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Exact Analytical Solutions of Nonlinear Differential Equation of a Large Amplitude Simple Pendulum

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ABSTRACT

The governing equation of large amplitude simple pendulum is a nonlinear equation which is very difficult to be solved exactly and analytically. However, the classical way for finding analytical solution is obviously still very much important. This is because an exact analytical solution serves as an accurate benchmark for numerical solution and provides a better insight into the significance of various system parameters affecting the phenomena than the numerical solution. Therefore, in this present work, exact analytical solutions for nonlinear differential equation of large amplitude simple pendulum is presented. Also, with the aid of the exact analytical solutions, parametric studies are carried out to study the effects of the model parameters on the dynamic behavior of the large-amplitude nonlinear oscillation system. The solutions can serve as benchmarks for the numerical solution or approximate analytical solution.

***Keywords*:** Large amplitude, Oscillation system, Strong nonlinearity, Exact Analytical solution

1. INTRODUCTION

Oscillatory systems have been widely applied in various engineering and industrial systems such as mass on moving belt, simple pendulum, pendulum in a rotating plane, chaotic pendulum, elastic beams supported by two springs, vibration of a milling machine [1–3] have made them to be extensively investigated. Because many practical physics and engineering

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components consist of vibrating systems that can be modeled using the oscillator systems [4–6], the importance of the systems has been established in physics and engineering. In order to simply describe the motion of the oscillation systems and provide physical insights into the dynamics, the nonlinear equation governing the pendulum motion has been linearized and solved [7-9]. The applications of such linearized model are found in the oscillatory motion of a simple harmonic motion, where the oscillation amplitude is small and the restoring force is proportional to the angular displacement and the period is constant.

However, when the oscillation amplitude is large, the pendulum's oscillatory motion is nonlinear and its period is not constant, but a function of the maximum angular displacement. Consequently, much attentions have been devoted to develop various analytical and numerical solutions to the nonlinear vibration of large-amplitude oscillation system. In one of the works, Beléndez et al. [10] developed an analytical solution to conservative nonlinear oscillators using cubication method while Gimeno and Beléndez [11] adopted rational-harmonic balancing approach to the nonlinear pendulum-like differential equations. Lima [12] submitted a trigonometric approximation for the tension in the string of a simple pendulum. Amore and Aranda [13] utilized improved Lindstedt–Poincaré method to provide approximate analytical solution of the nonlinear problems.

Momeni et al. [14] applied He's energy balance method to Duffing harmonic oscillators. With the aid of the He's energy balance method, Ganji et al. [15] provided a periodic solution for a strongly nonlinear vibration system. The He's energy balance method as well as He's Variational Approach was used by Askari et al. [16] for the nonlinear oscillation systems. The numerical method was also adopted by Babazadeh et al. [17] for the analysis of strongly nonlinear oscillation systems. Biazar and Mohammadi [18] explored multistep differential transform method to describe the nonlinear bahaviour of oscillatory systems while Fan [19] and Zhang [20] applied He's frequency–amplitude formulation for the Duffing harmonic oscillator.

The above reviewed works are based on approximate solutions. In order to gain better physical insight into the nonlinear problems, Beléndez et al. [21] presented the exact solution for the nonlinear pendulum using Jacobi elliptic functions. However, an exact analytic solution, as well as an exact expression for the period, both in terms of elliptic integrals and functions, is known only for the oscillatory regime ($0 < \theta_0 < \pi$). In this regime, the system oscillates between symmetric limits. Consequently, the quest for analytical and numerical solutions for the large-amplitude nonlinear oscillation systems continues. Therefore, with the aid of a modified variational iterative method, Herisanu and Marinca [22] described the behavior of the strongly nonlinear oscillators.

Kaya et al. [23] applied He's variational approach to analyze the dynamic behaviour of a multiple coupled nonlinear oscillators while Khan and Mirzabeigy [24] submitted an improved accuracy of He's energy balance method for the analysis of conservative nonlinear oscillator. A modified homotopy perturbation method was utlized by Khan et al. [25] to solve the nonlinear oscillators. In another work, a coupling of homotopy and the variational approach was used by Khan et al. [26] to provide a physical insight into the vibration of a conservative oscillator with strong odd-nonlinearity.

Wu et al. [27] presented analytical approximation to a large amplitude oscillation of a nonlinear conservative system while homotopy perturbation method was used by He [28] to provide an approximation analytical solution for nonlinear oscillators with discontinuous. Belato et al. [29] analyzed a non-ideal system, consisting of a pendulum whose support point is vibrated along a horizontal guide. Few years later, Cai et al. [30] used methods of multiple scales and Krylov-Bogoliubov-Mitropolsky to developed the approximate analytical solutions to a class of nonlinear problems with slowly varying parameters. In some years after, Eissa and Sayed [31] examined the vibration reduction of a three-degree-of-freedom spring-pendulum system, subjected to harmonic excitation. Meanwhile, Amore and Aranda [32] utilized Linear Delta Expansion to the Linstedt–Poincaré method and developed improved approximate solutions for nonlinear problems. The chaotic behavior and the dynamic control of oscillation systems has been a subject of major concern as presented in the literature. Therefore, with the aid of a shooting method, Idowu et al. [33] studied chaotic solutions of a parametrically undamped pendulum.

Also, Amer and Bek [34] applied multiple scales method to study the chaotic response of a harmonically excited spring pendulum moving in a circular path. In the previous year, Anh et al. [35] examined the vibration reduction for a stable inverted pendulum with a passive mass-spring-pendulum-type dynamic vibration absorber. Ovseyevich [36] analyzed the stability of the upper equilibrium position of a pendulum in cases where the suspension point makes rapid random oscillation with small amplitude.

In the above reviewed literatures, numerical methods or approximate analytical methods were applied to solve the oscillatory problems. However, the classical way for finding analytical solution is obviously still important since it serves as accurate benchmark for numerical solutions. The experimental data are useful to access the mathematical models, but are never sufficient to verify the numerical solutions of the established mathematical models. Comparison between the numerical calculations and experimental data fail to reveal the compensation of modelling deficiencies through the computational errors or unconscious approximations in establishing applicable numerical schemes. Additionally, analytical solutions for specified problems are also essential for the development of efficient applied numerical simulation tools.

Also, in practice, approximate analytical solutions with large number of terms are not convenient for use by designers and engineers. Analytical solutions, when available, are advantageous in that they provide a good insight into the significance of various system parameters affecting the transport phenomena. Also, analytical expression is more convenient for engineering calculation compare with experimental or numerical studies and it is obvious starting point for a better understanding of the relationship between physical quantities/properties. Analytical solution provides continuous physical insights than pure numerical/computation method. It helps to reduce the computation cost and task in the analysis of such problem. Also, it is convenient for parametric studies and accounting for the physics of the problem. It appears more appealing than the numerical solution. Inevitably, exact analytical expressions are required to exactly describe the behaviour of the oscillatory system. Therefore, in this present work, an exact analytical solution of nonlinear differential equation of large amplitude simple pendulum is presented. Also, with the aid of the exact analytical solution, parametric studies were carried to study the impacts of the model parameters on the dynamic behavior of the large-amplitude nonlinear oscillation system.

2. PLANE PENDULUM

Consider an oscillating system which freely oscillates in a vertical plane about the equilibrium position (P) on a vertical straight line which passes through the pivot point under the action of the gravitational force as shown in Fig. 1.

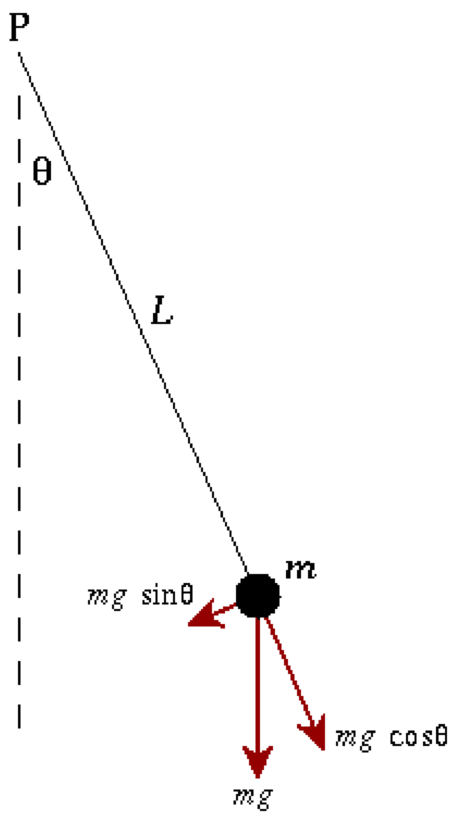

Fig. 1. A simple pendulum with large amplitude

The system consists of a particle of mass *m* attached to the end of an inextensible string, with the motion taking place in a vertical plane. Using Newton's law for the rotational system, the differential equation modelling the free undamped simple pendulum is

$$\tau = \mathrm{I}\alpha \quad \Rightarrow \quad -mgsin\theta L = mL^2 \frac{d^2\theta}{dt^2} \tag{1}$$

and after rearrangement, we arrived at the nonlinear equation of motion of the pendulum as

$$\frac{d^2\theta}{dt^2} + \frac{g}{L} sin\theta = 0 \tag{2}$$

the initial conditions are

$$\theta = \theta_0, \quad \frac{d\theta}{dt} = 0 \quad \text{when} \quad t = 0, \tag{3}$$

θ_0 is the initial angular displacement or the amplitude of the oscillations.

If the amplitude of the angular displacement is small, the equation of motion reduces to the equation of simple harmonic motion, which is given as

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0 \tag{4}$$

The exact analytical solution is given as

$$\theta = \theta_0 cos\sqrt{\frac{g}{L}}t = \theta_0 cos\omega t \tag{5}$$

where θ is the angular displacement, t is the time, L is the length inextensible string (the pendulum), g is the acceleration due to gravity and $\omega = \sqrt{\frac{g}{L}}$ is the natural frequency of the motion.

Our interest in the present study is the analysis of the nonlinear oscillations of the pendulum. It should be noted that the presence of the trigonometry function $sin\theta$ in Eq. (2) introduces a high nonlinearity in the equation. However, Beléndez et al. [21] presented the exact solution for the large-amplitude nonlinear equation using the complete and incomplete elliptic integrals of the first kind as

$$\theta(t) = 2arcsin\left\{sin\frac{\theta_0}{2} sin\left[K\left(sin^2\frac{\theta_0}{2}\right) - \omega_o t;\ sin^2\frac{\theta_0}{2}\right]\right\} \tag{6}$$

where

$\omega_0 = \sqrt{\frac{g}{L}}$ is the frequency for the small-angle regime.

and

K(m) is the complete elliptic integral of the first kind which is given as

$$K(m) = \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-m^2z^2)}} \tag{7}$$

where

$z = sin\varphi$

It should be pointed out that the exact analytic solution by Beléndez et al. [21], as well as the exact expression for the period, both in terms of elliptic integrals and functions, is only known for the oscillatory regime. In this regime ($0 < \theta_0 < \pi$), the system oscillates between symmetric limits. In fact, the developed exact solution provides a high accurate solution to the nonlinear oscillation system with initial amplitudes as high as 135°.

Therefore, in another work, Beléndez et al. [40] applied a rational harmonic representation to develop an approximation solution for the pendulum as:

$$\theta(t) = \frac{\left(960 - 49\theta_0^2\right)\theta_0 cos\omega t}{960 - 69\theta_0^2 + 20\theta_0^2 cos^2 \omega t} \tag{8}$$

where

$$\omega = \frac{1}{4}\sqrt{\frac{g}{l}}\left(1 + \sqrt{cos\frac{\theta_0}{2}}\right)^2 \tag{9}$$

After substitution of Eq. (9) into Eq. (8), we have

$$\theta(t) = \frac{\left(960 - 49\theta_0^2\right)\theta_0 cos\left[\frac{1}{4}\sqrt{\frac{g}{l}}\left(1 + \sqrt{cos\frac{\theta_0}{2}}\right)^2\right]t}{960 - 69\theta_0^2 + 20\theta_0^2 cos^2\left[\frac{1}{4}\sqrt{\frac{g}{l}}\left(1 + \sqrt{cos\frac{\theta_0}{2}}\right)^2\right]t} \tag{10}$$

From Eq. (9), it is also established that for a nonlinear oscillation system, the angular frequency of the oscillation depends on the amplitude of the motion. However, the developed approximate analytical expression in Eq, (10) provides a high accurate solution to the nonlinear oscillation system with initial amplitudes upto 170°. Therefore, the quest for an exact analytical solution for large-amplitude nonlinear oscillation systems over a wider and unlimited range.

3. EXACT ANALYTICAL SOLUTION OF NONLINEAR SIMPLE PENDULUM

The nonlinear equation (2) is solved as follow. Multiplying the differential equation by the factor $mL^2 \frac{d\theta}{dt}$ and rearrange, gives

$$\frac{d}{dt}\left[\frac{ml^2}{2}\left(\frac{d\theta}{dt}\right)^2 - mgLcos\theta\right] = 0 \tag{8}$$

On integration of Eq. (8), we have

$$\frac{ml^2}{2}\left(\frac{d\theta}{dt}\right)^2 - mgLcos\theta = R \tag{9}$$

where R is the constant of integration.

Using the initial conditions, we have

$$R = -mgLcos\theta_o \tag{10}$$

Then, Eq. (9) becomes

$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{L}(cos\theta - cos\theta_o)} \tag{11}$$

After rearrangement of the variable, we have

$$\sqrt{\frac{2g}{L}}dt = \frac{d\theta}{\sqrt{(cos\theta - cos\theta_o)}} \tag{12}$$

Using the trigonometric relation, $cos\theta = 1 - 2sin^2\left(\frac{\theta}{2}\right)$ in Eq. (12) and integrate both sides, we have

$$2\sqrt{\frac{g}{L}}\int dt = \int \frac{d\theta}{\sqrt{\left(sin^2\left(\frac{\theta_o}{2}\right) - sin^2\left(\frac{\theta}{2}\right)\right)}} \tag{13}$$

Eq. (13) can be written as

$$2\sqrt{\frac{g}{L}}\int dt = \int \frac{d\theta}{\sqrt{\left(k^2 - sin^2\left(\frac{\theta}{2}\right)\right)}} \tag{14}$$

where $k = sin\left(\frac{\theta_o}{2}\right)$

Making a change of variable and use

$$sin\left(\frac{\theta}{2}\right) = ksin\phi \tag{15}$$

Eq. (14) becomes

$$\sqrt{\frac{g}{L}}\int dt=\int\frac{d\phi}{\sqrt{\left(1-k^2sin^2\phi\right)}} \tag{16}$$

The RHS of Eq. (16) is called the first kind elliptic integral, denoted by

$$F\left(k,\phi\right)=\int_0^\phi\frac{d\xi}{\sqrt{\left(1-k^2sin^2\xi\right)}} \tag{17}$$

With the help of Newton's binomial,

$$\frac{1}{\sqrt{\left(1-k^2sin^2\xi\right)}}=\sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2n)!!}k^{2n}sin^{2n}\xi=\sum_{n=0}^{\infty}\frac{(1/2)_n}{n!}k^{2n}sin^{2n}\xi \tag{18}$$

The above shows double factorial and Pochhammer symbol. Substituting Eq. (18) into Eq. (17), gives

$$F\left(k,\phi\right)=\sum_{n=0}^{\infty}\frac{(1/2)_n}{n!}k^{2n}\int_0^\phi sin^{2n}\xi d\xi \tag{19}$$

But

$$\int_0^\phi sin^{2n}\xi d\xi=\frac{-cos\phi}{2n}\left\{sin^{2n-1}\phi+\sum_{l=0}^{n-1}\frac{(2n-1)(2n-3)...(2n+2k-1)}{2^l\left(n-1\right)\left(n-2\right)...\left(n-l\right)}sin^{2n-2l-1}\phi\right\}+\frac{(2n-1)!!}{2^l\,l!}\phi \tag{20}$$

Altenatively,

$$\int_0^\phi sin^{2n}\xi d\xi=\frac{1}{2^{2n}}\binom{2n}{n}\phi+\frac{(-1)^n}{2^{2n-1}}\sum_{l=0}^{n-1}(-1)^l\binom{2n}{l}\frac{sin\left(2n-2l\right)\phi}{\left(2n-2l\right)} \tag{21}$$

From Eqs. (20) and (21), Eq. (19) could be written as

$$F\left(k,\phi\right)=\sum_{n=0}^{\infty}\frac{(1/2)_n}{n!}k^{2n}\left[\frac{-cos\phi}{2n}\left\{sin^{2n-1}\phi+\sum_{l=0}^{n-1}\frac{(2n-1)(2n-3)...(2n+2k-1)}{2^l\left(n-1\right)\left(n-2\right)...\left(n-l\right)}sin^{2n-2l-1}\phi\right\}+\frac{(2n-1)!!}{2^l\,l!}\phi\right] \tag{22}$$

Altenatively,

$$F\left(k,\phi\right)=\sum_{n=0}^{\infty}\frac{(1/2)_n}{n!}k^{2n}\left[\frac{1}{2^{2n}}\binom{2n}{n}\phi+\frac{(-1)^n}{2^{2n-1}}\sum_{l=0}^{n-1}(-1)^l\binom{2n}{l}\frac{sin\left(2n-2l\right)\phi}{\left(2n-2l\right)}\right] \tag{23}$$

The above series in the RHS is uniformly convergent for all the values of ϕ
From Eq. (16), we have

$$t=\sqrt{\frac{L}{g}}\left\{\sum_{n=0}^{\infty}\frac{(1/2)_n}{n!}k^{2n}\left[\frac{-cos\phi}{2n}\left\{sin^{2n-1}\phi+\sum_{l=0}^{n-1}\frac{(2n-1)(2n-3)...(2n+2k-1)}{2^l(n-1)(n-2)...(n-l)}sin^{2n-2l-1}\phi\right\}+\frac{(2n-1)!!}{2^l l!}\phi\right]\right\} \tag{24}$$

Alternatively

$$t=\sqrt{\frac{L}{g}}\left\{\sum_{n=0}^{\infty}\frac{(1/2)_n}{n!}k^{2n}\left[\frac{1}{2^{2n}}\binom{2n}{n}\phi+\frac{(-1)^n}{2^{2n-1}}\sum_{l=0}^{n-1}(-1)^l\binom{2n}{l}\frac{sin(2n-2l)\phi}{(2n-2l)}\right]\right\} \tag{25}$$

Recall that

$$sin\left(\frac{\theta}{2}\right)=ksin\phi \quad \text{and} \quad k=sin\left(\frac{\theta_o}{2}\right)$$

Therefore

$$\phi=arcsin\left[\frac{sin\left(\frac{\theta}{2}\right)}{sin\left(\frac{\theta_o}{2}\right)}\right] \tag{26}$$

$$t=\sqrt{\frac{L}{g}}\left[\sum_{n=0}^{\infty}\frac{(1/2)_n}{n!}k^{2n}\left\{\begin{array}{l}\dfrac{-cos\left\{arcsin\left[\dfrac{sin\left(\frac{\theta}{2}\right)}{sin\left(\frac{\theta_o}{2}\right)}\right]\right\}}{2n}\left\{\begin{array}{l}sin^{2n-1}\left\{arcsin\left[\dfrac{sin\left(\frac{\theta}{2}\right)}{sin\left(\frac{\theta_o}{2}\right)}\right]\right\}\\+\displaystyle\sum_{l=0}^{n-1}\frac{(2n-1)(2n-3)...(2n+2k-1)}{2^l(n-1)(n-2)...(n-l)}sin^{2n-2l-1}\left\{arcsin\left[\dfrac{sin\left(\frac{\theta}{2}\right)}{sin\left(\frac{\theta_o}{2}\right)}\right]\right\}\end{array}\right\}\\+\dfrac{(2n-1)!!}{2^l l!}\left\{arcsin\left[\dfrac{sin\left(\frac{\theta}{2}\right)}{sin\left(\frac{\theta_o}{2}\right)}\right]\right\}\end{array}\right\}\right] \tag{27}$$

Alternatively,

$$t=\sqrt{\frac{L}{g}}\left[\sum_{n=0}^{\infty}\frac{(1/2)_n}{n!}k^{2n}\left[\frac{1}{2^{2n}}\binom{2n}{n}\left\{arcsin\left[\frac{sin\left(\frac{\theta}{2}\right)}{sin\left(\frac{\theta_o}{2}\right)}\right]\right\}+\frac{(-1)^n}{2^{2n-1}}\sum_{l=0}^{n-1}(-1)^l\binom{2n}{l}\frac{sin(2n-2l)\left\{arcsin\left[\frac{sin\left(\frac{\theta}{2}\right)}{sin\left(\frac{\theta_o}{2}\right)}\right]\right\}}{(2n-2l)}\right]\right] \tag{28}$$

4. RESULTS AND DISCUSSION

The simulated results of the nonlinear simple pendulum, also, parametric studies such the effects of length of pendulum and the initial angular displacement or the amplitude of the oscillations on the nonlinear vibration of the simple pendulum are presented.

4. 1. Effects of the length of pendulum on the angular displacement of the simple pendulum

Figs 2 and 3 show the effect of length of pendulum on the angular displacement of the nonlinear vibration of the simple pendulum when the initial angular displacement or amplitude of the system are 1.0 and 2.0 respectively. As expected, the results show that by increasing the length of the pendulum, the system frequency decreases i.e. the system oscillates with the lower frequency.

It could also be noted as the amplitude of the system increases, the nonlinear frequency of the pendulum decreases.

For high amplitudes, the periodic motion exhibited by a simple pendulum is practically harmonic but its oscillations are not isochronous i.e. the period is a function of the amplitude of oscillations.

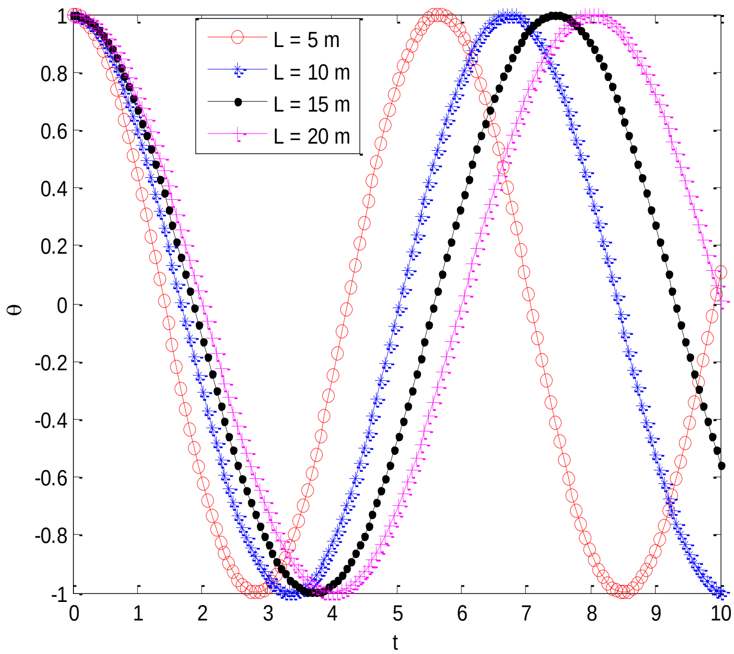

Fig. 2. Time response of the simple pendulum when $\theta_0 = 1.0$

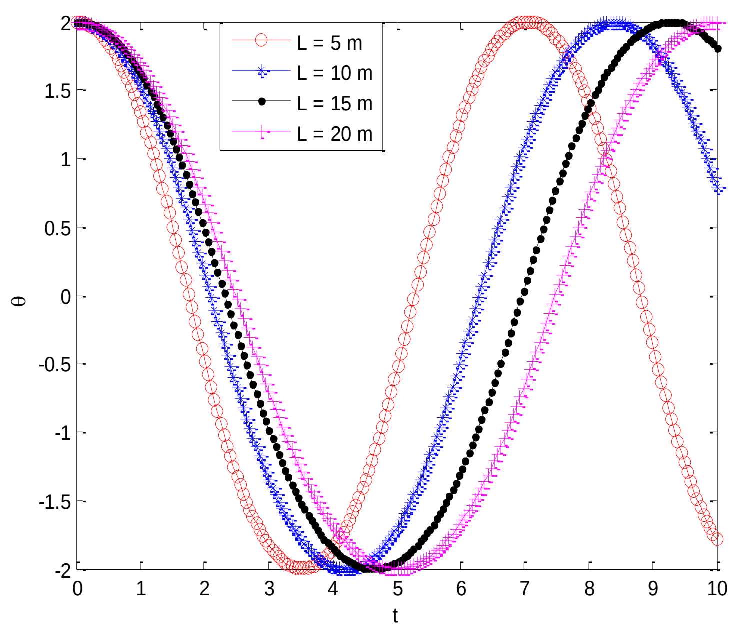

Fig. 3. Time response of the simple pendulum when $\theta_0 = 2.0$

4. 2. Effects of the length of pendulum and amplitude of oscillation on the phase plots of the system

The impacts of the pendulum length and the amplitude of the oscillations on the phase plots of the nonlinear simple pendulum are displayed in Figs. 4 and 5. The nearly circular curves around (0,0) in figures show that the pendulum goes into a stable limit cycle.

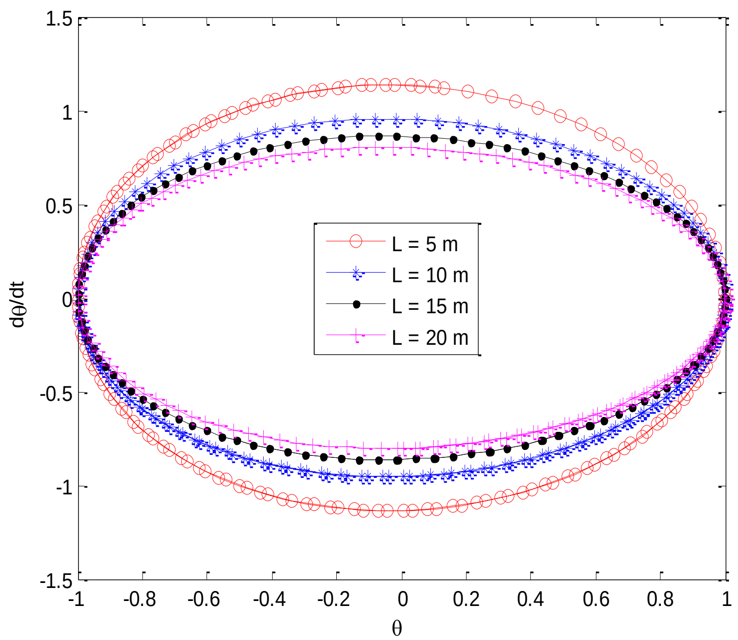

Fig. 4. Phase plot in the $(\theta(t), \mathrm{d}\theta / dt)$ plane when $\theta_0 = 1.0$

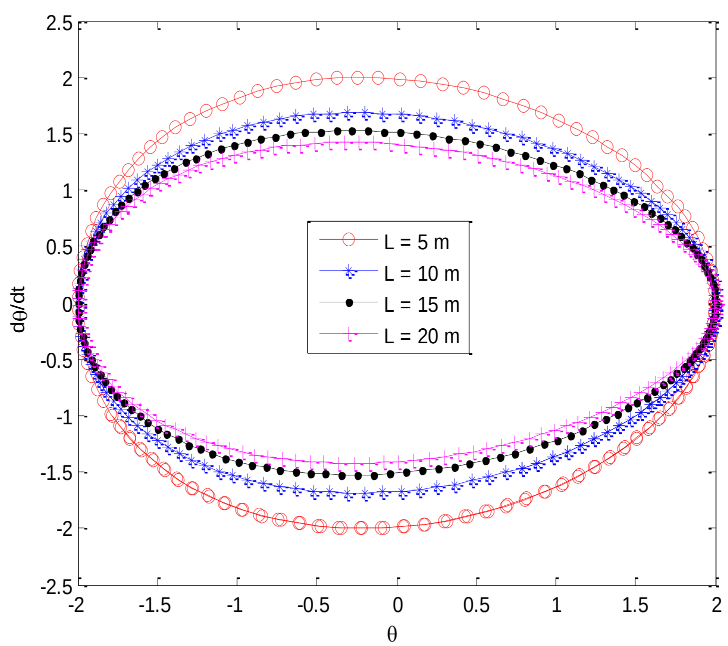

Fig. 5. Phase plot in the $(\theta(t), \mathrm{d}\theta / dt)$ plane when $\theta_0 = 2.0$

4. 3. Effects of the length of pendulum and amplitude of oscillation on the angular velocity

The impacts of the length of pendulum and the amplitude of the oscillations on the angular velocity of the nonlinear simple pendulum are displayed in Figs. 6 and 7 when amplitude of the system are 1.0 and 2.0. The results depict that the maximum velocity of the system decreases as the length of the pendulum increases.

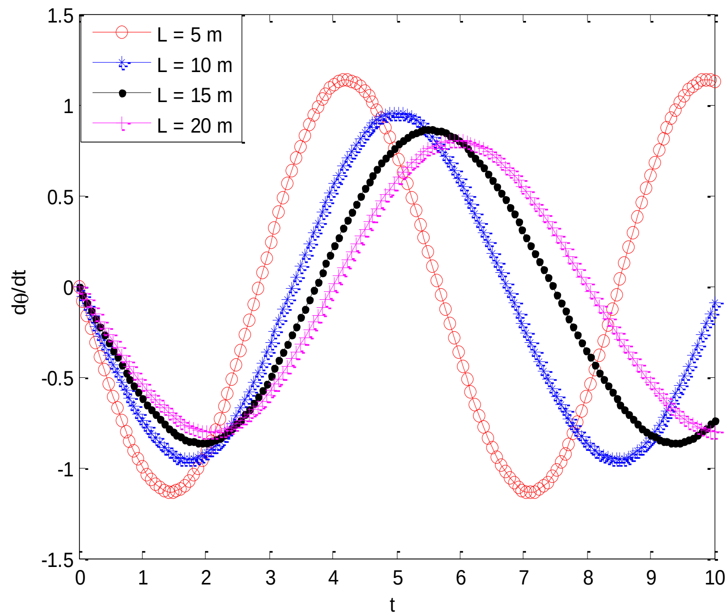

Fig. 6. Velocity variation with time when $\theta_0 = 1.0$

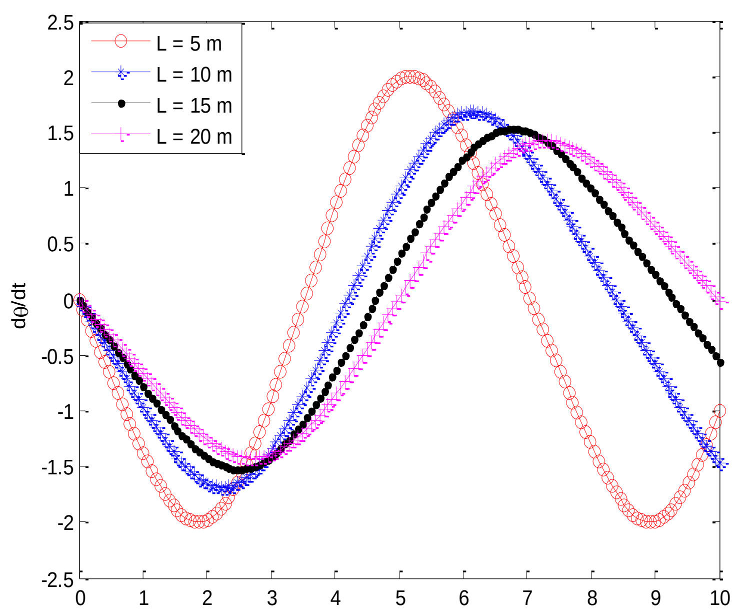

Fig. 7. Velocity variation with time when $\theta_0 = 2.0$

4. 4. Phase plane diagrams for different trajectories of the pendulum with a rotating plane

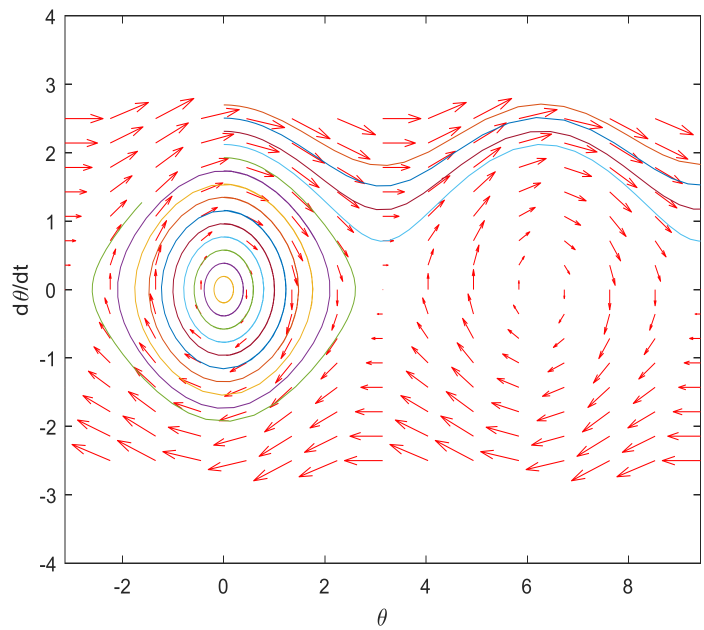

Fig. 8. Phase plot for different trajectories in the $(\theta(t), \mathrm{d}\theta / dt)$ plane for the simple pendulum

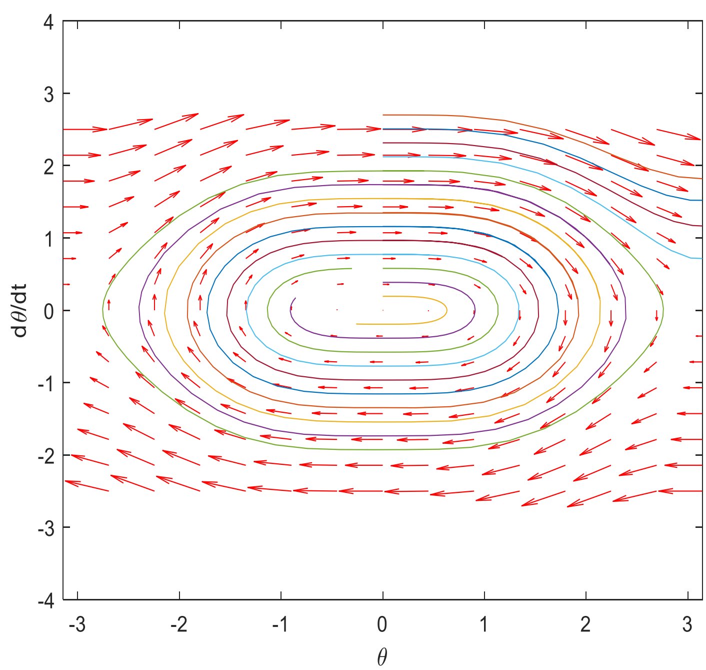

Fig. 9. Phase plot for different trajectories in the $(\theta(t), \mathrm{d}\theta / dt)$ plane for the pendulum with a rotating plane when $\beta = 1.0$

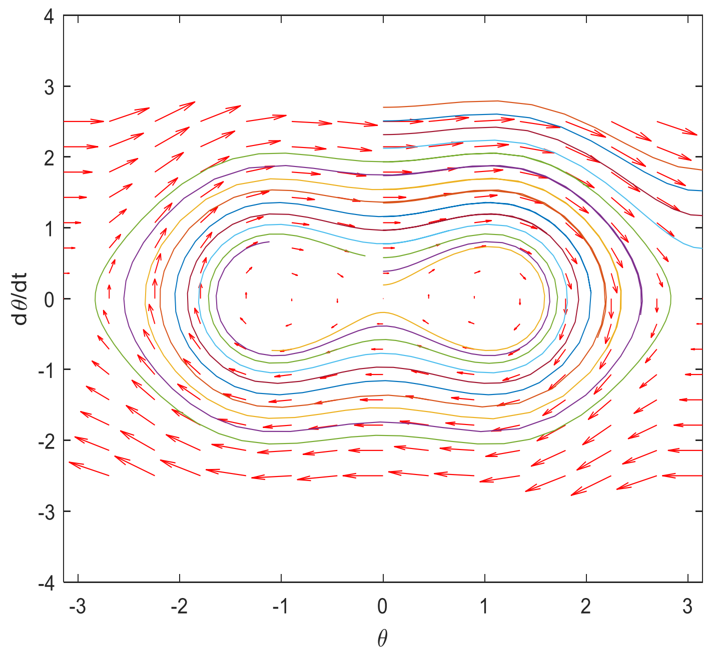

Fig. 10. Phase plot for different trajectories in the $(\theta(t), \mathrm{d}\theta / dt)$ plane for the pendulum with a rotating plane when $\beta = 2.0$

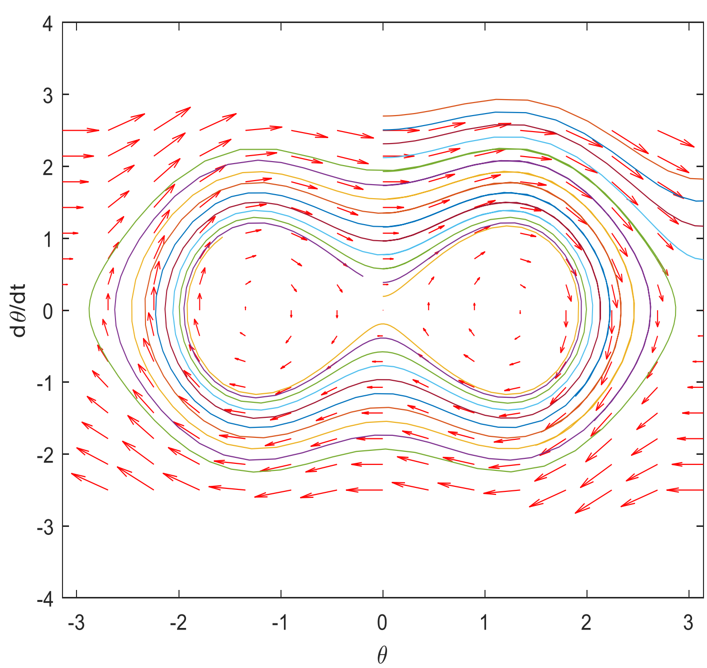

Fig. 11. Phase plot for different trajectories in the $(\theta(t), \mathrm{d}\theta / dt)$ plane for the pendulum with a rotating plane when $\beta = 3.0$

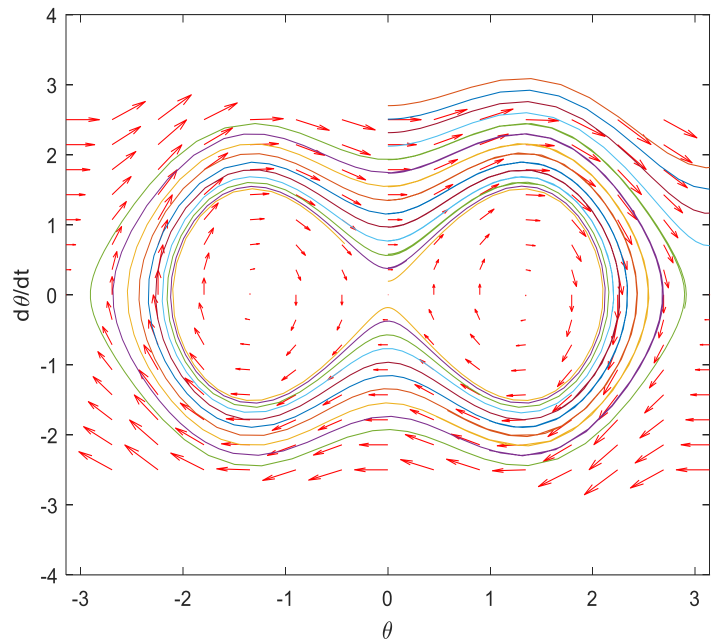

Fig. 12. Phase plot for different trajectories in the $(\theta(t),\, d\theta\,/\,dt)$ plane for the pendulum with a rotating plane when $\beta = 4.0$

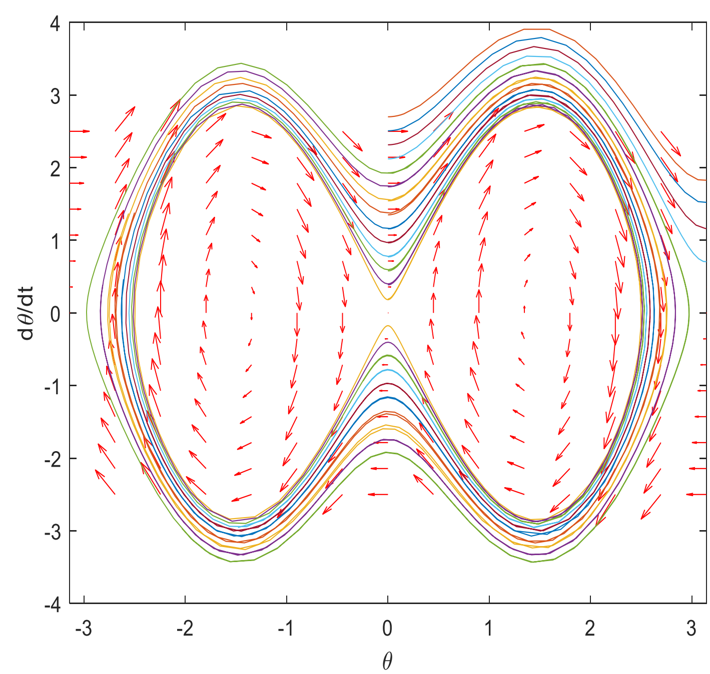

Fig. 13. Phase plot for different trajectories in the $(\theta(t),\, d\theta\,/\,dt)$ plane for the pendulum with a rotating plane when $\beta = 10.0$

The phase plane diagrams for different trajectories of the pendulum with a rotating plane are shown in Fig. 8-13. The arrows in the figures show the direction of motion of the two variables as the pendulum swings. The curves in the plots show particular ways that the pendulum swing. The circular curves around the points (π,0) and (–π,0) really represent the same motion. It is shown that when a pendulum starts off with high enough velocity at $\theta = 0$, it goes all the way around.

It is very obvious that its velocity will slow down on the way up but then it will speed up on the way down again. In such motion, the trajectory appears to fly off into θ space. θ is an angle that has values from $-\pi$ to $+\pi$.

The θ data in this case is not wrapped around to be in this range. In the absence of damping or friction, it just keeps spinning around indefinitely. The counterclockwise motions of the pendulum of this kind are shown in the graph by the wavy lines at the top that keep going from left to right indefinitely, while the curves on the bottom which go from right to left represent clockwise rotations.

However, if damping or friction, the pendulum will gradually slow down, taking smaller and smaller swings, instead of swinging with fixed amplitude, back and forth. As presented in the phase-plots, the motion comes out as a spiral. In each swing, the pendulum angle θ goes to a maximum, then the pendulum stops momentarily, then swings back gaining speed. But the speed when it comes back to the middle is slightly less.

The result is that on the phase plot, it follows a spiral, getting closer and closer to stopping at (0,0). This will be presented in our subsequent works.

5. CONCLUSION

In this work, exact analytical solutions of nonlinear differential equation of large amplitude simple pendulum have been developed also, parametric studies have been carried to study the impacts of the model parameters on the dynamic behavior of the large-amplitude nonlinear oscillation system. The solutions can serve as benchmark for the numerical solution or approximate analytical solution.

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