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D-countability axioms

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ABSTRACT

In this paper, we introduce new axioms using the concepts of the countability axioms via dense sets, namely dense countability axioms and they are denoted by D-countability axioms, where a topological space is called D-sequential (D-separable, D-first countable, D-Lindelöf, D-δ-compact or D-second countable) space if it has a dense sequential (separable, first countable, Lindelöf, δ-compact or second countable) subspace. We prove that D-separable spaces and D-second countable spaces are equivalent to separable spaces. Moreover, we study some properties of D-countability axioms; as their subspaces and their continuous images. In addition, we provide some inter-relations between D-countability axioms and countability axioms through some examples.

***Keywords*:** Countability axioms, sequential spaces, δ-compact spaces

AMS Subject Classification (2000): 54D70, 54D10, 54D55, 54D45

1. INTRODUCTION

Many studies have been made on defining new countability axioms, some of them are weaker than the classical countability axioms and some others are stronger. Where important classical countability axioms for topological spaces includes: sequential space, separable space, first countable space, second countable space, $\boldsymbol{\sigma}$-compact space and Lindelöf space [1].

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In 1974, Siwec [2] defined g-first countable spaces and g-second countable spaces via the concept of weak bases [3], where every g-first countable space is sequential, and a space is first countable iff it is g-first countable and Frechet [3]. Few years later, countability properties that are weaker than the first-countability property namely quasi and weakly quasi first countable spaces were studied in [4].

Quintana in 2007 [5] used alpha-semi open sets to defined alpha-semi countability axioms and studied some associated properties. A generalized countability axioms namely b-countability axioms (b-first countable, b-second countable, b-separable and b-Lindelöf) were defined and studied in finitely b-additive spaces using the class of b-open sets [6, 7]. In 1994, Beshimov [8] introduced weakly separable spaces and investigated some properties of weakly separable spaces; as their separable compactifications, also he stated that every weakly separable Hausdorff compact space is separable [9-12].

Some different types of Lindelöf spaces were given as; rc-Lindelöf and almost rc-Lindelöf spaces [13-15], and I-Lindelöf spaces [16]. In 2014 Sudip [17] introduced I-sequential space, which is strictly weaker than the first countable space. In this paper we define new axioms using the concepts of the countability axioms via dense sets, namely dense countability axioms and they are denoted by D-countability axioms, where a topological space X is called D-seqential (D-separable, D-first countable, D-Lindelöf, D-σ-compact or D-second countable) space if it has a dense sequential (separable, first countable, Lindelof, σ-compact or second countable) subspace.

We prove that D-separable spaces and D-second countable spaces are equivalent to separable spaces. Moreover, we study some properties of D-countability axioms; as their subspaces and their contiuous images. Finally, we provide the inter-relations between D-countability axioms and the countability axioms.

2. D-SEQUENTIAL SPACES

One of the countability axioms for topological spaces is sequential space, which is a space that satisfies very weak axioms of countability. The first formal definition of separable space is originally due to S. P. Franklin [18]. In this section we define D-sequential space via dense set, and study the properties of this class of spaces and how it relate to the sequential space.

Definition 2.1. [1] A topological space is said to be sequential space if given any subset of it which is not closed, there is sequence of points in the subset having a limit, which lies outside the subset.

Definition 2.2. A topological space X is said to be D-sequential space, if X has a sequential subspace which is dense.

Examples 2.1.

1 -The closed ordinal space $[0, w_1]$ is D-sequential, but it is not sequential space
2 -The countable space is not D-sequential, because there is no sequential subspace which is dense.

Corollary 2.1. Every sequential space is D-sequential, but not conversely; see example (2.1(1)).

Theorem 2.1. Clopen subspace of D-sequential space is D-sequential.

Proof: - Let A be a clopen subspace of D-sequential space X, then X has a sequential subspace B, which is dense in X, hence $A \cap B \neq \emptyset$, and $A \cap B \subset B$, since $A \cap B$ is closed in B and B is a sequential subspace of X, then we get $A \cap B$ is sequential space [1]. Suppose W is a non-empty open set in A, and A is open in X, then W is an open in X, hence $W \cap B \neq \emptyset$, i.e. $W \cap (A \cap B) \neq \emptyset$, then $A \cap B$ is dense in A. So A is D-sequential space.

Example 2.2. A subspace of D-sequential space need not be D-sequential, for example: Let X =$(0, \infty)$, with countable topology, then (X, τ) is not D-sequential, but if $Y = X \cup \{0\} = [0, \infty)$ with $\tau_2 = \{U \cup \{0\}: U \in \tau_1\} \cup \{\emptyset\}$, then $\{0\}$is a sequential subspace in Y which is dence, i. e. (Y, τ_2) is D-sequential space, but (X, τ_1) is a subspace of (Y, τ_2) which is not D-sequential.

Theorem 2.2. If a space X has D-sequential subspace which is dense subset in X, then it is D-sequential.

Proof: - Let A be a D-sequential subspace which is dense subset in X, then A has a sequential subspace B which is dense in A, hence B is a sequential subspace which is dense in X, so X is D-sequential.

Theorem 2.3. Image of D-sequential space under continuous and closed map is D-sequential.

Proof: - Suppose X is D-sequential space, and $F: X \longrightarrow Y$ is closed continuous map from a topological space X onto a topological space Y. Then X has a sequential subspace A which is dense in X, since F is closed continuous map, then F (A) is a sequential subspace of F(X), and since A is dense subset in X and F is continuous map, then F (A) is dense in F(X). Thus F(X) is D-sequential space.

Example 2.3. Continuous image of D-sequential space need not be D-sequential; for example: let $X = Y = \mathbb{R}$, and τ_1 be the discrete topology on X, τ_2 is the cocountable topology on Y. Then the identity map from $(\mathbb{R}, \tau_1)$ onto the space $(\mathbb{R}, \tau_2)$ is continuous but it is not closed. However the space $(\mathbb{R}, \tau_2)$ is not D-sequential space.

3. D-FIRST COUNTABLE SPACES

The class of spaces satisfying the first axiom of countability was defined by F. Hausdorff in 1914. Where the space is called first-countable space if every point has a countable local base. Here we define a new space that has a dense first countable subspace; namely D-first countable space.

Definition 3.1. [1] A topological space X is said to be first countable space if for every $x \in X$ there is a countable local base B_x of x.

Definition 3.2. A topological space X is said to be D-first countable space, if X has a first countable subspace, which is dense in X.

Examples 3.1.

1 -Let X be uncountable set and $x_0 \in$ X. Define $\tau = \{X\} \cup \{W \subseteq X: x_0 \notin W\}$, and Y = X-$\{x_0\}$, then Y is dense in X, and τ_Y is the discrete topology, i.e. Y is dense and first countable subspace of (X, τ), so (X, τ) is D-first countable space.
2 -The countable space X is not D-first countable, because there is no first countable subspace which is dense in X.
3 -The closed ordinal space $[0, w_1]$ is D-first countable space, but not first countable, since $[0, w_1)$ is dense first countable subspace of $[0, w_1]$.

Corollary 3.1.

1 -Every first countable space is D-first countable, but not conversely, see example (3-2(3)).
2 -Every D-First countable space is D-sequential.

Theorem 3.1. The open subspace of D-first countable space is D-first countable.

Proof: - Let A be an open subspace of D-first countable space X, then X has a first countable subspace B, which is dense in X, hence $A \cap B \neq \emptyset$, and $A \cap B \subset B$, since B is first countable subset of X, then $A \cap B$ is first countable space [1]. Suppose W is a non-empty open set in A, and A is open in X, then W is open in X, hence $W \cap B \neq \emptyset$, i.e. $W \cap (A \cap B) \neq \emptyset$, then $A \cap B$ is dense subset in A, so A is D-first countable subspace of X.

Example 3.2. A subspace of D-first countable space need not be D-first countable; see example (2.2)

Theorem 3.2. If a space X has a D-first countable subspace which is dense subset in X, then it is D-first countable.

Proof: - Let A be a D-first countable-subspace which is dense subset in X, then A has a first countable subspace B which is dense in A, hence B is a first countable-subspace which is dense in X, so X is D-first countable space.

Theorem 3.3. Image of D-first countable space under continuous and open map is D-first countable.

Proof: - Suppose X is D-first countable space, and $F: X \longrightarrow Y$ is an open continuous map of a topological space X onto a topological space Y. Then X has a first countable subspace A, which is dense in X, since F is an open continuous map, then F(A) is dense first countable subspace of F(X), then F(X) is D-first countable space.

Example 3.3. Continuous image of D-first countable space need not be D-first countable; see example (2.3): $(\mathbb{R}, \tau_1)$ is D-first countable, but $(\mathbb{R}, \tau_2)$ is not D-first countable.

4. D-SEPARABLE SPACES

Separable space is due to M. Frechet in 1906 [1], where a space is separable if it contains a countable dense subspace. In this section we use the concept of dense sets to define D-separable space, then we prove that separable spaces and D-separable spaces are equivalent.

Definition 4.1. [1] A topological space X is said to be separable space if there exist a countable dense subset of X.

Definition 4.2. A topological space X is said to be D-separable space if X has separable subspace, which is dense in X.

Theorem 4.1. Let (X, τ) be a topological space, then the following statements are equivalent:

1 -X is separable space.
2 -X is D-separable space.

Proof: - (1 ⇒ 2) Direct. (2 ⇒ 1) Let X be D-separable space, then X has a separable subspace A, which is dense, since A is separable subspace, then A has a countable dense subset B, since B is dense subset in A and A is dense subset in X, then B is dense in X. Hence X is separable space

Corollary 4.1.

1-Every separable space is D-sequential.
2-Every separable space is D-first countable.

Proof:

1-Suppose X is separable space, then X has a countable dense subset A, since every countable space is sequential, i.e. X has a sequential subspace A which is dense, thus X is D-sequential.
2-Suppose X is separable space, then X has a countable dense subset A, since every countable space is first countable space, i.e. X has a first countable subspace A which is dense, thus X is D-first countable.

Example 4.1. The uncountable discrete space X is D-sequential and D-first countable spaces, since it is sequential and first countable, but it is not separable.

5. D-LINDELOF SPACES

Lindelöf space was introduced by Alexandroff and Urysohn in 1929 [1], the Lindelöf property is a weakening of the more commonly used notion of compactness, which requires the existence of a finite subcover.

Here we introduce D-Lindelöf spaces and study their properties, and prove that in paracomact spaces, D-Lindelöf and Lindelöf are equivalent. In the end of this section, we consider d- Lindelöf space, which due to M. Ganster [19] in 1989, then we show that there is no general relation between d- Lindelöf spaces and D-Lindelöf spaces which we define.

Definition 5.1. [1] A topological space X is said to be Lindelöf space if every open cover of X has a countable sub cover.

Definition 5.2. A topological space X is said to be D-Lindelöf space if X has a Lindelöf subspace, which is dense in X.

Examples 5.1.

1 -Let X = ℝ, and τ = {U ⊆ ℝ: x_0 ∈ U} ∪ {∅}. Where x_0 is an element of ℝ. Then (X, τ) is not Lindelöf space, since {{x_0, x}, $x \neq x_0$} is an open cover of X, for all x∈ X, which has no subcover, now A = {x_0} is dense in X, and τ_A is the trivial topology, i.e. (A, τ_A) is Lindelöf subspace of (X, τ), which is dense, then (X, τ) is D-Lindelöf space.
2 -The uncountable discrete space X is D-sequential and D-first countable space,since it is sequential and first countable space, but not D-Lindelöf space, because there is no Lindelöf subspace of X, which is dense.
3 -The cocountable space X is D-Lindelöf, but it is not D-sequential, D-first countable nor separable.

Corollary 5.1. Every Lindelöf space is D-Lindelöf, but not conversely; (see example (5-2(1)).

Theorem 5.1. Clopen subspace of D-Lindelöf space is D-Lindelöf.

Proof: - Let A be a clopen subspace of D-Lindelöf space X, then X has a Lindelöf subspace B, which is dense in X, then A∩B ≠ ∅ , and A∩B⊂B, since A∩B is closed subspace of B, and B is Lindelöf space, then A∩B is Lindelöf space [1]. Now suppose N is a non-empty an open set in A, and A is an open in X, then N is an open in X, hence N∩B ≠ ∅, i.e. N∩ (A∩B) ≠ ∅, then A∩B is dense in A, so A is D-Lindelöf space.

Theorem 5.2. D-Lindelöf, paracompact space is Lindelöf.

Proof: - Let A be a Lindelöf subspace which is dence in X. And let $\varphi = \{U_\gamma\}_{\gamma\in I}$ be an open cover of X, since X is paracompact then therer exist a locally finite open cover $\{V_\gamma\}_{\gamma\in I}$ such that $\overline{V_\gamma} \subseteq U_\gamma$ for all γ , consider the set I_0 for all $\gamma \in I$ such that V_γ intersects A. This set is a countable since any locally finite cover of a Lindelöf space is countable [1], so we have A = union of {A intersect V_γ: $\gamma \in I_0$}, since A is dense.
So:

$$X = \overline{A} = \cup_{\gamma\in I_0} \overline{(A \cap V_\gamma)}$$

$$= \cup_{\gamma\in I_0} \overline{(A \cap V\gamma)} \quad \text{(by being locally finite)}$$

$$\subseteq \cup_{\gamma\in I_0} \overline{V_\gamma} \subseteq \cup_{\gamma\in I_0} U_\gamma .$$

Hence has a countable subcover.

Example 5.2. Let X = $\mathbb{R}$, and τ is the same topology as in example (5.1.(1)), let B = X – {x_0} be a subspace of X, since τ_B is the discrete topology space, then (X, τ) is D-Lindelöf space, but (B, τ_B) is not D-Lindelöf space.

Theorem 5.3. If a space X has a D-Lindelöf subspace which is dense subset in X, then it is D-Lindelöf.

Proof: - Let A be a D-Lindelöf subspace which is dense subset in X, then A has a Lindelöf subspace B which is dense in A, hence B is a Lindelöf subspace which is dense in X, so X is D-Lindelöf space.

Theorem 5.4. Image of D-Lindelöf space under continuous map is D-Lindelöf.

Proof: - Let X be a D-Lindelöf space, and F be a continuous map of space X onto a space Y, then X has a Lindelöf subspace A, which is dense in X, since F is a continuous map, then F (A) is dense and Lindelöf subspace of F(X) Thus F(X) is D-Lindelöf space.

Corollary 5.2. Every separable space is D-Lindelöf.

Proof: - Suppose X is separable space, then X has a countable dense subset A, since every countable space is Lindelöf, i.e. X has a Lindelöf subspace which is dense; hence X is D- Lindelöf.

Theorem 5.5. D-Lindelöf, regular space is has a dense normal subspace.

Proof: - Let X be a D-Lindelöf regular space, then X has a Lindelöf subspace A which is dense, and A is regular, since any subspace of regular space is regular, i.e. A is regular Lindelöf subspace of X which is dense, and since regular Lindelöf spaces are normal [1], we obtain that A is normal dense subspace of X.

Definition 5.3. [19] A topological space X is said to be d-Lindelöf space if every cover of X by dense subsets has a countable subcover.

Examples 5.3.

1 - Let X = $\mathbb{R}$, τ = {∅, $\mathbb{R}$, {0}} ∪ {A⊆ $\mathbb{R}$: 0 ∈ A, A^c finite}, then the family of closed sets is; $\mathscr{F}$ = {∅, $\mathbb{R}$, $\{0\}^c$} ∪ {F⊆ $\mathbb{R}$: 0 ∉ F, F is finite}. Since any set that contains 0 is dense, then $\mathbb{R} = \bigcup_{\substack{x\in\mathbb{R}\\ x\neq 0}} \{0, x\}$ with no countable subcover, i.e. X is Lindelöf space, but not d-Lindelöf.

2 -The uncountable discrete space X is d-Lindelöf, since X is the only dense subset in X, but X is not D-Lindelof.

6. D-σ-COMPACT SPACES

In this section we consider D-σ-compact space as a space with dense compact subspace, then we disuss their properties and its relations with the classical countability axioms.

Definition 6.1. [1] A topological space X is said to be a σ-compact space if it is the union of countable many compact subsets of X.

Definition 6.2. A topological space X is said to be D-σ-compact if X has a σ-compact subspace, which is dense in X.

Corollary 6.1. Every D-σ-compact space is D-Lindelöf.

Examples 6.1.

1 -The closed ordinal space $[0, w_1]$ is D-σ-compact, since it is σ-compact space, but not separable.
2 -The cocountable space X is D-Lindelöf, but not D-σ-compact, because there is no dense subset which is σ-compact.
3 -The uncountable discrete space X is D-sequential and D-first countable, but not D-σ-compact, because there is no dense subset which is σ-compact.

Corollary 6.2. Every σ-compact space is D-σ-compact, but not conversely; (see example (5.1 (1)).

Theorem 6.1. Clopen subspace of D-σ-compact space is D-σ-compact.

Proof: - Let A be a clopen subspace of D-σ-compact space X, then X has a σ-compact subspace B which is dense in X, then $A \cap B \neq \emptyset$, and $A \cap B \subset B$, since $A \cap B$ is a closed subspace of B, and B is σ-compact space, then $A \cap B$ is σ-compact space [1]. Let N be a non-empty open set in A, and A is an open in X, then N is an open in X, hence $N \cap B \neq \emptyset$, i.e. $N \cap (A \cap B) \neq \emptyset$, then $A \cap B$ is dense in A. So A is D-σ-compact space.

Example 6.2. If $X = \mathbb{R}$, and τ is the same topology as in example (3.1(1)), let $B = X - \{x_0\}$ be a subspace of X, since τ_B is the discrete topology space, then (X, τ) is D-σ-compact space, but (B, τ_B) is not D-σ-compact space.

Theorem 6.2. If a space X has a D-σ-compact subspace which is dense subset in X, then it is D-σ-compact.

Proof: - Let A be a D-σ-compact subspace which is dense subset in X, then A has a σ-compact subspace B which is dense in A, hence B is a σ-compact subspace which is dense in X, so X is D-σ-compact space.

Theorem 6.3. Image of D-σ-compact space under continuous map is D-σ-compact.

Proof: - Suppose X is D-σ-compact space, and $F: X \longrightarrow Y$ be a continuous map of a topological space X onto a topological space Y, then X has a σ-compact subspace A which is dense in X, since F is a continuous map, then F (A) is dense, and it is σ-compact space, then F(X) is D-σ-compact space.

Corollary 6.3. Every separable space is D-σ-compact.

Proof: - Suppose X is separable space, then X has a countable dense subset A, since every countable space is σ-compact, i.e. X has σ-compact subspace which is dense A, and then X is D-σ-compact space.

7. D-SECOND COUNTABLE SPACES

The class of spaces satisfying the second axiom of countability was defined by F. Hausdorff [1], and this axiom of countability has the strongest notion of countability axioms. In this section we give a definition of D-second countable space, then we prove that this axiom is equivalent to the axiom of separability.

Definition 7.1. [1] A topological space X is satisfies second countability axiom if X has a countable base B.

Definition 7.2. A topological space X is said to be D-second countable space if X has second countable subspace which is dense in X.

Corollary 7.1. D-Second countable space is D-sequential, D-first countable, separable, and D-Lindelöf.

Corollary 7.2. Let (X, τ) be a topological space, then the following statements are equivalent:

1 -X is D-second countable space.
2 -X is separable space.
3 -X is D-separable space.

Proof: - (1⇒ 2) Direct.
(2⇒ 3) From theorm (4-2).
(3⇒1) Let X be a D-separable space, then X has a separable dense subset A, since A is separable then A has a countable dense subset B, so B is also countable dense subset in X, and since every countable space is second countable, i.e. X has a second countable subspace B which is dense, then X is D-second countable.

8. CONCLUSIONS

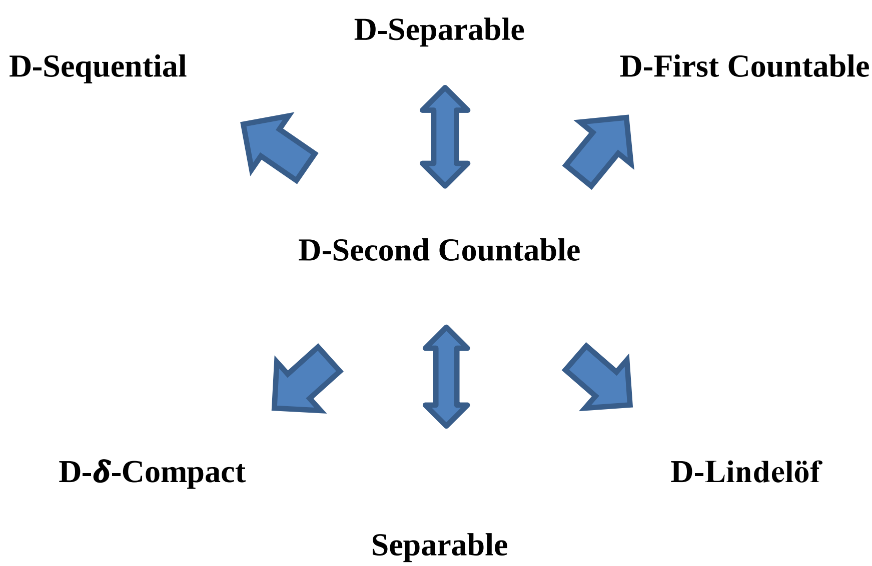
We introduce the notion of dense countability axioms; namely D-countability axioms. We study the basic properties of D-countability axioms, as their subspaces and their continuous images. In addition, we discuss the relations between D-countability axioms and countability axioms, and we prove that the axioms of separable, D-separable and D-second countable are equivalent.

Here we summarize some of our results:

- Sequential space is D-sequential, but not conversely.
- First countable space is D-first countable, but not conversely.

- Lindelöf space is D-Lindelöf, but not conversely.
- σ-Compact space is D-σ-compact, but not conversely.
- Separable, D-separable and D-second countable spaces are equivalent.
- Separable space is D-sequential, D-first countable, D-Lindelöf, D-σ-compact, and D-second countable space.
- D-first countable space is D-sequential.
- D-σ-compact space is D-Lindelöf.
- D-Lindelöf paracompact space is Lindelöf.
- D-Lindelöf regular space has a dense normal subspace.
- Clopen subspace of D-sequential (D-Lindelöf, D-σ-compact) is D-sequential (D-Lindelöf, D-σ-compact).
- Open subspace of D-first countable is D-first countable.
- Continuous image of D-σ-compact (D-Lindelöf) is D-σ-compact (D-Lindelöf).

This diagram shows the relations between D-countability axioms:



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