Available online at www.worldscientificnews.com

World Scientific News

An International Scientific Journal

WSN 143 (2020) 53-66 EISSN 2392-2192

Solutions of a class of singular linear systems of difference equations. Part 1

Asmaa M. Kanan

Department of Mathematics, Faculty of Science, Sabratha University, Sabratha, Libya

E-mail address: Asmaakanan20@gmail.com

ABSTRACT

We extend results of Campbell, Meyer, Jr. and Rose of applications of the Drazine inverse to linear systems of differential equations with singular constant coefficients to solutions of linear systems of difference equations $A\,x_{n+1} + B\,x_n = f_n$, $n \geq 0$ when A and B are $m \times m$ complex matrices and may both singular, under conditions that $rank(A) = 1$ and trace of A is not equal zero. f_n is an arbitrary function in $\mathbb{C}^m$, and $x_n \in \mathbb{C}^m$. We give a new closed form for all solutions of those systems when they are tractable, using the theory of the Drazin inverse, and a matrix $K \in \mathbb{C}^{m\times m}$.

***Keywords*:** Singular difference equations, Drazin inverse, Rank

1. INTRODUCTION

The linear singular systems of difference equations play an important role in mathematical models. They appear in many practical areas, such as the Leslie population growth model, backward population projection, discrete systems, discrete control problem, Leontief model.

There is a relation between linear difference equations and linear differential equations. Anyone who has made a study of differential equations will know that even supposedly elementary examples can be hard to solve. Thus many authors have studied classes of difference systems, and the stability of such systems, see [1-9].

(Received 07 February 2020; Accepted 26 February 2020; Date of Publication 27 February 2020)

Some of them found many results of singular systems of differential equations applied on singular systems of difference equations [10-15].

In this paper, we study the linear systems of first order difference equations with singular constant coefficients. We treat the linear system as a difference equation $A\,x_{n+1} + B\,x_n = f_n$, $n \geq 0$, in the case that $A, B \in \mathbb{C}^{m\times m}$, $x_n \in \mathbb{C}^m$. f_n is an arbitrary function in $\mathbb{C}^m$ defined for all n. A and B may both singular. We extend solutions notion for linear systems of differential equations with singular constant coefficients [16, 17] with some of notions in [18, 19] to the linear systems of differential equations. Hence we give closed form solutions for linear systems of difference equations with singular constant coefficients when the system is tractable, under the conditions $rank(A) = 1$ and trace of A is not equal zero.

Throughout this paper, the range of A is denoted by $R(A)$, the null space of A is denoted by $\mathcal{N}(A)$. I is an identity matrix, $\mathbf{0}$ is zero vector, O is zero matrix such that $O^0 = I$. Trace of A is denoted by $Tr(A)$.

2. PRELIMINARIES

Consider the following linear singular systems of difference equations

$$A\,x_{n+1} + B\,x_n = f_n, \qquad n \geq 0, \tag{1}$$

where $A, B \in \mathbb{C}^{m\times m}$ may both be singular, and x_n, $f_n \in \mathbb{C}^m$. $rank(A) = 1$ and $Tr(A) \neq 0$.

The homogeneous system of (1) is

$$A\,x_{n+1} + B\,x_n = \mathbf{0}, \qquad n \geq 0. \tag{2}$$

The initial value problem associated with (1) is

$$A\,x_{n+1} + B\,x_n = f_n, \qquad x_0 = c, \qquad n = 1, 2, \dots \tag{3}$$

The initial value problem associated with (2) is

$$A\,x_{n+1} + B\,x_n = \mathbf{0}, \qquad x_0 = c, \qquad n = 1, 2, \dots \tag{4}$$

Definition 2.1. Let $A \in \mathbb{C}^{m\times m}$. The Drazin inverse of A is the unique matrix $A^D \in \mathbb{C}^{m\times m}$ satisfies the following conditions:

$$(i) \quad A^D A A^D = A^D,$$

$$(ii) \quad A A^D = A^D A,$$

$$(iii) \quad A^{k+1} A^D = A^k,$$

where $k = Ind(A)$ is called the index of A, it is the smallest non-negative integer such that

$$rank(A^k) = rank(A^{k+1}).$$

Note that A^D always exists, and $A^D = A^{-1}$ for $Ind(A) = 0$. Some of Properties of the Drazin inverse can be found in [17, 20].

The Drazin inverse is preserved by similarity [20]. This means, if T is nonsingular, then

$$(T^{-1}AT)^D = T^{-1}A^D T,$$

(or $(TAT^{-1})^D = TA^D T^{-1}$). So, if A is singular with $Ind(A) = k > 0$, and is written as

$$A = T^{-1}\begin{bmatrix} C & O \\ O & N \end{bmatrix} T,$$

(or $= T\begin{bmatrix} C & O \\ O & N \end{bmatrix} T^{-1}$), where C is nonsingular and N is nilpotent of index k, then

$$A^D = T^{-1}\begin{bmatrix} C^{-1} & O \\ O & O \end{bmatrix} T,$$

(or $A^D = T\begin{bmatrix} C^{-1} & O \\ O & O \end{bmatrix} T^{-1}$). You can note that, for this A, there exist unique matrices C and N such that

$$A = C + N,\ \ NC = CN = \mathbf{0},$$

where N is nilpotent of index k,

$$N = A(I - A^D A), C = A^2 A^D, A^D = C^D.$$

Note that A, B, A^D, B^D all commute, since A and B commute. In this paper we assume that $AB = BA$.

Definition 2.2. A vector $c \in \mathbb{C}^{m\times m}$ is called a consistent initial condition for the difference equation (1), if the initial value problem (3), possesses at least one solution for x_n, where $A, B \in \mathbb{C}^{m\times m}$.

Note that, the set of consistent initial conditions of (3) is a linear subspace.

Definition 2.3. The system (1) is called a tractable if the initial value problem (3) possesses exactly one solution for every consistent initial condition c.

Definition 2.4. If $A \in \mathbb{C}^{m\times m}$, if $C_A = AA^D A = A^2 A^D = A^D A^2$ and if $N_A = A - C_A$, then for integers $m \geq -1$ we define

$$C_A^{(l)} = A^{l+1}A^D = \begin{cases} A^D, & if\ m = -1 \\ AA^D, & if\ m = 0 \\ C_A^l, & if\ m \geq 1 \end{cases}$$

$$N_A^{(l)} = \begin{cases} 0, & if\ m = -1 \\ A^l - C_A^{(l)}, & if\ m \geq 0 \end{cases} = \begin{cases} 0, & if\ m = -1 \\ I - AA^D, & if\ m = 0 \\ N_A^l, & if\ m \geq 1 \end{cases}.$$

Note that $A = C_A + N_A$.

The next lemma is very important in the main results.

Lemma 2.1. [16] If $A \in \mathbb{C}^{m\times m}$ is such that $rank(A) = 1$ and $A^D = \frac{1}{[Tr(A)]^2}A$ where $Tr(A) \neq 0$, then $AA^D - K = I$, where

$$K = \begin{bmatrix} \frac{-(Tr\,(A)-a_{11})}{Tr(A)} & \frac{a_{12}}{Tr(A)} & \cdots & \frac{a_{1m}}{Tr(A)} \\ \frac{a_{21}}{Tr(A)} & \frac{-(Tr(A)-a_{22})}{Tr(A)} & \cdots & \frac{a_{2m}}{Tr(A)} \\ \cdots & & \cdots & \cdots \\ \frac{a_{m1}}{Tr(A)} & \cdots & & \\ & \cdots & \cdots & \frac{-(Tr(A)-a_{mm})}{Tr(A)} \end{bmatrix} \in \mathbb{C}^{m\times m}.$$

Note that, in the least lemma,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix}.$$

The following lemma gives a necessary and sufficient condition for (2) to be tractable.

Lemma 2.2. The homogeneous difference equation (2) is tractable if and only if there exist a scalar $\lambda \in \mathbb{C}$ such that $(\lambda A + B)^{-1}$ exists.

Lemma 2.3. Let $A, B \in \mathbb{C}^{m\times m}$. Suppose there exists a $\lambda \in \mathbb{C}$ such that that $(\lambda A + B)^{-1}$ exists, and let

$$\hat{A}_\lambda = (\lambda A + B)^{-1}A \text{ and } \hat{B}_\lambda = (\lambda A + B)^{-1}B.$$

Then

$$\hat{A}_\lambda \hat{B}_\lambda = \hat{B}_\lambda \hat{A}_\lambda.$$

Lemma 2.4. [17] Suppose that $AB = BA$. And $\mathcal{N}(A) \cap \mathcal{N}(B) = \{0\}$.

Then

$$(I - AA^D)BB^D = (I - AA^D).$$

Remark 2.1. For the linear system of difference equations

$$A\,x_{n+1} = x_n\,,\ n \geq 0,$$

where A is singular, if x_n is a solution then

$$x_n = A\,x_{n+1} = A^2\,x_{n+2} = \cdots A^k x_{n+k},$$

as well as

$$x_{n+1} = A\,x_{n+2} = A^2\,x_{n+3} = \cdots A^k x_{n+k+1},$$

when $k = ind(A)$.

3. MAIN RESULTS

In this section, we extend results of Campbell, Meyer, Jr. and Rose in [17] to linear singular systems of difference equations, and using Lemma 2.1 (Lemma 1. in [16]) we get our main results.

3. 1. The equation $x_{n+1} + Ax_n = 0$

The general solution of equation

$$x_{n+1} + A\,x_n = \mathbf{0}\,, \qquad n \geq 0 \tag{5}$$

given by

$$x_n = -A^n q,$$

where q an arbitrary vector in $\mathbb{C}^m$.

If x_n^{P} is any particular solution of the complete equation (nonhomogeneous equation)

$$x_{n+1} + A\,x_n = f_n\,, \qquad n \geq 0, \tag{6}$$

and x_n^{H} is the general solution of the complete equation then $x_n^{\mathrm{H}} + x_n^{\mathrm{P}}$ is the general solution of (6). That is

$$x_n = x_n^{\mathrm{H}} + x_n^{\mathrm{P}} = -A^n q + x_n^{\mathrm{P}}.$$

3. 2. The equation $Ax_{n+1} + Bx_n = f_n\,,\ n \geq 0$ when $AB = BA$

This equation is the same equation (1), where A and B both singular. If A is nonsingular, then this equation may be written in the form (6). We study this equation under the conditions $rank(A) = 1$ and $Tr(A) \neq 0$. The homogenous equation of (1) is (2).

Let

$$x_n^1 = (I + K)x_n \ \text{ and } \ x_n^2 = (-K)x_n,$$

where K is given by Lemma 2.1, then (1) can be written in the following equation

$$(C+N)(x_{n+1}^1+x_{n+1}^2)+B(x_n^1+x_n^2)=f_n.$$

We multiply this equation first by C^DC and then by $(I-C^DC)$ and note that C^D satisfies the conditions $CC^DC=C$ and $C^DCC^D=C^D$, so we get that (2) is equivalent to

$$Cx_{n+1}^1+Bx_n^1=f_1\,, \tag{7}$$

$$Nx_{n+1}^2+Bx_n^2=f_2\,, \tag{8}$$

where $f_1=C^DCf$ and $f_2=(I-C^DC)f$.

Since C is nonsingular then we can rewrite (7) as following:

$$x_{n+1}^1+C^DBx_n^1=C^Df\,. \tag{9}$$

Note that the equation (9) is in the form (6).

Theorem 3.1. If A and B commute with $rank(A)=1$ and $Tr(A)\neq 0$, and $\mathcal{N}(A)\cap\mathcal{N}(B)=\{0\}$.

Then

$$x_n=(-A^DB)^n(I+K)q \tag{10}$$

where q an arbitrary, is the general solution of (2).

Proof. First, we prove that (10) is a solution for (2).

$$A\,x_{n+1}+B\,x_n=A[\,(-A^DB)^{n+1}(I+K)q]+B[(-A^DB)^n(I+K)q]$$

$$=-AA^DB(-A^DB)^n(I+K)q+B(-A^DB)^n(I+K)q\ =0.$$

Now, to prove that (10) is the general solution, we need to show that for each solution x_n there exists a vector q such that (10) holds. So, if x_n is a solution then (8) and (9) hold for $f_n=0$. Since $\mathcal{N}(A)\cap\mathcal{N}(B)=\{0\}$, we have that B is one-to-one on $R(N)$ by Lemma 2.4.

Let $ind(A)=k$, then

$$0=N^kx_{n+1}^2+BN^{k-1}x_n^2=BN^{k-1}x_n^2,$$

hence

$$N^{k-1}x_n^2=0\Longrightarrow N^{k-1}x_{n+1}^2=0\,,$$

So

$$0=N^{k-1}x_{n+1}^2=-BN^{k-2}x_n^2\,.$$

Continuing in this manner, we get that $Bx_n^2 = \mathbf{0}$, $Nx_n^2 = \mathbf{0}$, and $(-K)x_n^2 = x_n^2$. Hence $x_n^2 = \mathbf{0}$ and $x_n = x_n^1$. From (9) we get

$$x_n^1 = (-C^D B)^n q = (-A^D B)^n q.$$

Thus

$$x_n = x_n^1 = (I + K)x_n^1 = (-A^D B)^n (I + K)q.$$

Theorem 3.2. If $AB = BA$. with $rank(A) = 1$ and $Tr(A) \neq 0$, and $\mathcal{N}(A) \cap \mathcal{N}(B) = \{0\}$. Then

$$x_n = A^D \sum_{i=0}^{n-1} (-A^D B)^{n-i-1} f_i + (-K) \sum_{i=0}^{k-1} (-AB^D)^i B^D\, f_{n+i}, \tag{11}$$

is the particular solution of (1).

Proof. Suppose that $AB = BA$ with $rank(A) = 1$ and $Tr(A) \neq 0$. Suppose also that $\mathcal{N}(A) \cap \mathcal{N}(B) = \{0\}$. Let

$$x_n^1 = A^D \sum_{i=0}^{n-1} (-A^D B)^{n-i-1} f_i\,,$$

$$x_n^2 = (-K) \sum_{i=0}^{k-1} (-AB^D)^i B^D\, f_{n+i}.$$

We shall show that

$$A\, x_{n+1}^1 + B\, x_n^1 = (I + K) f_n \tag{12}$$

and

$$A\, x_{n+1}^2 + B\, x_n^2 = (-K) f_n. \tag{13}$$

To verify (12),

$$A\, x_{n+1}^1 = A(\, A^D \sum_{i=0}^{n} (-A^D B)^{n-i} f_i)$$

$$= AA^D (f_n + \sum_{i=0}^{n-1} (-A^D B)^{n-i} f_i)$$

$$= AA^D f_n + AA^D \sum_{i=1}^{n-1} (-A^D B)^{n-i+1} f_{i-1}$$

$$= AA^D f_n + AA^D (-A^D B) \sum_{i=1}^{n-1} (-A^D B)^{n-i} f_{i-1}$$

$$= AA^D f_n - AA^D B(A^D \sum_{i=0}^{n-1} (-A^D B)^{n-i-1} f_i)$$

$$= AA^D f_n - Bx_n^1$$

$$= (I + K) f_n - Bx_n^1 .$$

To verify (13),

$$A\, x_{n+1}^2 = A(-K) \sum_{i=0}^{k-1} (-AB^D)^i B^D\, f_{n+1+i}$$

$$= A(-K) \sum_{i=1}^{k-1} (-AB^D)^{i-1} B^D\, f_{n+i}$$

$$= (-K) \sum_{i=1}^{k-1} (-1)(-AB^D)^i\, f_{n+i}$$

$$= (-K)BB^D \sum_{i=1}^{k-1} (-1)(-AB^D)^i\, f_{n+i}$$

$$= (-K)B \sum_{i=1}^{k-1} (-1)(-AB^D)^i B^D\, f_{n+i}$$

$$= (-K)B(-x_n^2 + B^D\, f_n)$$

$$= -Bx_n^2 + (-K)BB^D f_n$$

$$= -Bx_n^2 + (-K) f_n .$$

The next theorem results from combining Theorem 3.1 and Theorem 3.2.

Theorem 3.3. If $AB = BA$. with $rank(A) = 1$ and $Tr(A) \neq 0$, and $\mathcal{N}(A) \cap \mathcal{N}(B) = \{0\}$. Then the general solution of (1) is given by

$$x_n = (-A^D B)^n (I + K) q + A^D \sum_{i=0}^{n-1} (-A^D B)^{n-i-1} f_i + (-K) \sum_{i=0}^{k-1} (-AB^D)^i B^D f_{n+i}. \quad (14)$$

3. 3. The equation $\boldsymbol{A x_{n+1} + B x_n = f_n}$, $\boldsymbol{n} \geq 0$.

In this section we give necessary and sufficient condition for uniqueness of solutions.

Theorem 3.4. The equation $A\, x_{n+1} + B\, x_n = \mathbf{0}$ (equation (2)), where $rank(A) = 1$ and $Tr(A) \neq 0$, has unique solution for consistent initial conditions if and only if there exists a scalar $\lambda \in \mathbb{C}$ such that $(\lambda A + B)^{-1}$ exists.

Proof. Suppose there exists a $\lambda \in \mathbb{C}$ such that that $(\lambda A + B)^{-1}$ exists.
Then

$$\mathcal{N}(A) \cap \mathcal{N}(B) = \{0\},$$

But

$$\mathcal{N}(A) = \mathcal{N}((\lambda A + B)^{-1} A) = \mathcal{N}(\hat{A}_\lambda),$$

and

$$\mathcal{N}((\lambda A + B)^{-1} B) = \mathcal{N}(\hat{B}_\lambda) = \mathcal{N}(B).$$

Thus, the equation

$$(\lambda A + B)^{-1} A\, x_{n+1} + (\lambda A + B)^{-1} B\, x_n = \mathbf{0},$$

or

$$\hat{A} x_{n+1} + \hat{B}\, x_n = \mathbf{0}, \quad (15)$$

has unique solutions by Theorem 3.1. Note that the equation (15) is equivalent to (2).

Conversely, suppose that (2), has unique solutions. We want to show that there is a $\lambda \in \mathbb{C}$ such that $(\lambda A + B)^{-1}$ exists. For that, suppose that this is not true. Then $(\lambda A + B)$ is singular for all $\lambda \in \mathbb{C}$. This means that for every λ there exist a vector $u_\lambda \in \mathbb{C}^m$ such that

$$(\lambda A + B) u_\lambda = \mathbf{0}, \quad u_\lambda \neq \mathbf{0}.$$

but

$$x_n^{(\lambda)} = \lambda^n u_\lambda\,,$$

is a solution of (2), because

$$A\, x_{n+1}^{(\lambda)} + B\, x_n^{(\lambda)} = A\, \lambda^{n+1} u_\lambda + B\, \lambda^n u_\lambda = -B\, \lambda^n u_\lambda + B\, \lambda^n u_\lambda = \mathbf{0}.$$

Let $\{u_{\lambda_1}, u_{\lambda_2}, \ldots, u_{\lambda_s}\}$ be a finite linearly dependent set of such vectors. Let

$$x_n^{(\lambda_i)} = \lambda_i^{\mathrm{n}} u_{\lambda_i},$$

and let $\{c_1, c_2, \ldots, c_s\} \subseteq \mathbb{C}$ be such that

$$\sum_{i=1}^{s} c_i\, u_{\lambda_i} = \mathbf{0},$$

where not all the c_i's are 0.

Let

$$z_n = \sum_{i=1}^{s} c_i\, x_n^{(\lambda_i)}.$$

Then z_n is not identically zero and it is a solution of (2). Thus, there are two different solutions of (2), that is, z_n and 0 which both satisfy

$$A\, x_{n+1} + B\, x_n = \mathbf{0}, \; x_0 = c.$$

Therefore, (2) does not have unique solutions for consistent initial conditions which contradicts our hypothesis. Hence $(\lambda A + B)^{-1}$ exists for some $\lambda \in \mathbb{C}$.

Theorem 3.5. Suppose that $A\, x_{n+1} + B\, x_n = \mathbf{0}$ (equation (2)), with $rank(A) = 1$ and $Tr(A) \neq 0$, is tractable. Then the general solution is given by

$$x_n = \begin{cases} -(I+K)q, & if\ n = 0 \\ (\hat{A}^D \hat{B})^n q, & if\ n = 1, 2, 3, \ldots \end{cases}$$

where $q \in \mathbb{C}^m$, K is given by Lemma 2.1.,

$$\hat{A}_\lambda = (\lambda A + B)^{-1} A = \begin{bmatrix} \hat{a}_{11} & \hat{a}_{12} & \ldots & \hat{a}_{1m} \\ \hat{a}_{21} & \hat{a}_{22} & \ldots & \hat{a}_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ \hat{a}_{m1} & \hat{a}_{m2} & \ldots & \hat{a}_{mm} \end{bmatrix},$$

$$\hat{B}_\lambda = (\lambda A + B)^{-1} B.$$

Proof. Suppose (2) is tractable, then there exists a scalar $\lambda \in \mathbb{C}$ such that $\lambda A + B$ is nonsingular we multiply (2) by $(\lambda A + B)^{-1}$, we get the equivalent equation

$$\hat{A}_\lambda\, x_{n+1} + \hat{B}_\lambda\, x_n = \mathbf{0}.$$

Taking a similarity (see [20]) we may write

$$\hat{A}_\lambda = \begin{bmatrix} C & O \\ O & N \end{bmatrix}, \quad \hat{B}_\lambda = \begin{bmatrix} I-\lambda C & O \\ O & I-\lambda N \end{bmatrix}, \quad x_n = \begin{bmatrix} x_n^1 \\ x_n^2 \end{bmatrix},$$

since $\lambda \hat{A}_\lambda + \hat{B}_\lambda = I$. So the difference system becomes

$$\begin{bmatrix} C & O \\ O & N \end{bmatrix} \begin{bmatrix} x_{n+1}^1 \\ x_{n+1}^2 \end{bmatrix} + \begin{bmatrix} I-\lambda C & O \\ O & I-\lambda N \end{bmatrix} \begin{bmatrix} x_n^1 \\ x_n^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus

$$C\, x_{n+1}^1 + (I-\lambda C)\, x_n^1 = 0,$$
$$N\, x_{n+1}^2 + (I-\lambda N)\, x_n^2 = 0.$$

Since C is nonsingular then the first equation is tractable. Using Remark 2.1 we get

$$x_n^1 = (-1)^n\, C^{-n}(I-\lambda C)^n\, x_0^1,$$

and the second equation becomes

$$x_n^2 = -\,(I-\lambda N)^{-k} N^k\, x_{n+k}^2 = 0,$$

since N is nilpotent of index k.

Thus if $n = 0$, then using definition 2.4 and then using Lemma 2.1 we get

$$x_n = -(I+K)q, \quad q \in \mathbb{C}^m,$$

as desired.

If $n = 1$ we get

$$x_n = -C^{-1}(I-\lambda C)q = -\hat{A}^D \hat{B} q,$$

If $n = 2$ then

$$x_n = (C^{-1}(I-\lambda C))^2 q = (\hat{A}^D \hat{B})^2 q,$$

continuing in this manner, we get that

$$x_n = \left(\hat{A}^D \hat{B}\right)^n q, \qquad n = 1, 2, 3, \ldots.$$

as desired.

Corollary 1. Suppose that the following conditions hold

(i) There exists a $\lambda \in \mathbb{C}$ such that $(\lambda A + B)^{-1}$ exists, and $\hat{A}\hat{B} = \hat{B}\hat{A}$.

(ii) $rank(A) = 1,\ Tr(A) \neq 0$,
(iii) $c \in \mathbb{C}^m$ is a consistent initial vector for (2) with $x_0 = c$.

Then the unique solution of the homogeneous initial value problem (4) is given by

$$x_n = \left(\hat{A}^D \hat{B}\right)^n c, \qquad n = 0, 1, 2, 3, \ldots.$$

Theorem 3.6. Suppose that $A\, x_{n+1} + B\, x_n = f_n\,,\ n \geq 0$, with $rank(A) = 1$ and $Tr(A) \neq 0$, is tractable. then the particular solution is given by

$$x_n = \hat{A}^D \sum_{i=0}^{n-1} \left(-\hat{A}^D \hat{B}\right)^{n-i-1} \hat{f}_i + (-K) \sum_{i=0}^{k-1} \left(-\hat{A}\hat{B}^D\right)^i \hat{B}^D\, \hat{f}_{n+i}$$

where $\hat{f}_i = (\lambda A + B)^{-1} f_i$, and K is given by Lemma 2.1.

The proof of this theorem is the same proof of Theorem 3.2.

Combining Theorem 3.5. and 3.6 we get the next

Theorem 3.7. Suppose that $A\, x_{n+1} + B\, x_n = f_n\,,\ n \geq 0$, with $rank(A) = 1$ and $Tr(A) \neq 0$, is tractable.

Then the general solution is given by

$$x_n = \left(-\hat{A}^D \hat{B}\right)^n (I + K) q + \hat{A}^D \sum_{i=0}^{n-1} \left(-\hat{A}^D \hat{B}\right)^{n-i-1} \hat{f}_i$$

$$+(-K) \sum_{i=0}^{k-1} \left(-\hat{A}\hat{B}^D\right)^i \hat{B}^D\, \hat{f}_{n+i}$$

Corollary 2. Suppose that the following conditions hold

(i) There exists a $\lambda \in \mathbb{C}$ such that $(\lambda A + B)^{-1}$ exists, and $\hat{A}\hat{B} = \hat{B}\hat{A}$.
(ii) $rank(A) = 1,\ Tr(A) \neq 0$,
(iii) c lies in the set $\{\widehat{W} + R(\hat{A}^D)\}$,

where

$$\widehat{W} = -K \sum_{i=0}^{k-1} \left(-\hat{A}\hat{B}^D\right)^i \hat{B}^D\, \hat{f}_{n+i}$$

Then the unique solution of the nonhomogeneous initial value problem (3) is given by

$$x_n = \left(-\hat{A}^D \hat{B}\right)^n (I + K) x_0 + \hat{A}^D \sum_{i=0}^{n-1} \left(-\hat{A}^D \hat{B}\right)^{n-i-1} \hat{f}_i$$

$$+(-K)\sum_{i=0}^{k-1}\left(-\hat{A}\hat{B}^{D}\right)^{i}\hat{B}^{D}\,\hat{f}_{n+i}.$$

4. CONCLUDING REMARKS

Our study showed how to find solutions for linear systems of difference equations with singular constant coefficients in the case that the coefficients matrices A and B are square, under the conditions that $rank(A) = 1$ and $Tr(A) \neq 0$. Also, it showed that the solutions were uniquely determined by consistent initial conditions. We hope that the ideas of this paper can be applied to the same systems but in the case that A and B are rectangular.

References

[1] I. K. Dassios, On non-homogeneous linear generalized linear discrete time systems, *Circuits Systems and Signal Processing*, 31 (2012) 1699-1712

[2] I. Dassios, G. Kalogeropoulos. On a non-homogeneous singular linear discrete time system with a singular matrix pencil. *Circuits Syst Signal Process*, 32 (2013) 1615-1635

[3] I. K. Dassios, On a boundary value problem of a class of generalized linear discrete-time systems, *Advances in Difference Equations*, 2011: 51 (2011)

[4] A. A. Soliman, Stability analysis of difference systems via cone valued Liapunov`s function method, *J. Math. Kyoto Univ.* 46 (2006) 65-74.

[5] S. Zhang, Stability analysis of delay difference systems, *Comput. Math. Appl.* 33 (1997) 41-52.

[6] P. K. Anh, L. C. Loi, On multipoint boundary-value problems for linear implicit non-autonomous systems of difference equations, *Vietnam of Mathematics*, 29 (2001) 281-286

[7] C. Peng, Z. Cheng-Hui, Indefinite linear quadratic optimal control problem for singular linear discrete-time system: Krein space method. *Acta Automatica Sinica* 33 (2007) 635-639

[8] I. K. Dassios, D. I. Baleanu, On a singular system of fractional nabla difference equations with boundary conditions, *Boundary Value Problems*, 2013: 148 (2013)

[9] I. K. Dassios, On stability and state feedback stabilization of singular linear matrix difference equations, *Advances in Differential Equations*, 2012: 75 (2012)

[10] S. L. Campbell, Limit Behavior of Solutions of Difference Equations, *Linear Algebra and its Applications*, 23 (1979) 167-178

[11] P. K. Anh, H. T. N. Yen, On the Solvability of Initial-Value Problems for Nonlinear Implicit Difference Equations. *Advances in Difference Equations* volume 2004, Article number: 390451 (2004)

[12] P. K. Anh, N. H. Du, and L. C. Loi, Connections Between Implicit Difference Equations and Differential -Algebraic Equations. *Acta Mathematica Vietnamica* 29 (2004) 23-39

[13] P. K. Anh, D. S. Hoang, Stability of a Class of Singular Difference Equations, *International Journal of Difference Equations*, 1 (2006) 181-193

[14] P. K. Anh, N. H. Du, and L. C. Loi, Singular Difference Equations: An Overview, *Vietnam Journal of Mathematics*, 35 (2007) 339-372

[15] S. Hristova, A. Golev, K. Stefanova, Quasilinearization of the initial value problems for difference equations with "maxima". *J. Appl. Math*. 2012 (2012), 17 pages.

[16] A. M. Kanan, K. Hassan, Solution of linear systems of differential equations with singular constant coefficients by the Drazin inverse of matrices, *World Scientific News* 137 (2019) 229-236

[17] S. L. Campbell, C. D. Meyer, Jr., and N. J. Rose, Applications of the Drazin inverse to Linear Systems of differential equations with Singular Constant Coeffficients, *SIAM J. Appl. Math*. 31 (1976) 411-425

[18] P. Wang, X. Liu, The quadratic convergence of approximate solutions for singular difference systems of differential equations with "maxima". *J. Math. Computer Sci.* 16 (2016) 227-238

[19] P. Wang, X. Liu, T. Li, The rapid convergence for nonlinear singular differential systems with "maxima", *J. Nonlinear Sci. Appl.* 10 (2017) 5402-5421

[20] S. L. Campbell, The Drazin inverses of an infinite matrix, *SIAM J. Appl. Math.* 31 (1976) 492-503