but frequently inconsiderate former mathematics.

The encircled plus sign ⊕ signifies interspatial sum for addition involving two or more incompatible in general terms, each of which reside in different spaces. Recall that reciprocals are particular examples of the incompatible variables. It also symbolizes pairing of dual reciprocal spaces, because the second integral belongs in the dual reciprocal space paired with the given by default primary space – see [1] for more comprehensive explanations. In the sense the regular plus sign is reserved only for the regular, intraspatial sums. The intraspatial operations are performed entirely within a single primary 3D space, in which all the active as well as inactive variables are directly represented in the native algebraic basis of the space.

Recall that although reciprocal variables, which belong in the secondary, paired reciprocal space, can be indirectly represented in the primary space upon conversion, they do not really dwell in the primary space permanently and thus are not represented directly therein. In other words: reciprocal variables are not housed in the primary space but can be hosted in it upon conversion of their native direct representation to their indirect representation in the native basis of the primary space.

Although it is certainly not an illegitimate operation to try to integrate any compounded product of functions, the integral of a product that did not emerge from prior legitimate differentiation has no direct relevance to the compounded product, but is related to the usually undisclosed functions from multiplication of which the product might have emerged if the omitted differentiation were actually performed.

Hence, I am not talking about absolutely wrongful operations but about some notoriously undisclosed, deceptively inappropriate derivations when the conclusion drawn from such an integration was presented as if it was characterizing the compounded product of functions, as it was often the case.

Arbitrary compounding or making multiplicative composition of scalar functions is widespread and was usually uncontrolled in the past, insofar as I can tell. Not only simple composition but even quadratic and bilinear composition was usually declared as natural algebraic operation – see [26], for example, but if the operation is not examined also in terms of differentials as well as in terms of prospective integrals, which can then be confronted with results of pertinent physical experiments, then we cannot be sure what is actually produced by such – often quite arbitrary – multiplicative compounding.

4. DIVISION BY ZERO IMPLEMENTED AS MULTIPLICATION BY INFINITY

I expect that algebraic multiplication of <u>any</u> positive integer n by the ascending infinity (which operation is equivalent to an algebraic division of the integer n by zero) should yield

$$N_{n\uparrow}^{n\cdot\infty} \coloneqq n \cdot \infty = \frac{\mathrm{n}}{0} \eqqcolon N_{n\uparrow}^{n/0} \Leftrightarrow \sum_0^n \{[\int_1^\infty tv'([x],t)dt] \,|\, \text{ȸ} \oplus \left[\int_0^1 [\frac{1}{v'([x],t)dt}]\right] |\varphi\} \qquad (7)$$

which is operationally similar to adding the primary integral resembling formwise incomplete elliptic integral of the second kind to the paired inverted integral resembling incomplete elliptic integral of the first kind, respectively [2], [1]. Identifying infinity with the inverse of zero, such as $n \cdot \infty = \frac{\mathrm{n}}{0}$ and the resulting $0 \cdot \infty = 1$ and even their divisions, such as $\frac{\infty}{\infty}, \frac{0}{0}$ is not entirely new, but it was rather avoided and usually advised to first take their logarithms – see [27], as the SSR paradigm is unsuitable for any meaningful direct algebraic operations on infinity. Yet under the MSR paradigm one may operate directly on infinity – see [7], [8]. Recall that $\frac{0}{0} = \frac{\infty}{\infty} = 1$ for the conventional division by zero in compliance with the logical intuition that dividing any entity by itself must yield unity.

Since multiplication by infinity does designate a point at infinity with respect to the positive integer n and to the influence function v(x,t), the whole operation (7) can be denoted in shorthand notation as $N_{n\uparrow}^{n\cdot\infty} \coloneqq n \cdot \infty = \frac{\mathrm{n}}{0} \eqqcolon N_{n\uparrow}^{n/0}$ by analogy to the point at infinity that is conventionally denoted by A_∞^0 when it is used in reference to affine spaces [21] p.28. The up arrow means that for now the positive integer n is progressing strictly upwards. Expanding the scope of operations presented here shall be done elsewhere on as needed basis.

The resulting two integrals on the RHS of (7) are (formwise) equivalent just like the preliminary conceptual evaluation of the intuitively clear symbolic implication

$$\infty + \infty = 2\cdot\infty \Rightarrow 2/0 \qquad (8)$$

seems to suggest. Their interspatial equivalence has been operationally confirmed in [1] after being conceptually introduced and theoretically prepared in [2].

Although the order of the variables in the prospective derivative v'([x],t) of a background/influence function v(x,t) is immaterial, it signifies the fact that the set of temporarily frozen background variables [x] is just a functional during the operations performed in the prototype procedure shown in the formula (7) supplying problem-specific background for the actively varying auxiliary variable t. The background is usually specified by a function of the conventional spatial variables x,y,z, in 3D spaces.

The formula (7) conforms to the chain of expressions extracted from the formula (2) for n = 2. As usual, the encircled plus sign ⊕ that signifies interspatial addition when the sum involves some incompatible terms. It is used when the singlevalued zero and the essentially multivalued ascending infinity cannot be directly represented in the very same space [1]. Yet the value of the interspatial plus sign ⊕ can effectively become negative too, in which case it would actually turn into interspatial subtraction, in which case it can be denoted by the symbol ⊖, as it depends on evaluation of the functions involved.

Nevertheless, having both zero and the ascending infinity as its inverse: $\infty=1/0$ in the same operational formula permits us to operate on them as if they were placed in the same (operational) basket; and that is what ultimately counts when it comes to realistic applications. The operational basket is not necessarily identical with space, even though it does correspond to a certain geometric structure commonly called space, and thus sometimes it may be informally identified with the particular space that the operational basket corresponds to. The spaces (i.e. geometric or quasigeometric spatial structures) on the other hand, are very important for interpreting the operational results and for making conceptual derivations. At this stage of the development I am not pushing for making *a priori* axiomatic identification of various algebraic and structures, however. It shall be done elsewhere.

The problem-specific derivative v'[t] of a generic functional v[t] is treated as if it were a function of a certain actively varying/running variable t pertinent to the problem at hand. This prototype formula (7) assumes that ascending infinity dwells in its own dual reciprocal space Ɐ|ȹ that is denominated in units of the inverse, dual reciprocal basis ȹ which is native to the dual reciprocal space Ɐ|ȹ i.e. ∞∈Ɐ|ȹ; Ɐ is a dual set reciprocal to the primary set A that is equipped with its own native primary basis ȸ to form the primary space A|ȸ wherein the zero dwells: 0∈A|ȸ provided that $(1/0) = \infty$ holds true. However, before the interspatial operation can be performed, the foreign (to the primary space) variable ∞ needs to be converted into the native algebraic basis of the chosen destination space (that may be the primary space too), in which both operands are supposed to be operated on – see [7], [8] for examples.

This and subsequent prototype formulas to be offered in what follows were inspired by the treatment of infinite series [28] and integrals, especially by the unsolvable in general nonlinear differential equations involving incomplete elliptic integrals. Their treatment was expounded in numerous academic textbooks, especially in [29], [30], [31], [32], [33], [34]; compare also [35], [36], [37].

The form of the above integrands is inspired by many, of course, but mainly by works of George Pólya on complex descending singularities that have already been discussed briefly in [2]. His rigor is strict and his ingenuity is astonishingly deep and is truly inspirational and was conceptually very impressive to me. The most persuading operational (as opposed to conceptual) reason for pairing of dual reciprocal spaces was supplied by his ingenious paper on decomposition of series [38] – compare also brief explanation in [2]. See also his other papers related to the extremely important issue, especially [39], [40], [41], [42].

If his work on this topic and especially the aforementioned papers were not followed immediately it is not because they are deficient. Quite on the contrary. Even though he did not question the SSR paradigm, to the best of my knowledge, he has outrun conceptually everyone in the field and outclassed everything else I did read on the topics. Only when I saw and reread all his papers written on the subject collected in just one volume, I have realized how great was his mathematical intuition and how far he might have advanced the wobbling traditional mathematics. Unfortunately, his genius was constrained by the imperceptible confines of the unspoken SSR paradigm; or else, he could have written this paper too.

5. ESTIMATING THE CONCEPTUAL VALIDITY OF THE PROPOSED ABOVE UNRESTRICTED DIVISION BY ZERO

Incidentally or not, the formula (7) resembles the final outcome of Frullani integral:

$$I = \int_0^\infty \frac{[f(Bx)-f(Cx)]}{x}\,dx = [f(0) - f(\infty)]\ln\frac{C}{B} \tag{9}$$

provided the functional f(∞) exists. Here B and C denote constant functionals, see [43], [44], [45], [46], [47]. Joseph Edwards has derived the Frullani integral from an area integral and gave very good intuitive explanation even though with the final functionals reversed [48], while Arias-de-Reyna also followed the historical development of the topic [49], and so did also Ostrowski who generalized it [50]; compare also [51]. The integral is also briefly mentioned in [52], without any conceptual explanation though.

Why have I chosen the Frullani integral for estimating the degree of validity of the unrestricted division by zero that is implemented as multiplication by infinity? My primary reason for doing that is that the integrand of this particular integral complies formwise with the integral kernel k() shown in the formula (4), whose numerator contains a certain influence function K(t,[x]) and the denominator contains the auxiliary function f(t) = f(1/x) that happens to be the inverse of the preliminarily "frozen" variable – i.e. of the functional [x] – of the influence function. The similarity is apparently only formwise, but that is exactly what is needed now to properly understand the new interpretation of the concept of representing scalar functions in either covariant or contravariant form, as I have proposed in [53] and then furthered their distinction in [2].

Finally, I have deprecated the – fake but still celebrated – controversy of covariance versus contravariance (which has reached its utterly confusing peak under the SSR paradigm) in [1], where I have shown that although the distinction between these representations is necessary, the virtual pronouncement that covariance is "good" and contravariance is "bad" is yet another flop of traditional mathematics. Both of them are fine as long as they are housed within their proper spaces, but not when compounded within the very same space. Einstein was fighting shadows, when he demanded covariance, because when you place each representation in the native basis of the (properly chosen) space, all such representations are covariant by default. Einstein would not have to fight the battle if the MSR paradigm would have been known and espoused. I am not faulting Einstein, whose great intuition and imagination I admire, but he was presumably just as confused by the infantile traditional approach to mathematics as anyone else back then. So, for the sake of intellectual sanity, let us forget – for a moment or two – the unwarranted prohibition of division by zero and review the options that Einstein, and many other realistic ingenious thinkers, were denied.

If the Frullani integral (9) would be rewritten in terms of a dual reciprocal space that is native to the reciprocal variable t=1/x then it would have the following yet identical form

$$\int_{1_C=t\cdot 0}^{1_B=t\cdot\infty} \frac{\left[f\left(\frac{B}{t}\right)-f\left(\frac{C}{t}\right)\right]}{\frac{1}{t}} d\left(\frac{1}{t}\right) = \left[\left(f(\infty) - f(0)\right)\ln\frac{C}{B}\right]_{t\cdot 0=t/\infty}^{t\cdot\infty} = \left[f(t\cdot\infty) - f\left(\frac{t}{\infty}\right)\right]\ln U \quad (10)$$

when 1/0 = ∞ provided the functional f(∞) exists. Notice that I am not substituting t=1/x which would imply dx = – (dt/t^2), but I am merely rewriting the Frullani integral as if it were evaluated within a certain reciprocal space Q that is dual to the primary space P in which the variable x is being natively housed and thus directly represented.

Notice that – given the fact that from the functional f(∞) – the upper boundary of the integration in (10) actually evaluates to a certain value $\frac{1}{t} = \infty \Rightarrow 1 = t\cdot\infty \coloneqq 1_B$ and from the value of the functional f(0) the lower boundary of the integration (10) actually evaluates to a certain value $\frac{1}{t} = 0 \Rightarrow 1 = t\cdot 0 = \frac{t}{\infty} \coloneqq 1_C$. These evaluations are due to the fact that the reciprocal variable 1/t in (10) actually varies inversely to the variable x used in (9). This means that the integral (10) is negative with respect to the integral (9) and therefore, its boundaries of integration are reversed, so that the resulting functional (understood as noun, not adjective) in the formula (9): [f(0) – f(∞)] is effectively reversed in the formula (10): [f(∞) – f(0)]. The reader can consult [54] p.14f for very concise explanation, if refreshing of calculus is desired. Perplexing as it may seem, I really like the kind of mathematics that can handle infinity too.

As the logarithm lnU = const in our rendition of the Frullani integral, its value is irrelevant to our explanations at this time; it shall be discussed elsewhere for it pertains to evaluations of scalar potentials. The traditional approach to radial scalar potentials is unacceptable because of successful physical considerations based upon my reconciliation of few formerly unanticipated results of some previously unreconciled experiments [55], [56], [57].

The issue of further splitting ranges of the integrals, namely: $\int_0^\infty = \int_0^1 + \int_1^\infty$ has already been concisely discussed in [54] p.14f, and was briefly exemplified also in [1].

It is easy to see that the reformulated Frullani integral (10) appears to be in compliance with the proposed above prototype formula (7) for realistic algebraic operation involving the previously dreaded unrestricted division by zero that is clearly equivalent to multiplication by ascending infinity, even though Frullani – as almost everyone else until quite recently – operated under auspices of the previously unspoken SSR paradigm. It is important, however, to emphasize the fact that the MSR paradigm does not invalidate everything that was developed under the SSR paradigm, but it surely expands the latter conceptually as well as operationally (i.e. algebraically/procedurally) and also structurally/geometrically.

6. DIVISION BY INFINITY EQUATED TO MULTIPLICATION BY ZERO

By analogy to the prototype formula (7) that equated division by zero with multiplication by the ascending infinity I can also propose prototype of its inverse formula, namely multiplication by zero implemented via (and equated to) division by the ascending infinity:

$$N_{n\uparrow}^{n\cdot 0} \coloneqq n\cdot 0 = \frac{\mathrm{n}}{\infty} \eqqcolon N_{n\uparrow}^{n/\infty} \Leftrightarrow \sum_0^n\{\, 1/\, [\int_1^\infty tv'([x],t)dt]|\text{ȸ} \oplus \left[\int_0^1 [\frac{1}{v'([x],t)dt}]\right] |\varphi] \,\} \quad (11)$$

with the above conditions in place. The change of boundaries of integration in the above integrals is pretty well explained and exemplified in [54] p.14f. Since the proposed new multiplication by zero equates to division by infinity, it too does designate yet another point at infinity with respect to the positive integer n and the influence function v(x,t). Hence it can also be denoted in shorthand notation as $N_{n\uparrow}^{n\cdot 0} \coloneqq n \cdot 0 = \frac{\mathrm{n}}{\infty} \eqqcolon N_{n\uparrow}^{n/\infty}$ just as it was done above for multiplication by infinity.

Despite the unfounded claim of traditional mathematics that any number multiplied by zero succumbs to zero, the formula (11) suggests otherwise and rightly so. If it were true that $n \cdot 0 = 0$ then the assumption that zero is inverse of infinity and vice versa would generate paradox, for it would hold only for n=1, contrary to my original assertion that it should hold for any positive integer number n. The absurdity of the traditional claim that $n \cdot 0 = 0$ is clear:

$$\left\{\frac{n}{\infty} = 0\right\} \Rightarrow \left\{\mathrm{n} = 0 \cdot \infty = \frac{0}{0} = 1\right\} \Rightarrow \{n = 1\} \tag{12}$$

which is indefensible by any logical standard. The misguided traditional assumption $n \cdot 0 = 0$ is just a tendency or an approximation of pseudomathematical reasoning, even though it is still endorsed all the way up to doctoral seminars. The formula (12) might belong to pathology of mathematical reasonings, if such a department would be actually created to study pervasive misconceptions perpetuated in traditional mathematical theories.

This folly of traditional mathematics is operationally disgusting not only to me but it was also troubling others. Louis Couturat, for one, has already questioned the validity of the thoughtless traditional assumption that $n \cdot 0 = 0$ on logical grounds [58]. Compare also his formal yet concise discussion of the relevant topics in [59] p.22. It would also defy the deduction theorem which states that in order to prove the implication P⇒Q we will generally assume P and then prove Q [60]. Therefore only the assumption (0/0) = (∞/∞) = 1 is logically admissible because anything divided by itself equals to unity by default. The thoughtless traditional assertion that a·0 = 0 for any real number a∈ℝ is thus untenable. The conceptually invincible assertion that 1 = (0/0) = 0·(1/0) = 0·∞ = 1 is the most logical foundation for the offered above algebraic operations on zero and infinity.

For similar evaluation of the LHS of the formula (11) implies quite correct implication:

$$\left\{\frac{\mathrm{n}}{\infty} = n \cdot 0\right\} \Rightarrow \left\{\mathrm{n} = n \cdot 0 \cdot \infty = n \cdot \frac{0}{0} = n \cdot 1 = \mathrm{n}\right\} \Leftrightarrow \{n = n\} \tag{13}$$

without generating an obvious paradox, provided that the conventional division by zero (CDbZ) – as opposed to the unconventional one [61] – is explicitly assumed, in which case (0/0) = (z/z) = 1 wherein z stands for any imaginable algebraic number or any algebraic entity whatsoever. This assumption is always true because anything divided by (or compared with) itself must yield 1 or identity. For if it were not, then we would have either to reinvent logic from scratch, or see a shrink treating dissociative identity disorder (also known as split- or multiple personality disorder in the past). For if you, for example, compared with yourself would not be quite identical, then you may have undiagnosed mental disorder. The evaluation (13) demonstrates thar the CDbZ is quite correct and yields logically consistent implications.

Note that the unconventional division by zero (UDbZ) that declares (0/0) = (z/z) = 0 for any complex number z would not appear abnormal if it were cast within two distinct abstract mathematical (i.e. operational and/or structural) realities folded upon each other. For if the zero in numerator would reside in one reality and the zero in denominator would sit in the other reality then the zeros would not be directly/immediately comparable as not being seated in the same operational basket. Yet in the latter case the UDbZ should read $\frac{0_A}{0_B} = Null$, thus clearly indicating that the zeros sit/belong in distinct and presumably different spaces, in order to avoid possible confusion.

Hence, I am not discarding entirely the idea of UDbZ, but I ought to point out its yet still unfinished allocation of the operational entities it tries to deal with. To me, the UDbZ might also reveal the need for presence of a certain twofold mathematical reality just as the CDbZ does (see [1], [2], [10], for details). But at present, the UDbZ theory is still too immature to be taken quite seriously, unfortunately, despite its great conceptual possibility to rectify some still unaddressed questions of traditional mathematics. Yet because of the UDbZ's almost statutory disregard for infinity, which entity is virtually folded upon zero, the theory effectively throws out its proverbial unborn yet baby (i.e. the ascending infinity or whatever alternative to the infinity will be eventually conceived therein) with the proverbial dirty mathematical bathwater. Besides, both these theories, CDbZ and UDbZ, challenge the inconsiderate prohibition of division by zero, each on different theoretical grounds, though.

7. OPERATIONS INVOLVING ZERO AND UNCOUNTABLE REAL INFINITY

By analogy to the above formulas (7) and (11) for positive integer n, for a real number a∈ℝ and an integer functional E[a]≤[a] (i.e. the value of the Entire/Floor of real functional [a] of the real number 'a') and for the excess Ɛ≔ (a-E[a]) over [a] I am offering the expression

$$Ɛ_{n\uparrow}^{n\cdot\infty} \coloneqq Ɛ\cdot\infty = \frac{Ɛ}{0} \eqqcolon Ɛ_{n\uparrow}^{n/0} \Leftrightarrow \int_0^{Ɛ} \{[\int_1^{\infty} tv'([x],t)dt]\,|ȸ \oplus \left[\int_0^1 [\frac{1}{v'([x],t)dt}]\right]|ȹ\}dh\,|ȸ \qquad (14)$$

and an inverse/reciprocal prototype formula for the excess Ɛ over [a] can also be offered as

$$Ɛ_{n\uparrow}^{n\cdot 0} \coloneqq Ɛ\cdot 0 = \frac{Ɛ}{\infty} \eqqcolon Ɛ_{n\uparrow}^{n/\infty} \Leftrightarrow \int_0^{Ɛ} 1/\{\int_1^{\infty}[tv'([x],t)dt]\,|ȸ \oplus [\int_0^1 \frac{1}{v'([x],t)dt}]|\,ȹ]\}\,dh|ȸ \qquad (15)$$

with the same operational conditions as above. The independent external variable h is yet another parameter i.e. running/dummy variable just like t, in the chain of functionals serving as the integrand of the outermost integral. Notice that the inversible operands should be converted from the dual reciprocal basis |ȹ to the primary algebraic basis |ȸ that is common to all the expressions evaluated above in order to fit into the same operational basket.

The equations (7), (11) and (14)-(15) reveal (and somewhat justify, if you will) the operational reason for deploying the method of pairing of dual reciprocal spaces. For unlike the differential v'(t)dt which makes sense as being covariant, the reciprocal differential dt/v'(t) would not make operational sense if it would be evaluated in the same basis |ȸ in which the differential v'(t)dt is being evaluated without prior conversion, because without specifying the algebraic basis in which it is portrayed it would appear as being not really covariant. Therefore, the reciprocal differential, which dwells in the dual reciprocal space Ɐ|ȹ equipped with the dual reciprocal basis |ȹ must be recast/converted into the primary basis |ȸ of the primary space A|ȸ in order for the operations to be meaningful and thus succeed.

Compare also brief discussion of the issue of covariance versus contravariance given in [53] for transformation of contravariant representations of vectors. Notice that vectors should not be labelled as covariant or contravariant, but their formal representations can. Since there is a lot of confusion when it comes to the issue of covariance versus contravariance, let me briefly outline some origins of the confusion that arose at the time when Einstein tried to reconcile his concept of 4D spacetime with the Kaluza's proposal that 5D theory can merge gravitation with electromagnetism into one, classical unified theory, at least mathematically. Einstein & Mayer proposed that at each point of a 4D vector space there is also a 5D linear vector space built such that its contravariant vectors are determined by five numbers [62].

I shall discuss this issue elsewhere, because it is too extensive and rather irrelevant to this presentation. In any case, their abstract proposal could be in preliminary informal agreement with the concept of paired dual reciprocal spaces that is espoused here, if they had dumped the stifling SSR paradigm.

Note that to evaluate the full expression $a\cdot\infty$ the formulas (7) and (14) must be added and for the full expression $a\cdot 0$ the formulas (11) and (15) must be added. As before, the interspatial scalar multiplication indicates the need for conversion of one operand into the native basis of the destination space before the integrals involved there can be meaningfully evaluated. The conversion is not an option but an unavoidable necessity, because ∞ is setvalued entity whereas 0 is singlevalued real number.

For the conversion is not really about evaluation but rather about the unambiguity of each representation of the two algebraic entities. The evaluations are supplied by the prototype formulas (7), (11) and (14)-(15) expressed in the context of a certain problem-specific influence function K() or v(), which effectively should be cast in the <u>differential</u> <u>operational</u> framework of the MSR paradigm. A merely algebraic operational framework is not always satisfactorily precise because it usually permits only very shallow evaluations without the geometric intricacies inherent in differentials. Nevertheless, it is important to realize that the operational ascending infinity is not an absolute concept. Its procedural (as opposed to structural) implementation is always relative to a particular formulation of the problem at hand [63], [64].

Just as solutions of equations in nonstandard analysis depend on choice of infinitesimal [65], which is essentially the descending infinity, so also prospective solutions to problems involving the ascending infinity depend mainly on one's choice of the influence function. This situation is similar to the hyperreal numbers whose set includes infinite integers [66], which are used in the nonstandard analysis that attempted to rescue the traditional mathematics from itself, yet without actually pinpointing or just admitting the presence of the fundamental follies of the latter.

That is why the prospective prototype formulas are evaluated with respect to an influence function v(t,x) that virtually outlines the domain and the range to which the ascending infinity is being applied. Analogous formulas for complex and hypercomplex numbers systems can also be offered, because the latter numbers are actually supplying extra multilinear extensions of the real numbers system into additional algebraic dimensions.

The prototype formulas proposed above are not the only possible, of course, but are simplified examples offered just in order to make them easily comprehensible even if perhaps not quite well understood yet. Perhaps at this present stage of the mathematical reasoning the conceptual distinction between comprehension and understanding could be explained as follows: one needs comprehension to talk about a subject matter without necessarily understanding it; I am paraphrasing Richard P. Feynman who said that nobody really understands quantum mechanics even though people talk about it and some even give lectures on the subject.

A mathematically advanced and alert reader might have already realized that the integrals used in the above eqs. (7), (11) and (14)-(15) are formwise quite analogous to incomplete elliptic integrals of the 1st and 2nd kind – see [2] for details. If the running/dummy variable or parameter t is being accumulated along a 2D curve then the function f(t) could contain square root, in which case the analogy to incomplete elliptic integrals, which are unsolvable in general, would be very wideranging indeed. This means that – with some exceptions – the ellipticlike integrals present in the eqs. (7), (11) and (14)-(15) are perhaps unsolvable in general within the SSR setting if the virtually infinite accumulation takes place along 2D line segments or 2D curves and involve some incompatible/reciprocal variables.

The insolvability (in general case) of the elliptic and formwise ellipticlike integrals is the main reason why – at present – it is not easy to fathom a direct example of comprehensive operations involving zero and infinity in the traditional SSR setting. Therefore, more practical example of feasibility of the proposed equations (7), (11) and (14)-(15) shall be presented elsewhere on the basis of their applications to solving problems in physical sciences.

However, the apparent affinity of the operations involving ascending infinity (and unrestricted division by zero) to those elliptic integrals, indicates that both: the operationally viable infinity and the unrestricted conventional division by zero are indispensable for the development of realistic future mathematics, that is mathematics that is both meaningful and relevant to nontraditional physical applications. Nevertheless, the reader can realize by now that at present, only the division by zero implemented as multiplication by infinity (and vice versa) performed in dual reciprocal spaces, is truly realistic and thus can generate conclusions that are meaningful for both theoretical physics and to the underlying it applied mathematics.

8. REALS DIVIDED AND MULTIPLIED BY ZERO AND INFINITY

Total divisions and multiplication by zero and infinity for a real number $a \in \mathbb{R}$ and an integer functional n = E[a]≤[a] (i.e. when n equals to the Entire/Floor of real functional [a] of the real number 'a') and for the real excess Ɛ≔ (a–E[a]) over [a] can now be expressed as

$$(T_{a\uparrow}^{a\cdot\infty} \coloneqq a \cdot \infty = \frac{\mathrm{a}}{0} \eqqcolon T_{a\uparrow}^{a/0}) \Leftrightarrow (N_{n\uparrow}^{n\cdot\infty} + \mathcal{E}_{n\uparrow}^{a\cdot\infty}) \Leftrightarrow (N_{n\uparrow}^{n/0} + \mathcal{E}_{n\uparrow}^{a/0}) \qquad (16)$$

and an inverse/reciprocal prototype formula for the excess Ɛ over [a] and n = E[a]≤ [a] becomes

$$(T_{a\uparrow}^{a\cdot 0} \coloneqq a \cdot 0 = \frac{\mathrm{a}}{\infty} \eqqcolon T_{a\uparrow}^{a/\infty}) \Leftrightarrow (N_{n\uparrow}^{n\cdot 0} + \mathcal{E}_{n\uparrow}^{a\cdot 0}) \Leftrightarrow (N_{n\uparrow}^{n/\infty} + \mathcal{E}_{n\uparrow}^{a/\infty}) \qquad (17))$$

with the same operational conditions as above. The logical equivalences are thus templates for formally equivalent – or operationally alternative, if you will – algebraic operations.

9. CONCLUSIONS

It has been shown that unrestricted division by zero can be implemented via algebraic multiplication by the neverending ascending infinity and vice versa, at least in principle. Hence, the notorious yet heretofore almost never challenged prohibition of division by zero has been exposed as an unprecedented conceptual and operational nonsense, which is, nevertheless, thoughtlessly perpetuated everywhere since times immemorial.

References

[1] Czajko J. New product differentiation rule for paired scalar reciprocal functions. *World Scientific News* 144 (2020) 358-371

[2] Czajko J. Pairing of infinitesimal descending complex singularity with infinitely ascending, real domain singularity. *World Scientific News* 144 (2020) 56-69

[3] Cheng E. Beyond infinity. An expedition to the outer limits of mathematics. New York: Basic Books, 2017, p.13ff.

[4] Rudin W. Real and complex analysis. New Delhi: McGraw-Hill, 2006, p.18f.

[5] Maurin K. Analysis II: Integration, distributions, holomorphic functions, tensor and harmonic analysis. Dordrecht: Reidel, 1980, pp.65,726.

[6] Page A. Mathematical analysis and techniques II. Oxford: Oxford Univ. Press, 1974, p.230.

[7] Czajko J. Algebraic division by zero implemented as quasigeometric multiplication by infinity in real and complex multispatial hyperspaces. *World Scientific News* 92(4) (2018) 171-197

[8] Czajko J. Quaternionic division by zero is implemented as multiplication by infinity in 4D hyperspace. *World Scientific News* 94(4) (2018) 190-216

[9] Czajko J. Finegrained 3D differential operators hint at the inevitability of their dual reciprocal portrayals. *World Scientific News* 132 (2019) 98-120

[10] Czajko J. Dual reciprocal scalar potentials paired via differential operators in Frenet frames make the operators to act simultaneously in each of two paired 3D reciprocal spaces. *World Scientific News* 137 (2019) 96-118

[11] Czajko J. Mathematical gateway to complementary hidden variables in macrophysics. *International Letters of Chemistry, Physics and Astronomy* 50 (2015) 117-142

[12] Czajko J. Operational constraints on dimension of space imply both spacetime and timespace. *International Letters of Chemistry, Physics and Astronomy* 36 (2014) 220-235

[13] Czajko J. Operational restrictions on morphing of quasi-geometric 4D physical spaces. *International Letters of Chemistry, Physics and Astronomy* 41 (2015) 45-72.

[14] Stein S.K. & Barcellos A. Calculus and analytic geometry. New York: McGraw-Hill Book, 1992, pp.87,134,228.

[15] Jeffrey A. Handbook of mathematical formulas and integrals. San Diego: Academic Press, 2000, p.87.

[16] Stroud K.A. & Booth D. Advanced engineering mathematics. New York: Industrial Press, 2003, p.275.

[17] Thomas G.B., Jr. & Finney R.L. Calculus and analytic geometry II. Reading, MA: Addison-Wesley, 1996, p.998.

[18] Gellert W. et al (Eds.). The VNR concise encyclopedia of mathematics. Second edition. New York: Van Nostrand Reinhold, 1989, p.704f.

[19] Guenther R.B. & Lee J.W. Partial differential equations of mathematical physics and integral equations. New York: Dover, 1995, p.233ff.

[20] Roach G.F. Green's functions. Cambridge: Cambridge Univ. Press, 1982, pp.1ff,79f,141.

[21] Iyanaga S. & Kawada Y. (Eds.) Encyclopedic dictionary of mathematics 1: A-M. Cambridge, MA: The MIT Press, 1980, p.28,604f.

[22] Lichnerowicz A. Linear algebra and analysis. San Francisco: Holden-Day, 1967, pp.256ff,260,275,279.

[23] Andrews L.C. Special functions for engineers and applied mathematicians. New York: Macmillan, 1985, p.108ff.

[24] Widom H. Lectures on integral equations. Mineola, NY: Dover, 2016, p.1ff.

[25] Muskhelishvili N.I. Singular integral equations. New York: Dover, 1992, p.214.

[26] Faulkner J.R. Octonion planes defined by quadratic Jordan algebras. *Memoirs of the AMS* 104. Providence, RI: AMS, 1970, p.6.

[27] Dwight H.B. Tables of integrals and other mathematical data. New York; Macmillan, 1961, p.20.

[28] Abbott S. Understanding analysis. New York: Springer, 2001, p.62ff.

[29] Tricomi F.G. Integral equations. New York: Interscience, 1957, p.161ff.

[30] Krylov V.I. Approximate calculation of integrals. Mineola, NY: Dover, 2005, p.277ff.

[31] Muskhelishvili N.I. Singular integral equations. New York: Dover, 1992, p.113ff.

[32] Zwillinger D. Handbook of integration. Boston: Jones and Bartlett Publishers, 1992, p.95ff.

[33] Nahin P.J. Inside interesting integrals (with an introduction to contour integration). New York: Springer, 2015, p.323ff.

[34] Snider A.D. Partial differential equations. Sources and solutions. Mineola, NY: Dover, 2006, p.42ff.

[35] Davis H.T. Introduction to nonlinear differential and integral equations. New York: Dover, 1962, p.413ff.

[36] Corduneanu C. Integral equations and applications. Cambridge: Cambridge Univ. Press, 1991, pp.56f,171f,140f,151f.

[37] Corduneanu C. Principles of differential and integral equations. Boston: Allyn and Bacon, 1971, p.155ff.

[38] Pólya G. Untersuchungen über Lücken und Singularitäten von Potennzreihen II. Zweite Mitteilung. [pp.543-592 in: Boas R.P. (Ed.) George Pólya collected papers I: Singularities of analytic functions. Cambridge, MA: The MIT Press, 1974, see p.550f].

[39] Pólya G. Über die Potenzreihenentwicklung gewisser mehrdeutigen Funktionen. [pp.593-613 in: Boas R.P. (Ed.) George Pólya collected papers I: Singularities of analytic functions. Cambridge, MA: The MIT Press, 1974, see p.596f].

[40] Pólya G. On polar singularities of power series and of Dirichlet series. [pp.521-537 in: Boas R.P. (Ed.) George Pólya collected papers I: Singularities of analytic functions. Cambridge, MA: The MIT Press, 1974].

[41] Pólya G. Untersuchungen über Lücken und Singularitäten von Potennzreihen. [pp.363-454 in: Boas R.P. (Ed.) George Pólya collected papers I: Singularities of analytic functions. Cambridge, MA: The MIT Press, 1974, see p.394f].

[42] Plancherel M. & Pólya G. Fonctions entières et integrals de Fourier multiples. [pp.619-643 in: Boas R.P. (Ed.) George Pólya collected papers I: Singularities of analytic functions. Cambridge, MA: The MIT Press, 1974, see pp.624f, 640ff].

[43] Boros G. & Moll V. Irresistible integrals. Symbolics, analysis and experiments in the evaluation of integrals. Cambridge: Cambridge Univ. Press, 2006, p.98.

[44] Weisstein E.W. Frullani's integral. MathWorld.

[45] Bravo S. et al. Integrals of Frullani type and the method of brackets. *Open Math*. 15 (2017) 1-12

[46] Albano M. et al. The integrals in Gradshteyn and Ryzhik. Part 15: Frullani integrals. arXiv.

[47] Jefferys H. & Jeffreys B.S. Methods of mathematical physics. Cambridge: At The Univ. Press, 1972, p.406f.

[48] Edwards J. A treatise on the integral calculus II. New York: Chelsea, 1922, p.337f.

[49] Arias-de-Reyna J. On the theorem of Frullani. *Proc. Am. Math. Soc*. 109 (1990) 165-175

[50] Ostrowski A.M. On some generalizations of the Cauchy-Frullani integral. *PNAS USA* 35 (1949) 612-616

[51] Ostrowski A.M. On Cauchy-Frullani integrals. *Comment. Math. Helvetici* 51 (1976) 57-91

[52] Spiegel M.R. Schaum's outline series. Mathematical handbook of formulas and tables. New York: McGraw-Hill Book, 1968, p.100.

[53] Czajko J. Finegrained 3D differential operators hint at the inevitability of their dual reciprocal portrayals. *World Scientific News* 132 (2019) 98-120

[54] Nahin P.J. Inside interesting integrals (with an introduction to contour integration). New York: Springer, 2015, pp.14f,15,16.

[55] Czajko J. On Conjugate Complex Time II: Equipotential Effect of Gravity Retrodicts Differential and Predicts Apparent Anomalous Rotation of the Sun. *Chaos Solit. Fract.* 11 (2000) 2001-2016

[56] Czajko J. Equipotential energy exchange depends on density of matter. *Stud. Math. Sci.* 7(2) (2013) 40-54

[57] Czajko J. Galilei was wrong: Angular nonradial effects of radial gravity depend on density of matter. *Int. Lett. Chem. Phys. Astron.* 30 (2014) 89-105

[58] Couturat L. De l’infini mathématique. New York: Burt Franklin, 1969, pp.99, 254,

[59] Couturat L. The algebra of logic. Mineola, NJ: Dover, 2005, pp.20f,22f.

[60] Shoenfield J.R. Mathematical logic. Reading, MA: Addison-Wesley, 1967, p.33.

[61] Czajko J. On unconventional division by zero. *World Scientific News* 99 (2018) 133-147

[62] Einstein A. & Mayer W. Einheitliche Theorie von Gravitation und Elektrizität. *Sitz.-ber. Preuss. Ak. Wiss.* 1931(25) 541

[63] Czajko J. On Cantorian Spacetime over Number Systems with Division by Zero. *Chaos Solit. Fract.* 21 (2004) 261-271

[64] Czajko J. Cantor and Generalized Continuum Hypotheses may be false. *Chaos Solit. Fractals* 21 (2004) 501-512

[65] Robert A.M. Nonstandard analysis. Mineola, NY: Dover, 2003, p.113f.

[66] Davis M. Applied nonstandard analysis. Minneola, NY: Dover, 2005, p.55.