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Pairing of infinitesimal descending complex singularity with infinitely ascending, real domain singularity

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ABSTRACT

Pairing of infinitesimal descending singularity of the 2D domain of complex numbers with an infinitely ascending singularity deployed in the 1D domain of real numbers, where the real singularity can be equated operationally with never-ending, whether countable or not, infinity, requires the employment of a pair of mutually dual reciprocal spaces in order for each of the spaces of the twin quasigeometric structure to be truly operational. Creation of twin quasigeometric structures comprising paired dual reciprocal spaces that are really operational and truly invertible, is the necessary condition for making the notion of operationally sound infinity viable. Although acceptance of the multispatial reality paradigm seems optional, it is shown that even performing legitimate scalar differentiation (in accordance with product differentiation rule) can yield either incomplete or incorrect evaluations of compounded scalar functions. This curious fact implies inevitable need for awareness of conceptual superiority of the multispatial reality paradigm over the former, unspoken and thus unchallenged in the past, single-space reality paradigm, in order to prevent even inadvertent creation of formwise illegitimate, or just somewhat incomplete, pseudodifferentials, which can be obtained even with the use of quite legitimate operational rules of scalar differential calculus.

***Keywords*:** Singularity, descending infinitesimal complex singularity, infinitely ascending real singularity, covariant differential, contravariant differential, paired dual reciprocal space

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1. INTRODUCTION

There are two basic operationally different and conceptually distinct types of singularity: the complex infinitesimal (descending) singularities and the infinite (ascending) singularities, whether they are countable or not. Although both of these singularities can be encountered in either the domain of real numbers $\mathbb{R}$ or in the domain of complex numbers $\mathbb{C}$ or in domains of hypercomplex numbers, the infinitesimal singularities are successfully handled (or detoured, if you will) in the domain of complex numbers, whereas the infinitely ascending singularities tend to annoy some deductive reasonings made in the real domain. Transition between the two singularities is investigated in the present paper.

The never-ending, ascending singularity is tied to operational notion of infinity, whose concept often generates pesky paradoxes in the "real" domain. In her book [1] Eugenia Cheng exposed many notorious (yet unsurprisingly rarely revealed) paradoxes, such as 1=0, which generate various veiled yet conceptually very harmful mathematical nonsenses, the open discussion of which was tacitly inhibited through the infamous prohibition of division by zero. Having said that I shall assert that those paradoxes are not really paradoxes of the concept of infinity itself, as some claim, but they are due to some tacitly veiled misconceptions of the traditional mathematics, which topic shall be further elaborated elsewhere.

Yet on transition from the two-dimensional (2D) complex domain to the 1D domain of real numbers, the fairly predictable behavior of the infinitesimal complex singularity tends to create not only conceptual but also operational troubles for traditional mathematical handling of the apparently limitless, infinite singularity. However, some topics using differential calculus cannot live without unambiguous operational notion of the ascending infinity too.

This sort of downward transition from $\mathbb{C}$ to $\mathbb{R}$ demands a certain reciprocal conversion. Hence it must not be direct transit from the complex infinitesimal to the real (as the term is being used in mathematics) infinity via a certain mapping, but an intermittent transition, which implies that simply mapping of sets that are understood as mere selections – which is the workhorse of many abstract reasonings in traditional mathematics – is definitely untenable. Mappings are helpful as a kind of shorthand notation and thus could indeed be meaningful within a single universal set but they could become conceptually problematic when deployed in a multispatial framework. Hence the prospective coexistence of these two types of singularities would require an extra, distinct algebraic structure at first (which then would have to be expanded into geometric or quasigeometric representation of the algebraic one) in addition to the conversion performed in transit between the two abstract structures.

Traditional mathematics operated under the formerly unspoken and thus unchallenged single-space reality (SSR) paradigm, which tacitly assumed that the abstract mathematical reality is just one, single point-set kind of an abstract mathematical universe and spaces are understood as mere selections from the single set of number points. Various selections can be made under different conditions which then become identified with attributes of the sets and of the spaces build upon the selected subsets. In the traditional SSR setting functions are often understood also as mere mappings from the input set/selection called domain to an output set/selection called range. In the sense, selections, subsets and spaces were usually treated as being interchangeable even if not always explicitly identified as such.

This interchangeability clouded many abstract reasonings that allowed making simplistic deductions which, in turn, permitted drawing conclusions that were often unrecognized as being faulty.

The multispatial reality (MSR) paradigm that I endorse regards spaces as certain abstract views – akin to the views of tables containing data stored in most commercial relational databases – of the set (which in the MSR setting is reminiscent of database table) upon which the spaces are erected. The views are as if filtered through the bases of the spaces. For genuine distinction between sets and spaces requires an algebraic basis, in which the objects viewed within the given space are natively represented. The very same set can thus be used to span various spaces over it when each space is equipped with distinct – even if not always substantially different – algebraic and/or geometric basis. I should mention that in order to grasp the authentic (i.e. both operational and structural) conceptual understanding of the MSR paradigm, familiarity with finegrained differential (and some nondifferential/algebraic) operators introduced in [2] and further explained in [3] may help too.

2. SINGULARITY AND RECIPROCITY IN THE COMPLEX DOMAIN

The commonly accepted statement that residue of a holomorphic function (also known as analytic/regular function) f(z) of complex variable z at a singular point z = ∞ cannot in general be computed by finding the residue of F(w) = f(1/z) at w = 0 [4] is unlikely to be disputed by mathematicians. But the question why cannot we come up with some kind of unambiguous representation of functions at or near the point at infinity is troubling many practicing scientists. It is not just an issue of estimating or approximating the value of a function at or near the given singularity but sometimes it virtually becomes the *sine qua non* theoretical question that challenges everything we have learned thus far, especially if it generates logically inconsistent derivations or if it contradicts some experimentally confirmed though previously quite unanticipated physical phenomena.

Thorough and ingenious investigations of abstract singularities in the domain of complex numbers have been conducted by George Pólya. He has demonstrated that the integral formula, which is also called Borel's case of Laplace transform of the given complex function F(Z), comprises the following two reciprocal equations

$$F(Z) = \frac{1}{2\pi i}\oint f(z)e^{zZ}dz \qquad (1)$$

$$f(z) = \int_0^{\infty} F(Z)e^{-zZ}dZ \qquad (2)$$

that are reminiscent of Fourier's integral theorem with the function f(z) defined at the point at infinity z = ∞ [5]; compare also his discussion of the reciprocal integral formulas in terms of infinite singular points in [6] and further discussion of the nature of the function f(z) at z = ∞ in [7]. Recall that the minus sign in the exponential term e^{-zZ} in (2) signifies complex inversion in the 2D complex domain, which fact makes the apparently coupled equations (1) and (2) as if doubly inversive, if you will.

Recall that complex inversion is only conventional, for it is actually defined as multiplicatively additive 2D inversion, with multiplicatively inverted modulus coupled with the argument/angle/phase only additively reversed (i.e. not really multiplicatively inverted). Complex inversion is not really multiplicatively multiplicative inversion. I am not saying that the argument's inversion is faulty or that it is not actually inverted, but merely that one could envision also a truly multiplicatively multiplicative (or perhaps doubly multiplicative, if you will) inversion imposed on hypercomplex numbers in four or higher number of dimensions. While the complex inversion is of necessity squeezed to fit the 2D complex domain, its impact should not affect adversely our mathematical imagination by restricting our brainpower to just the – essentially planar – confines of the complex numbers' domain.

The multiplicatively additive complex inversion is a conventional inversion scaled down in order to fit the only two dimensions available in the complex numbers domain and thus it is admissible and as such is surely valid. But this validity does not mean that there could not be truly multiplicative inversion in a domain of some hypercomplex numbers where the number of their native dimensions would allow such an abstract kind of true inversion and thus enable the resulting from it true reciprocity. The advent of the (more comprehensive than SSR) MSR paradigm does not preclude imagining and demanding that kind of an abstract inversion, at least in principle.

Pólya has recognized the possibility of presence of such an abstract, multiplicatively multiplicative (i.e. truly inversive) duality for he has explicitly remarked that the two above formulas also reveal curious reciprocity [5]. Yet he worked solely under the former SSR paradigm. Compare also his treatment of the pair of reciprocal integral formulas, which he further evaluated and extensively discussed in terms of power series in [8], [9], and [10].

Moreover, he has also realized that the ultimate decomposition of series into two complementary subcomponents in vicinity of the point z = 0 should proceed according to the following pattern within the complex domain $\mathbb{C}$

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma}\frac{f(w)dw}{w-z} + \frac{1}{2\pi i}\int_{C^*}\frac{f(w)dw}{w-z} \quad \Rightarrow \quad f(z) \coloneqq \Phi(z) + f^*(z) \qquad (3)$$

which utilizes Cauchy integral – see [11] p.551f, in the traditional SSR setting though.

He has also underscored the fact that value of the [primary] decomposed function becomes zero at infinity: $\Phi(\infty) = 0$ where the original contour C splits into two subcontours, namely: $C = \Gamma + C^*$ so that the conjugate function $f^*(z)$ is regular within the circle whose radius is greater than one. Notice that in absence of the MSR paradigm, the circle with radius > 1 could be interpreted as implying that the range of the function under consideration should be decomposed into primary subrange and a reciprocal subrange that is dual to the primary one.

The decomposition he has proposed in (3) seems to be straightforward but, nevertheless, some of its prospective – though perhaps not always foreseeable – abstract consequences can appear as truly mindboggling at that time. He also recognized the fact that the complex function F(Z) is a function of exponential type (via the transformation $w=e^{-z}$)

$$F(Z) = \frac{1}{2\pi i}\int \Phi(e^{-z})e^{zZ}dz \qquad (4)$$

as it appears again in [11] on p.554. Hence, for |z| greater than the radius of the given circle one can write the following evaluation of the primary function $\Phi(z)$:

$$\Phi(z) \coloneqq \frac{1}{2\pi i}\int_{\Gamma}\frac{f(w)dw}{w-z} = -\frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\int_{\Gamma} f(w)\, w^n dw \qquad (5)$$

that again somewhat justifies his aforesaid conclusion that $\Phi(\infty) = 0$ which thus maps infinity onto zero - see [11] p.552. Compare also the additional discussion of the function $f^*(z)$ given in [12]. Various types of inversions are concisely discussed and simply illustrated in [13]. A special case of an integral evaluated via certain power series can thus be written as follows:

$$\int_1^{\infty} w(u)u^{-s}du \qquad (6)$$

which might appear as being formwise more general, is also discussed in [14].

While the complex decomposition formula (3) is extremely important for conceptual understanding of the issues discussed in the present paper, it was largely ignored, presumably due to the – rarely admitted – fact that if the proposed decomposition would be taken really seriously then one could perhaps demand that – for its realistic implementation – we should eventually abandon the formerly unspoken and thus left unchallenged single-space reality (SSR) paradigm and eventually embrace the new, considerably more comprehensive, multispatial reality (MSR) paradigm.

Although the above formulas Pólya's has offered are operationally correct and quite unambiguous when considered in the complex domain, their prospective transfer into the domain of real numbers may seem rather problematic. The realization of this issue appears somewhat contrary to the common sense expectation that direct transition from the – more complicated, as it is – 2D domain $\mathbb{C}$ of complex numbers to the simpler 1D domain $\mathbb{R}$ of real numbers should be easier than would be the other way around. As straightforward as the decomposition formula (3) is in the complex domain, its transfer over to the real domain could become somewhat tricky – or even faulty – if the dual function f*(x) of a real variable x is compounded with a function other than the exponential function whose first derivative is identical with the function itself, namely $(e^x)' = e^x$; for it is unique feature if the transition is considered in the traditional SSR setting. Under auspices of the former SSR paradigm, the issue is similar to that of kernels of inverse integral operators in the real numbers' domain.

However, the above exposition of complex singularities apparently involved both: the notion of reciprocity and the – extremely difficult to evaluate in the traditional SSR setting – notion of operational infinity. Hence a review of incomplete elliptic integrals, which too are impossible to be solved in general, is advisable. A close look at two – mutually reciprocal – incomplete elliptic integrals may help us to grasp the theoretical immensity of the ingenious decomposition formula.

3. RECIPROCAL INTEGRALS WITH ZERO AND INFINITY ARE SIMILAR TO INCOMPLETE ELLIPTIC INTEGRALS

As long as functions involving angles are treated in the usual length-based space alone, most differential and integral operations on them performed via trigonometric functions are fairly well handled with the exception of some inhomogeneous ratios of the angular and length-based variables such as (sin x)/x, for example.

However, when differential and/or integral equations involve both: angles as indirect functions of the elapsing time parameter (that is also not always directly representable among length-based variables), such equations may not always be expressible – in general – in terms of integrated functionals, even though the equations can be approximately evaluated by numerical computations.

In the study of pendulum in physics, its nonlinear differential equation:

$$\frac{d^2\theta}{dt^2} + sin\theta = 0 \quad (7)$$

leads to an equation for period of the pendulum's oscillations expressed by the – reciprocal/inverse by "nature", or by design, if you will – elliptical integrals; see [15] p.96 for explanation of the variables involved there. Among such mutually reciprocal integrals, the incomplete elliptic integral of the first kind, namely:

$$M(k,\phi) = \int_0^{\phi} \frac{d\theta}{\sqrt{(1-k^2\sin^2\theta}} \quad (8)$$

which is reciprocal to the incomplete elliptic integral of the second kind, namely:

$$E(k,\phi) = \int_0^{\phi} \sqrt{(1-k^2\sin^2\theta}\ d\theta \quad (9)$$

see [15] p.98, [16], are of interest to us; compare also calculations and comprehensive explanations with colorful illustrations with Equator (i.e. the Atlas functions calculator) in [17]. Recall that Jacobi elliptic functions are usually defined also as inverses of the elliptical integral of the first kind [15] p.99; compare also [18] for a explanation and [19] for annotations and more references. The swinging time rate dt of the elapsing time interval t is given as:

$$dt = \frac{d\theta}{\sqrt{2(\cos\theta - \cos\theta_0)}} \quad (10)$$

which also stays in reciprocal/inverse relation to the rate dθ of the swinging angle θ [15] p.96. Although trigonometric functions made relationships between angles and curves or line segments easy, variables involving elapsing-time-based rates of change of these magnitudes (i.e. differentials) are not always quite simple. As a matter of fact, Laplace has already recognized the incompatibility of angular variables and line- or arcs-based intervals for he used the imaginary symbol $i = \sqrt{(-1)}$ particularly for circular arcs [20].

The reciprocal relations are yet another hint suggesting the need for multispatial approach to multiplicatively inversive reciprocal variables. Since the auxiliary variable ϕ goes from 0 to π/2 when θ ranges from 0 to θ_0 [15] p.97, one may view the variable ϕ as an operational equivalent of the varying value of infinity, which is not single-valued but set-valued varying operational entity. Hence the variable ϕ in the elliptic integrals can be replaced with the infinity symbol ∞ in the real domain. Reciprocity is also perceptible in the formula for the period T of the swinging pendulum, which is also reciprocal to the elapsed time interval t

$$T = 4\int_0^{\pi/2} \frac{d\theta}{\sqrt{(1-k^2\sin^2\theta}} \quad (11)$$

that too can be expressed with the use of the incomplete elliptical integral of the first kind – compare [15] p.97. Notice also its resemblance to very simplified differential equation of transverse displacement of a vibrating string [21] p.1ff, which corresponds to the angle of swinging pendulum. An illustration of a point belonging to complement is shown in [22] p.22f. Reciprocity of integral representations involving infinities makes thus the equations with incomplete elliptic integrals impossible to solve in general in the former SSR setting.

4. ANALOGY TO KERNELS OF INVERSE INTEGRAL OPERATORS

If an inverse differential operator L^{-1} exists, it takes the form of an integral operator, the kernel of which is Green's function of the operator L [21] p.141. By analogy to so-defined kernel, let us consider now a scalar integral kernel g(x) = K(t,x)▫f(x) of an abstract integral transformation K compounded with the simple function f(t) = t, for the sake of simplicity

$$g(x) = K \cdot f(x) = \int_0^{\infty} K(t,x) \cdot f(t) \cdot dt \quad with\, f(t) = t \qquad (12)$$

which will be used to show some formerly unnoticed issues to be highlighted in this paper. I am using tentatively the extra symbol ▫ in order to make absolutely clear the composition of the scalar terms/functions. In addition to depicting composition, the extra symbol also signifies multiplication of the scalar functions involved therein. My reason for using the symbol ▫ in the formulas is to emphasize the usually ignored fact that we do not always really know whether or not the operand under the integral is legitimately obtained differential, and if it is then of which function. For brief introduction to kernels of integral operators see [23]; compare also [21] p.79f, or [24] for integral kernels with an influence function K(x,t). For various methods of solving integral equations see [25], [26], [27], [28].

Since the integral transformation K() and the prospective differential transformation are defined in the traditional SSR framework but they are to be used also in the MSR setting of paired dual reciprocal spaces, let me evaluate the compound integrand K(t,x)▫f(t)▫dt according to the product differentiation rule (PDR) for scalar functions with respect to the formally independent variable t

$$\{f(t) \cdot K(t,x)\}'_t = f(t) \cdot K'(t,[x]) + f'(t) \cdot K(t,[x]) =$$

$$= t \cdot \frac{dK(t,[x])}{dt} + [K(t,[x])] \cdot dt \qquad (13)$$

where f(t) = t while [x] and all other possible variables/functions in square brackets denote certain provisionally fixed functionals, including such as might have resulted from certain previously performed, but often unmentioned explicitly, integrations.

Notice that without the extra separator ▫ the term [K(t,[x])▫dt on the far right-hand side (RHS) of (13) could be mistaken for properly obtained differential even though only the term dt represents differential of the properly differentiated function f(t) = t so that df = dt. Although for most practical purposes such a mistaken identity may seem immaterial, when it comes to inversions the distinction is of paramount operational and conceptual importance.

The notoriously lousy and often inconsiderate traditional mathematics sometimes made mistakes of that kind because of its lack of clarity that resulted in ambiguous evaluations, some of which generated then tacitly veiled nonsenses. Although some authors do not care about whether or not the terms in the formula (12) qualify for use as operands to be integrated, I think that it is important, that the terms are properly obtained and also properly formed as legitimate differentials. The point I am trying to make is that although the result [K(t,[x])]▫dt standing on the far RHS of the formula (13) has been properly obtained in accordance with the PDR rule, but it is not really legitimate differential, i.e. [K(t,[x])]▫dt ⇎ [K(t,[x])]dt as it might be mistaken, in absence of the extra separating symbol ▫. By the way, I am not trying to be rigorous for show.

I just want to demonstrate that lack of rigor in this particular case of mistaken identity virtually conceals presence of an unsuspected extra reality, which was not recognized in the former mathematics and thus the concealed extra reality was often invisible in the SSR framework.

Note that despite its outer appearance, the expression [K(t,[x])] on the far RHS of (13) is just a fixed functional (i.e. as if “frozen” function) in both variables (x and t, but in t only temporarily). In general, the term [K(t,[x])]▫dt is not properly obtained differential, neither with respect to the variable t nor x. Therefore the term [K(t,[x])]▫dt is not really truly covariant differential but might be a compound contravariant expression that has not been explicitly identified as being contravariant at its core due to the lousy traditional notation. This is because [K(t,[x])] = const, i.e. the functional [K(t,[x])] is actually fixed with a certain constant value, even though the variable t may not always be recognized as not really varying therein.

If it were constant while being represented in the same primary space, its derivative would be equal zero. That is why the term was routinely neglected as if it was equal to zero. Yet because it is actually contravariant or mixed expression that should belong in a reciprocal space, it cannot belong to the same space in which the proper derivative $\frac{dK(t,[x])}{dt}$ dwells.

In traditional mathematics the PDR was abused in at least two ways: either by omitting some of its inconvenient terms (see section 2 in [29], for example) or by virtually assuming that undifferentiated functions could be compounded and thus treated as if they were legitimately obtained and properly formed differentials. In either case, the frequently disrespected PDR rule can – and sometimes did – lead to drawing nonsensical conclusions.

As reciprocal to covariant representations, contravariant representations can be expressed in terms of inverse differential operators [30], [31], [3]. Hence, for the compounded scalar functions f() and K(), the eq. (13) can be legitimately expressed as

$$\{f(t) \,\text{▫}\, K(t,x)\}'_t \coloneqq t \,\text{▫}\, \frac{dK(t,[x])}{dt} \oplus \frac{1}{K([t],[x])} \,\text{▫}\, \frac{1}{dt} =$$

$$= t \,\text{▫}\, K'(t,[x])dt \oplus \frac{1}{K([t],[x])} \,\text{▫}\, \frac{1}{dt} \tag{14}$$

where the symbol $\oplus$ emphasizes here the formal inappropriateness of adding the illegitimate – even though legitimately obtained – expression $\frac{1}{K([t],[x])} \,\text{▫}\, \frac{1}{dt}$ which does not really form legitimate covariant derivative but a reciprocal contravariant expression.

Since the expression $\frac{1}{K([t],[x])} \,\text{▫}\, \frac{1}{dt}$ in (14) is not properly formed covariant differential, it cannot be used as legitimate integrand in the prospective integral formed of the legitimate differentiation that had created it, it must definitely be replaced with an operationally legitimate inverted/reciprocal differential. Yet we can turn the contravariant expression into legitimate covariant differential and integrate it as:

$$\frac{1}{K([t],[x])} \,\text{▫}\, \frac{1}{dt} \Rightarrow \int\{1 \,/\, \frac{\partial K(t,[x])}{\partial t}\} \,\text{▫}\, \frac{1}{dt} = \int\{\frac{1}{\frac{\partial K(t,[x])}{\partial t}} \,\text{▫}\, \frac{1}{dt}\} = \int\{\frac{\partial t}{K\prime([t],[x])dt} \,\text{▫}\, \frac{1}{dt}\} =$$

$$= \int\{\frac{1}{K\prime([t],[x])dt}\} \tag{15}$$

which yields operationally correct compound function g(x) of (12) now expressed as

$$g(x) = K(t,x) \cdot f(x) = \int_0^\infty K(t,x) \cdot f(t) \cdot dt =$$

$$= \int_1^\infty t \cdot K'(t,x)dt \oplus \int_0^1 1 \,/\, K'(t,[x])dt \qquad (16)$$

where the term K'(t,x)dt in the operands of both integrals is now legitimately obtained and properly formed differential of the integral transformation K with respect to the formally independent variable t. The operational formula (15) makes the formula (16) admissible. We may compound valid (i.e. legitimately obtained and correctly formed) distinct differentials provided their composition forms properly composed covariant differential or derivative.

Nevertheless, the inverted operand in the integral on the far RHS of the formula (16) belongs in dual reciprocal space paired with the given primary space in which the first integral standing on the LHS of the final expression dwells, and therefore the special sum symbol ⊕ is necessary in order to avoid possible confusion as to what belongs where. The inverted integral is infinitesimal and thus refers to the infinitesimal descending infinity, which belongs to interval (0,1). Hence the original primary interval (0,∞) now becomes (1,∞) to which the primary integral belongs. Inversions were discussed in simple geometric terms in [32].

The result (16) demands explanation of topics involving tensor calculus, especially the apparent vindication of the formerly undesirable contravariant derivatives, which now can be made acceptable, though only in dual reciprocal spaces. The issues related to tensor calculus shall be discussed elsewhere.

5. VALIDITY OF COMPOUND ALGEBRAIC AND DIFFERENTIAL OPERATIONS DEPENDS ON THE PARADIGMS ESPOUSED

I have demonstrated above that quite legitimate process of differentiation of two scalar functions can result in rather inadmissible compounded expression – even when the scalar product is quite legitimately obtained – but with its syntax possibly malformed. The mishap happened in the traditional SSR setting. Therefore, it was necessary to reformulate the whole operation of the product differentiation rule of compounded scalar functions in the MSR framework and cast the reformulated operation in an abstract dual reciprocal structure paired with the primary spatial structure.

In the MSR setting the integral kernels can contain two equitable terms, which appear as being analogous to the – unsolvable in general – incomplete elliptic integrals of the first and second kind. The simple operation of differentiation in the SSR setting requires thus multistaged operational procedure cast in the MSR framework.

Notice that I am not criticizing the theory of integral kernels, but am emphasizing the fact that we should be mindful of what is theoretically admissible and what is not quite acceptable unless it has been successfully formalized in the MSR setting. It is obvious that whether we like it or not, the abstract mathematical reality (as well as the – corresponding to it – physical reality) apparently favors multispatial representations of operational procedures and of the – corresponding to them – geometric and quasigeometric spatial structures. The refusal to admit pairing of dual reciprocal spaces in traditional mathematics tacitly created cover-up for denying the possibility of effective operations involving zero and infinity as the natural reciprocal of zero, see [33], [34].

The misunderstood infinity adversely affected not only abstract thinking in mathematics, but also abstract thinking in soome physical sciences such as quantum electrodynamics, for instance, which implied a nonsense, namely that infinite amounts of energy are emitted [35]. An example of the – sometimes thoughtlessly enforced yet conceptually rather unclear – ways of thinking in applied traditional mathematics (grounded in the SSR paradigm) is supplied by Nahin, who admits that some beginning students ask how can a vector rotate with negative frequency: $e^{-i\omega t} = e^{i(-\omega)t}$ see [36] p.36.

One can see that the negative frequency could be as well interpreted as actually positive frequency once it is considered in reciprocal space because we can obviously reformulate the rotating vector as $e^{-i\omega t} = 1/e^{i\omega t}$ with the imaginary unit being interpreted as an operator pointing to yet another, reciprocal space while the minus sign signifies reciprocal inversion. As a matter of fact, just a few pages later Nahin admits that motion can be evaluated either in the space domain as well as in the time domain [36] p.40f. As the time domain could be interpreted as being dual reciprocal to the space domain, when considered from the standpoint of the MSR paradigm, the aforesaid confusion is avoidable within the MSR framework.

The idea of pairing of dual reciprocal spaces is not only compliant with many otherwise contentious issues of traditional mathematics, but it is forward-looking as well. Gisin has remarked that local variables cannot describe the quantum correlations observed in tests of Bell's inequalities and he also showed that nonlocal variables cannot describe quantum correlations in a relativistic time-order invariant way either [37]. He emphasized the fact that if there were local hidden variables, quantum systems could not win the Bell's game – as Gisin called it – see [38], [39], the actual existence of which has been experimentally corroborated beyond reasonable doubt.

The variables housed in the dual reciprocal spaces are definitely not local, for they cannot be represented directly in the given primary space. Local variables are housed in the primary space and thus represented in the native basis of the primary space, whereas nonlocal variables are housed in the dual reciprocal space and thus represented in the reciprocal basis.

From the standpoint of the MSR paradigm, the pairing of reciprocal spaces apparently supplies the theoretical vehicle for representing the nonlocal interactions without any extraordinary physical assumptions. The mathematics that is necessary for representing the nonlocal happenings but previously was not recognized as capable of supporting nonlocality.

Complementary hidden variables are already available in the conventional differential calculus [40]. Nonlocality, however, would require some kind of supplementary hidden variables for the quantum nonlocality idea to work. It is fairly easy to see that paired dual reciprocal spatial structures supply such supplementary variables, whose presence had not been recognized before, and therefore such variables could be considered as being previously hidden in the sense that the possibility of their actual occurrence was entirely unsuspected in the former mathematics and physics.

Nevertheless, as John S. Bell remarked, the nonlocality does not seem to be related to the de Broglie-Bohm reconstruction of classical mechanics [41]. Therefore, it would require a new kind of approach to the already known mathematics, I think. The demonstrated above operational necessity of pairing of dual reciprocal spaces makes thus the abstract idea of the – experimentally confirmed – quantum nonlocality both: conceptually sustainable and mathematically necessary.

6. CONCLUSIONS

Transition from the infinitesimal descending complex singularity to the infinite ascending singularity to be deployed in the real domain implies the need for some conceptual modifications of the twin operational structure in which the two singularities can unobtrusively coexist. I have showed that effective coupling of these two conceptually distinct types of singularities requires pairing of dual reciprocal spaces.

It has also been demonstrated that properly conducted differentiation of certain compound scalar functions – performed according to the product differentiation rule – can yield expressions comprising legitimately obtained yet mathematically illegitimate differentials, which may appear as containing contravariant-looking derivatives when evaluated from the standpoint of the formerly unspoken – and thus unquestioned – single space reality paradigm.

It has also been shown that the mistakenly identified (and thus often just unapologetically discarded) contravariant expressions can be converted and turned into legitimate covariant derivatives and differentials resulting in properly formed covariant differentials which can then serve as operands in prospective integrals, provided that the integrations of operands formed from the converted differentials are going to be performed in a dual reciprocal space paired with the given primary space.

The latter conclusion implies the necessity to make paradigm shift from the traditionally undisclosed but virtually accepted as true by default – and thus never challenged – single-space reality paradigm to the new, multispatial reality paradigm, which appears in compliance with unambiguous operations involving both zero and infinity, as the "natural" reciprocal to zero. Furthermore, the new multispatial reality paradigm complies with some demands imposed on mathematics by the quantum nonlocality phenomena, which makes it not only backward-compatible with classical physics but also forward-looking toward certain new mathematical evaluations of the –experimentally confirmed – quantum nonlocality.

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