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Chebyshev Spectral Collocation Method to Micropolar Fluid Flow through a Porous Channel driven by Suction/Injection with High Mass Transfer

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ABSTRACT

This paper presents the application of Chebyshev spectral collocation method to flow analysis of a micropolar fluid conveyed through porous channel driven by suction or injection with high mass transfer. Effects of flow and rotation parameters such as Reynolds number and micro rotation parameters on the flow characteristics of the micropolar fluid are investigated using the developed approximate analytical solutions through the method. Comparing the results of the numerical solutions obtained in this study with the other results of the other methods in literature, very good agreements are established. The results obtained from this work can be used to further the study of the behavior of micropolar fluids in applications such as lubricants, blood flow porous media, micro channels and flow in capillaries.

***Keywords*:** Micropolar fluids, Porous media, Mass Transfer, Chebyshev spectral collocation method

1. INTRODUCTION

In his bid to model the flow characteristics of some non-Newtonian fluid whose micro constituents rotate during fluid flow process, Erigen [1] introduced the theory of micropolar fluid. He went further and developed the constitutive relation to include more material

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parameters and micro rotation vectors making the usual equations for Newtonian flow non-linear. The usefulness of his theory in explaining the characteristics of certain fluids such as liquid crystals, suspensions and animal blood has been well established in literature. Consequently, in the recent studies of non-Newton fluid flow, the investigations into the flow behaviour or characteristics of micropolar fluid have attracted quite a number of research papers. Idris [2] studied the effect of non-uniform temperature gradient on micropolar fluids under convective heat transfer while Yuan [3] investigated the behavior of micropolar fluids under laminar flow condition within a porous channel. Kelson [4-5] presented the effect of surface conditions on micropolar fluid flow over a stretching sheet with strong suction and injection. The flow of viscous fluid along a porous wall during suction was studied by Zaturska et al. [6]. Power law variations were adopted by Cheng [7] to study micropolar fluid from a vertical truncated cone under natural convection. Joneidi et al. [8] applied the differential transformation method (DTM) to heat transfer problems of nonlinear equations while Hassan [9] adopted the DTM in solving Eigen value problems. Magyari and Keller [10] studied boundary layer flows induced by permeable walls using exacts solutions. Natural convective flow over horizontal plate was investigated by Murthy and Singh [11] presenting the thermal effects with surface mass flux on convection. In another work, the flow of micropolar fluid with a fully developed natural convection in a vertical channel was analyzed by Chamkha et al. [12] and Abdulaziz and Hashim [13].

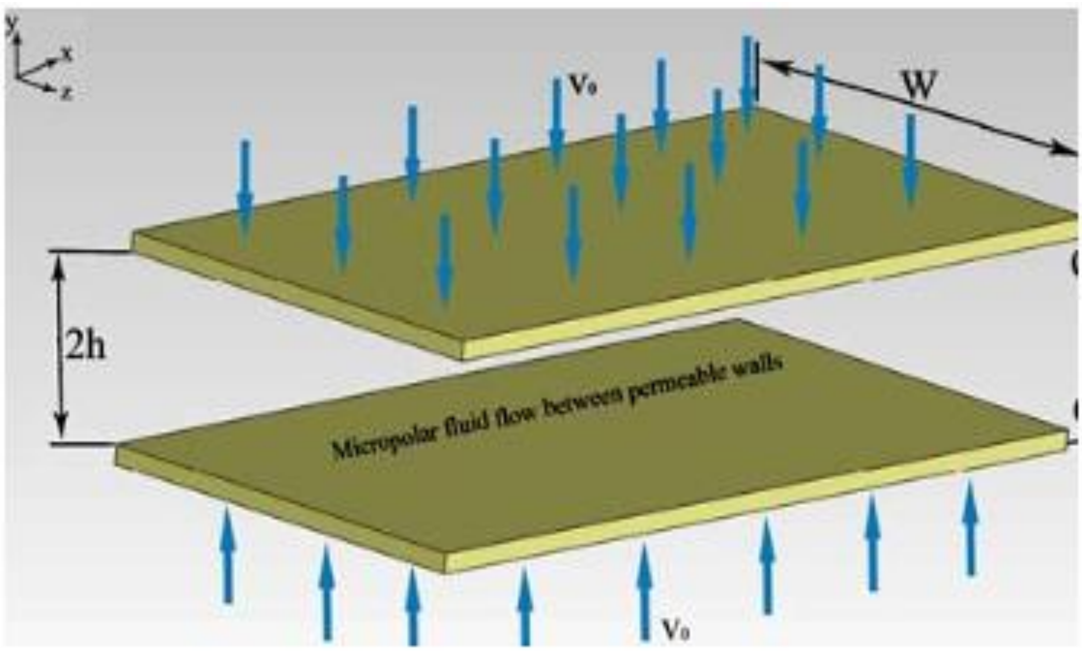
Si et al. [14] and Beg et al. [15] adopted homotopy analysis method was used to investigate the flow pattern of micropolar fluid flow through a porous medium. However, Rashidi et al. [16] applied a modified differential transformation method to study the free convection fluid flow and heat of micropolar fluid under the influence of magnetic field. An insight into the boundary-layer flow of a micropolar fluid through a porous channel was presented by Rashidi et al. [17] using semi-analytical method. Further studies on the influence of magnetic field, slip velocity, thermal radiation and internal heat generation on the flow behaviour of micropolar were presented by Narayana et al. [18], Oahimire and Olajuwon [19], Olajuwon et al. [20], Prakash and Muthamilselvan [21], Mahmoud et al. [22] and Borrelli et al [23]. The squeezing film characteristics of micropolar fluid between porous parallel stepped plates was focus of the paper submitted by Siddangoudaa [24]

Sequel to the reviewed works, it can be concluded that different approximate analytical methods have been used to analyzed the different nonlinear models developed for the flow process studied in the past works. However, the large number of expressions in the approximate analytical solutions of these methods made them in most cases, unsuitable for practical uses and for designers. Therefore, in recent times, a new numerical method called Chebyshev spectral collocation method was developed.

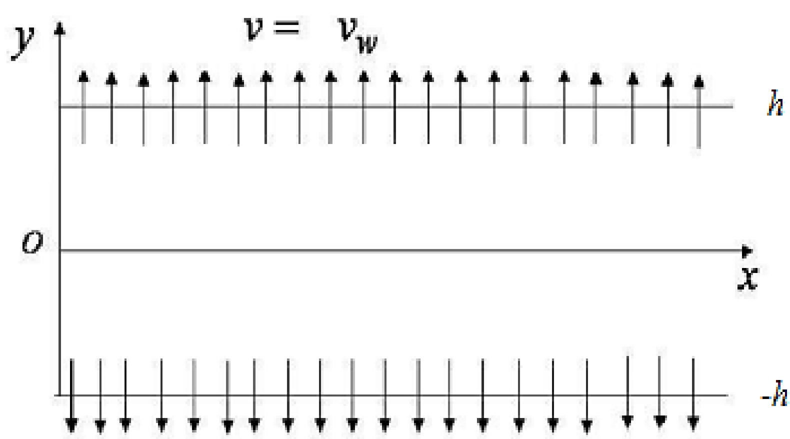
The method has been shown to be relatively simple, more convenient, reliable and practicable [25-38] due to its large converging speed and the fast rate of convergence over most of the commonly used numerical methods. The wide range of application of this method ranges from computational fluid dynamics [25, 26], electrodynamics [27], magnetohydrodynamics [28, 29] to numerical solutions for non-linear differential equations [30-38] has been established. Therefore, in this study, Chebychev spectral collocation method is applied to analyze the flow and rotation of micropolar fluids transported through porous channels with high mass transfer is analyzed. Effects of flow and rotation parameters such as Reynolds number and micro rotation parameters on the flow characteristics of the micropolar fluid are investigated.

2. MODEL DEVELOPMENT AND ANALYTICAL SOLUTION

Consider the laminar, incompressible and isothermal flow of a micropolar fluid through a channel with porous walls where fluid undergoes suction or injection with speed v_w. The channel wall are parallel to the x axis as described using cartesian co-ordinate with width of distance 2h and located at a reference $y = \pm h$. The formulation of the model development of the micropolarfluid is developed with respect to the above conditions following the assumptions that the fluid is incompressible, flow is steady and laminar. Also radiation heat transfer is negligible.



(a)



(b)

Fig. 1(a,b). Geometry of micropolar fluid flow through a porous channel with suction and injection walls (Permeable walls) [29]

Following the assumptions, the governing equations of the channel flow are given as

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0 \tag{1}$$

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial P}{\partial x}+(\mu+\kappa)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\kappa\frac{\partial N}{\partial y} \tag{2}$$

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial P}{\partial y}+(\mu+\kappa)\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+\kappa\frac{\partial N}{\partial x} \tag{3}$$

$$\rho\left(u\frac{\partial N}{\partial x}+v\frac{\partial N}{\partial y}\right)=-\frac{\kappa}{j}(\mu+\kappa)\left(2N+\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+\frac{\mu_s}{j}\left(\frac{\partial^2 N}{\partial x^2}+\frac{\partial^2 N}{\partial y^2}\right) \tag{4}$$

The governing equations expressed in Eqs. (1)-(4) include micro rotation or angular velocity and material parameters which direction is in the xy - plane consistent with other micropolar fluid studies. In this study, material parameters are taken as independent and constant.

$$\begin{aligned}&u(x,\pm h)=0, v(x,\pm h)=\pm v_w,\\&N(x,\pm \mathrm{h})=-\mathrm{s}\left.\frac{\partial u}{\partial y}\right|_{x=\pm h}\end{aligned} \tag{5}$$

Fluid flow is assumed symmetric about y = 0

$$\frac{\partial u}{\partial y}(x,0)=v(x,0)=0 \tag{6}$$

The value of s depicts various flow situation of the micropolar fluid. When s = 0 the microelement close to the porous wall surface are unable to rotate while when s = 0.5 the microrotation is same as the fluid vorticity at the boundary. Similarly fluid injected or removed from the stream is depicted by the value of q. Given that suction is the condition when $q>0$ and injection is the situation when $q<0$. The governing equation is therefore simplified by including micropolar effects by assuming stream functions and micropolar to the Berman's similarity solution [26]:

$$\psi=-v_w xF(\eta) \tag{7}$$

$$N=\frac{qx}{h^2}g(\eta) \tag{8}$$

where

$$\eta = \frac{y}{h}, u = \frac{\partial \psi}{\partial y} = -\frac{v_w x}{h} F^{'}(\eta), v = -\frac{\partial \psi}{\partial x} = v_w F(\eta) \tag{9}$$

Dimensionless micropolar parameters and non-zero cross flow Reynolds number are introduced as

$$N_1 = \frac{\kappa}{\mu}, N_2 = \frac{v_s}{\mu h^2}, N_3 = \frac{j}{h^2}, \mathrm{Re} = \frac{\rho v_w h}{\mu} \tag{10}$$

With the aid of Eqs.(7)-(10) the Eqs (1)-(4) may be reduced to ordinary nonlinear differential equations as stated below:

$$(1+N_1)\frac{d^4F}{d\eta^4} - N_1\frac{d^2G}{d\eta^2} - Re\left(F\frac{d^3F}{d\eta^3} - \frac{dF}{d\eta}\frac{d^2F}{d\eta^2}\right) = 0 \tag{11}$$

$$N_2\frac{d^2G}{d\eta^2} + N_1\left(\frac{d^2F}{d\eta^2} - 2G\right) - N_3 Re\left(F\frac{dG}{d\eta} - G\frac{dF}{d\eta}\right) = 0 \tag{12}$$

With the appropriate boundary conditions defined as

$$F(\pm 1) = 1, F^{'}(\pm 1) = 0, G(\pm 1) = sF^{"}(\pm 1) \tag{13}$$

Symmetry of fluid flow through the porous channel is assumed therefore boundary condition takes the form

$$\begin{aligned} &F(0) = F^{"}(0) = F^{'}(1) = 0,\ F(1) = 1 \\ &G^{'}(0) = 0,\ G(1) = \mathrm{s}\,F^{"}(1) \end{aligned} \tag{14}$$

3. APPLICATION OF THE CHEBYCHEV SPECTRAL COLLOCATION METHOD

The basic definition and the procedure of the method can be found in any of the previous publications [25-39]. Making a suitable transformation to map the physical domain [0, 1] to a computational domain [-1,1] to facilitate our computations. Eqs. (12) are transformed to the following equations

$$(1+N_1)\frac{d^4\tilde{F}}{d\eta^4} - N_1\frac{d^2\tilde{G}}{d\eta^2} - Re\left(\tilde{F}\frac{d^3\tilde{F}}{d\eta^3} - \frac{d\tilde{F}}{d\eta}\frac{d^2\tilde{F}}{d\eta^2}\right) = 0 \tag{15}$$

$$N_2\frac{d^2\tilde{G}}{d\eta^2}+N_1\left(\frac{d^2\tilde{F}}{d\eta^2}-2\tilde{G}\right)-N_3Re\left(\tilde{F}\frac{d\tilde{G}}{d\eta}-\tilde{G}\frac{d\tilde{F}}{d\eta}\right)=0 \tag{16}$$

and the appropriate boundary conditions of Eq. (14) become

$$\begin{aligned}&\tilde{F}(0)=\tilde{F}^{''}(0)=\tilde{F}^{'}(1)=0,\ \tilde{F}(1)=1\\&\tilde{G}^{'}(0)=0,\ \tilde{G}(1)=s\,\tilde{F}^{''}(1)\end{aligned} \tag{17}$$

Invoking CSCM to Eq. (15) and the boundary conditions in Eq. (17), one arrives at a system of nonlinear algebraic equations as

$$\begin{aligned}&(1+N_1)\sum_{j=0}^{M}d_{k,j}^{(4)}\tilde{F}\left(\eta_j\right)+N_1\left(\sum_{j=0}^{N}d_{k,j}^{(2)}\tilde{G}\left(\eta_j\right)\right)\\&-Re\left\{\left(\sum_{j=0}^{N}\tilde{F}\left(\eta_j\right)d_{k,j}^{(3)}\tilde{F}\left(\eta_j\right)\right)-\left(\sum_{j=0}^{N}d_{k,j}^{(1)}\tilde{F}\left(\eta_j\right)\right)\left(\sum_{j=0}^{N}d_{k,j}^{(2)}\tilde{F}\left(\eta_j\right)\right)\right\}=0\end{aligned} \tag{18}$$

For *k = 2, 3,...M-1*

$$\begin{aligned}&N_2\sum_{j=0}^{M}d_{k,j}^{(2)}\tilde{G}\left(\eta_j\right)+N_1\left\{\left(\sum_{j=0}^{M}d_{k,j}^{(2)}\tilde{F}\left(\eta_j\right)\right)-\tilde{F}\left(\eta_j\right)\right\}\\&-N_3Re\left\{\sum_{j=0}^{M}\tilde{F}(\eta_j)d_{k,j}^{(1)}\tilde{G}\left(\eta_j\right)-\sum_{j=0}^{M}\tilde{G}\left(\eta_j\right)d_{k,j}^{(1)}\tilde{F}\left(\eta_j\right)\right\}=0\end{aligned} \tag{19}$$

For *k=1, 2, 3,...M-1*
and the following boundary conditions in Eq. (17) become

$$\begin{aligned}&\tilde{F}(-1)=\tilde{F}^{''}(-1)=0,\tilde{G}^{'}(-1)=0,\\&\tilde{F}^{'}(1)=0,\ \tilde{F}(1)=1,\tilde{G}(1)=s\,\tilde{F}^{''}(1)\end{aligned} \tag{20}$$

The unknowns in the above system of nonlinear algebraic equation for the unknown $\tilde{F}\left(\eta_j\right), i=1,2,3,...,N$ and $\tilde{G}\left(\eta_j\right), i=1,2,3,...,N-1$ are found using Newton's iterative method.

4. RESULTS AND DISCUSSION

The effect of micropolar fluid parameters at various values on the velocity and rotation profile is presented. The Fig. 2 shows the effect of the Reynolds number (Re) on velocity profile. It can be depicted that the velocity distribution decreases as Re increases when fluid is undergoing suction and during injection the velocity profile increases for increasing values of Re.

Table 1. Comparison of Numerical and homotopy perturbation solution.
When $N_1 = N_2 = 1$, $N_3 = 0.1$ and Re = -1.

η	G(η)			
	NS[13]	Present work	NS[13]	Present work
0.00	0.00000	0.00000	0.00000	0.00000
0.05	0.07518	0.07515	-0.02019	-0.02019
0.10	0.14996	0.14996	-0.04010	-0.04010
0.15	0.22403	0.22403	-0.05946	-0.05946
0.20	0.29694	0.29694	-0.07798	-0.07798
0.25	0.36834	0.36834	-0.09537	-0.09537
0.30	0.43784	0.43784	-0.11135	-0.11135
0.35	0.50506	0.50506	-0.12561	-0.12561
0.40	0.56962	0.56962	-0.13785	-0.13785
0.45	0.63116	0.63116	-0.14774	-0.14774
0.50	0.68925	0.68925	-0.15497	-0.15497
0.55	0.74354	0.74354	-0.15919	-0.15919
0.60	0.79366	0.79366	-0.16006	-0.16006
0.65	0.83921	0.83921	-0.15721	-0.15721
0.70	0.87984	0.87984	-0.15026	-0.15026
0.75	0.91516	0.91516	-0.13884	-0.13884
0.80	0.94483	0.94483	-0.12253	-0.12253
0.85	0.96846	0.96846	-0.10093	-0.10093
0.90	0.98578	0.98578	-0.07361	-0.07361
0.95	0.99639	0.99639	-0.04012	-0.04012
1.00	1.00000	1.00000	0.00000	0.00000

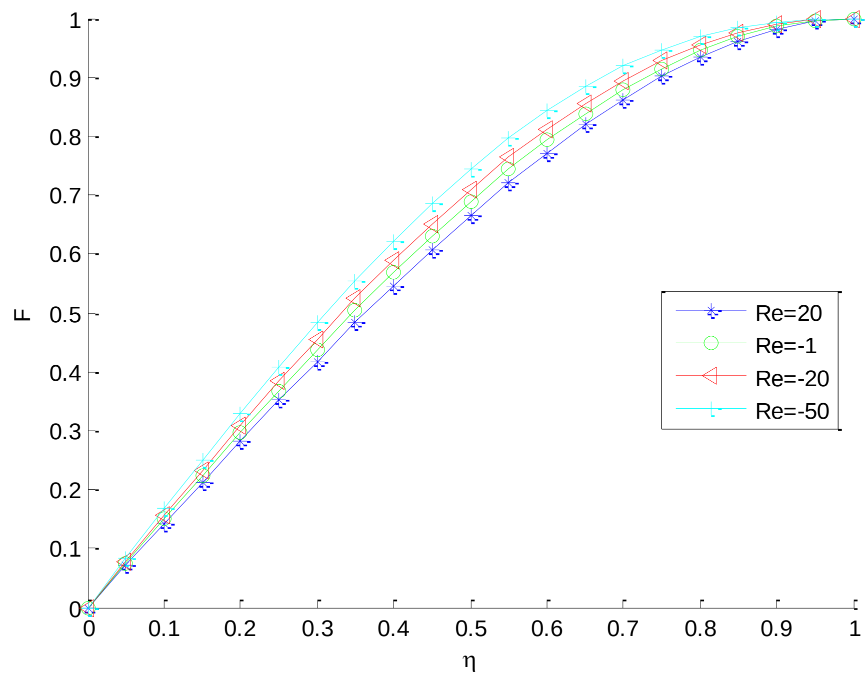


Fig. 2. Effect of Reynold's number (Re) on velocity profile when $N_1 = N_2 = 1$ and $N_3 = 0.01$

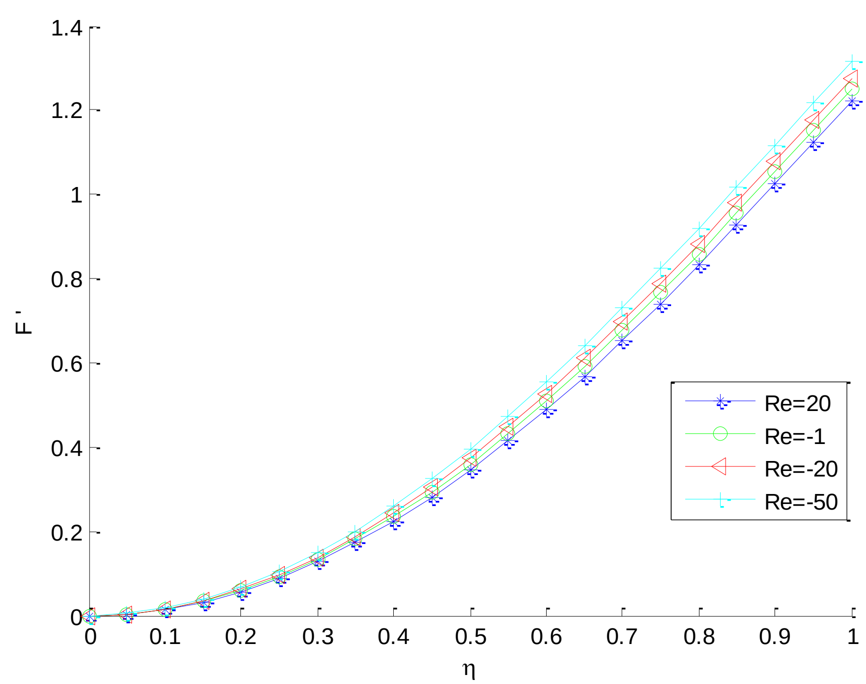


Fig. 3. Effect of Reynold's number (Re) on velocity profile when $N_1 = N_2 = 1$ and $N_3 = 0.01$.

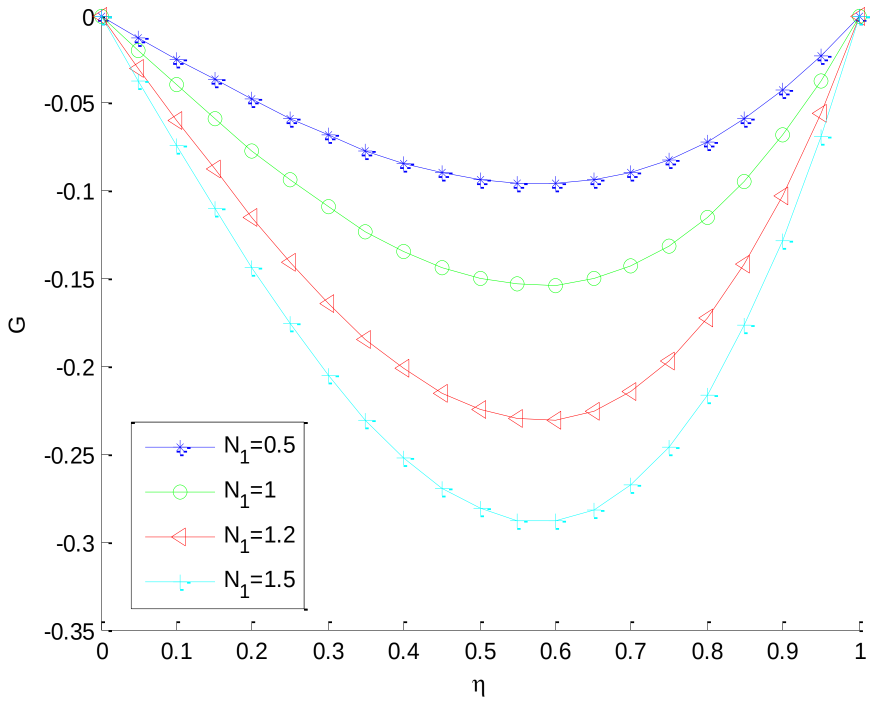


Fig. 4. Effect of micro rotation parameter, N_1 on rotation profile when $-Re = N_2 = 1$ and $N_3 = 0.01$

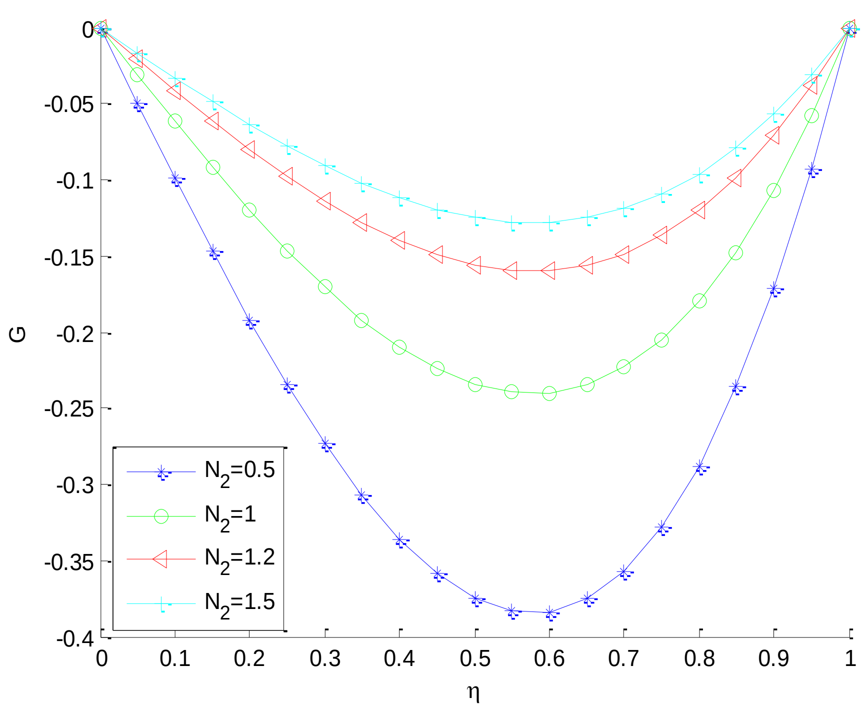


Fig. 5. Effect of micro rotation parameter, N_2 on rotation profile when $-Re = N_1 = 1$ and $N_3 = 0.01$.

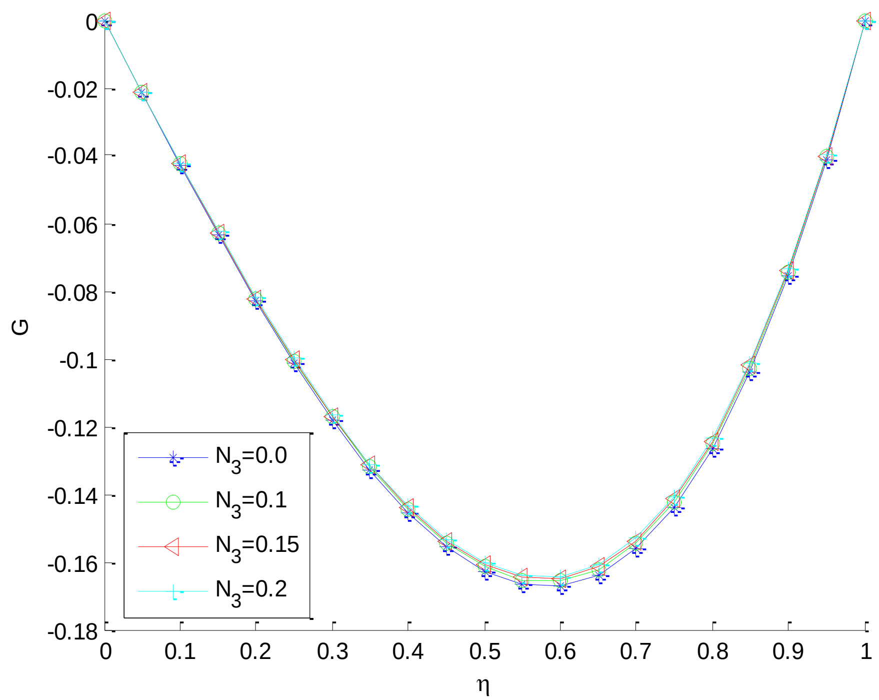


Fig. 6. Effect of micro rotation parameter, N_3 on rotation profile when $-Re = N_2 = N_1 = 1$.

Effect of Reynolds number (Re) on velocity profile $F'(\eta)$ is observed in Fig. 3 which shows at increasing values of the Reynolds number, the rotation distribution increases. This can be physically explained that at increasing Reynolds number, the minimum point of micropolar fluid rotation is still retained at the origin. Fig. 4 shows the effect of microrotation parameter (N_1). From the figure, increasing values of N_1 parameter, the velocity profile decreases slightly which is as a result of an increase in rate of shear at the wall causing a decrease in boundary layer thickness. As micro rotation parameter N_1 increases for suction flow, the rotation profile decreases till $\eta = 0.58$ (not accurately determined). A reverse case is observed for injection as depicted in Fig. 4 illustrating that there is an increase from suction to injection. During suction flow at increasing values of microrotation parameter N_2 as shown in Fig. 5 that rotation profile increases for suction thereafter reduces during injection. Also the effect of microrotation parameter N_3 on rotation profile is seen in Fig. 6, as observed increasing values of N_3 parameter shows an increasing rotation distribution for suction till point $\eta = 0.6$ (not accurately determined). Thereafter rotation distribution decreases for injection flow.

5. CONCLUSIONS

In this paper, flow of a micropolar fluid conveyed through porous channel driven by suction or injection with high mass transfer has been analyzed using Chebychev spectral collocation method to solve the coupled nonlinear ordinary equations arising from the mechanics of fluid. The numerical solutions were used to investigate the effects of flow and rotation parameters such as Reynolds number and micro rotation parameters. The effect of parameters such as Reynold's number and microrotation parameter was established. The results obtained can be used to advance the study of micropolar fluid in processes such as blood flow, turbulent shear flow, micro channel and porous channel.

Nomenclature

F	dimensionless streamfunction
G	dimensionless microrotation
H	width of channel (m)
j	micro-inertia density
N	microrotation/angular velocity (S^{-1})
$N_{1,2,3}$	dimensionless parameter
p	embedding parameter
q	mass transfer parameter (ms^{-1})
Re	Reynolds number
s	microrotation boundary condition
u,v	Cartesian velocity components (ms^{-1})
x,y	Cartesian coordinate parallel and normal to channel (m)

Greek symbols

η	dimensionless normal distance
μ	dynamic viscosity ($kgm^{-1}s^{-1}$)
κ	coupling coefficient ($kgm^{-1}s^{-1}$)
ρ	fluid density (kgm^{-3})
ψ	stream function (m^2s^{-1})
ν_s	microrotation/spin gradient viscosity (m kg s)$^-$

References

[1] A. C. Eringen. Theory of micropolar fluids. *Journal of Mathematics and Mechanics* 16, 1-18, 1966

[2] R. Idris, H. Othman, I. Hashim. On the effect of non-uniform basic temperature gradient on Benard –Marangoni convection in micropolar fluids. *International Communication in Heat and Mass Transfer* 36(3), 203-209, 2009

[3] S. W. Yuan. Further investigation of laminar flow in channels with porous wall. *Journal of Applied Physics*, 27, 267-269, 1956

[4] N. A. Kelson, A. Desseaux. Effect of surface conditions on flow of a micropolar fluid driven by a stretching sheet. *International Journal of Engineering Science*, 39(16), 1881-1897, 2001

[5] N. A. Kelson and T. W. Farrell. Micropolar flow over a porous stretching sheet with strong suction or injection. *International Communications in Heat and Mass Transfer* 28(4), 479-488, 2001

[6] M. B. Zaturska, P. G. Drazin, W. H. H. Banks. On the flow of a viscous fluid driven along a channel by suction at porous walls. *Fluid Dynamics Research,* 4, 151-178, 1988

[7] C. Ching –Yang. Natural convection of micropolar fluid from a vertical truncated cone with power – law variation in surface temperature. *International Communications in Heat and Mass Transfer,* 35(1), 9-46, 2008

[8] S. W. Yuan. Further Investigation of laminar flow in channels with porous walls. *Journal of Applied Physics,* 27, 267-269, 1956

[9] I. H. Abdel-Halim. On solving some eigen- value problems by using a differential transformation. *Applied Mathematical Computation,* 127, 1-22, 2002

[10] E. Magyari and B. Keller. Exact solutions for self-similar boundary layer flows induced by permeable stretching wall. *European Journal of Mechanics,* 19, 109-122, 2000

[11] P. S. V. N. Murthy, P. Singh. Thermal dispersion effects on non-Darcy natural convection over horizontal plate with surface mass flux. *Archive of Applied Mechanics,* 67, 487-495, 1997

[12] Chamkha Ali J, Grosant T, Pop I. Fully developed free convection of a micro polar fluid in a vertical channel. *Int Commun Heat Mass Transfer* 29, 1119-1127, 2002

[13] Abdulaziz O, Hashim I. Fully developed free convection heat and mass transfer of a micro polar fluid between porous vertical plates. *Numer Heat Transfer A* 2009;55:270–88

[14] Si Xin-yi, Si Xin-hui, Zheng Lian-cun, Zhang Xin-xin. Homotopy analysis solution for micro polar fluid flow through porous channel with expanding or contracting walls of different permeability. *Appl Math Mech* 2011; 32(7): 859-774

[15] Beg O Anwar, Rashidi MM, Beg TA, Asadi M. Homotopy analysis of transient magneto-bio-fluid dynamics of micropolar squeeze film in a porous medium: a model for magneto-biorheological lubrication. *J Mech Med Biol* 2012; 12: 1250051

[16] Rashidi MM, Laraqi N, Basiri Parsa A. Analytical modeling of heat convection in magnetized micropolar fluid by using modified differential transform method. *Heat Transfer Asian Res* 2011; 40(3), 187-204

[17] Rashidi Mohammad Mehdi, Laraqi Najib, Sadri Seyed Majid. Semi analytical solution of boundary-layer flow of a micropolar fluid through a porous channel. *Walailak J Sci Tech* 2012; 9(4): 381-393

[18] Narayana PV Satya, Venkateswarlu B, Venkataramana S. Effects of hall current and radiation absorption on MHD micropolar fluid in a rotating system. *Ain Shams Eng J* 2013; 4: 843-854

[19] Oahimire JI, Olajuwon BI. Effect of hall current and thermal radiation on heat and mass transfer of a chemically reacting MHD flow of a micropolar fluid through a porous medium. *J King Saud Univ Eng Sci* 2014; 26: 112-121

[20] Olajuwon BI, Oahimire JI, Ferdow M. Effect of thermal radiation and hall current on heat and mass transfer of unsteady MHD flow of a viscoelastic micropolar fluid through a porous medium. *Eng Sci Technol Int J* 2014; 17: 185-193

[21] Prakash D, Muthtamilselvan M. Effect of radiation on transient MHD flow of micropolar fluid between porous vertical channels with boundary conditions of the third kind. *Ain Shams Eng J* 2014; 5: 1277-1286

[22] Mahmoud Mostafa AA, Waheed Shimaa E. MHD flow and heat transfer of a micropolar fluid over a stretching surface with heat generation (absorption) and slip velocity. *J Egypt Math Soc* 2012; 20: 20-27

[23] Borrelli A, Giantesio G, Patria MC. Magnetoconvection of a micropolar fluid in a vertical channel. *Int J Heat Mass Transfer* 2015; 80 (January): 614-625

[24] Siddangoudaa A. Squeezing film characteristics for micro polar fluid between porous parallel stepped plates. *Tribol Ind* 2015; 37: 97-106

[25] R. Peyret, Spectral Methods for Incompressible Viscous Flow, Springer Verlag, New York, 2002.

[26] F.B. Belgacem, M. Grundmann, Approximation of the wave and electromagnetic diffusion equations by spectral methods, *SIAM Journal on Scientific Computing* 20 (1) (1998) 13-32

[27] X.W. Shan, D. Montgomery, H.D. Chen, Nonlinear magnetohydrodynamics by Galerkin-method computation, *Physical Review A* 44 (10) (1991) 6800-6818

[28] X.W. Shan, Magnetohydrodynamic stabilization through rotation, *Physical Review Letters* 73 (12) (1994) 1624-1627

[29] J.P. Wang, Fundamental problems in spectral methods and finite spectral method, *Sinica Acta Aerodynamica* 19 (2) (2001) 161-171

[30] E.M.E. Elbarbary, M. El-kady, Chebyshev finite difference approximation for the boundary value problems, *Applied Mathematics and Computation* 139 (2003) 513-523

[31] Z.J. Huang, and Z.J. Zhu, Chebyshev spectral collocation method for solution of Burgers’ equation and laminar natural convection in two-dimensional cavities, Bachelor Thesis, University of Science and Technology of China, Hefei, 2009.

[32] N.T. Eldabe, M.E.M. Ouaf, Chebyshev finite difference method for heat and mass transfer in a hydromagnetic flow of a micropolar fluid past a stretching surface with Ohmic heating and viscous dissipation. *Applied Mathematics and Computation* 177 (2006) 561-571

[33] A.H. Khater, R.S. Temsah, M.M. Hassan, A Chebyshev spectral collocation method for solving Burgers'-type equations. *Journal of Computational and Applied Mathematics* 222 (2008) 333-350

[34] C. Canuto, M.Y. Hussaini, A. Quarteroni, T.A. Zang, Spectral Methods in Fluid Dynamics, Springer, New York, 1988

[35] E.H. Doha, A.H. Bhrawy, Efficient spectral-Galerkin algorithms for direct solution of fourth-order differential equations using Jacobi polynomials, *Appl. Numer. Math.* 58 (2008) 1224-1244

[36] E.H. Doha, A.H. Bhrawy, Jacobi spectral Galerkin method for the integrated forms of fourth-order elliptic differential equations. *Numer. Methods Partial Differential Equations* 25 (2009) 712-739

[37] E.H. Doha, A.H. Bhrawy, R.M. Hafez, A Jacobi–Jacobi dual-Petrov–Galerkin method for third- and fifth-order differential equations. *Math. Computer Modelling* 53 (2011) 1820-1832

[38] E.H. Doha, A.H. Bhrawy, S.S. Ezzeldeen, Efficient Chebyshev spectral methods for solving multi-term fractional orders differential equations. *Appl. Math. Model.* Volume 35, Issue 12, 2011, 5662-5672. doi:10.1016/j.apm.2011.05.011