Available online at www.worldscientificnews.com

World Scientific News

An International Scientific Journal

WSN 143 (2020) 127-138 EISSN 2392-2192

Solutions of a class of singular linear systems of difference equations. Part 2

Asmaa M. Kanan

Department of Mathematics, Faculty of Science, Sabratha University, Sabratha, Libya

E-mail address: Asmaakanan20@gmail.com

ABSTRACT

We extend results of Campbell of the linear systems of differential equations $A\,\dot{x} + Bx = f$ when A and B are rectangular, and results of Kanan of solution of a class of singular linear systems of difference equations $A\,x_{n+1} + Bx_n = f_n$ when A and B are square, to such systems of difference equations when A and B are rectangular. Explicit solutions of the last one are derived for several cases. One such is, when the matrix $(\lambda A + B)$ is one-to-one, another case is when such matrix is onto, for a scalar $\lambda \in \mathbb{C}$. Also, explicit solutions are derived for the case that A is onto, and for the case that B is onto.

***Keywords*:** Singular linear difference systems, Moore-Penrose generalized inverse, Drazin inverse

1. INTRODUCTION

For many applications it is necessary to solve the linear systems of difference equations

$$A\,x_{n+1} + Bx_n = f_n, \qquad n \geq 0$$

when A and B are rectangular matrices, x_n is a column vector and f_n is an arbitrary function.

(Received 10 February 2020; Accepted 28 February 2020; Date of Publication 05 March 2020)

The system of difference equations is called over-determined if $A, B \in \mathbb{C}^{m\times r}$ and $m > r$. This means that, more equations than unknowns. And it is called under-determined if $A, B \in \mathbb{C}^{m\times r}$ and $r > m$, this means that, more unknowns than equations.

In this paper, we extend results of Campbell on linear systems of differential equations with singular coefficients [1], and results of Kanan [2] to the linear systems of difference equations when the coefficients matrices are rectangular. So we will use the Moore-Penrose generalized inverse of the matrices [3-7], and the Drazin inverse the matrices [8-10]. For some concepts and studies of linear singular difference systems you can see [11-22].

We shall consider an important special case of each type. That is, for each case, the general solution will be given.

We will investigate the equation $A\,x_{n+1} + Bx_n = f_n$ when the matrix $\lambda A + B$ is one-to-one and when it is onto. Also, we will consider the case when A is one-to-one and when it is onto. The same thing for the matrix B. We give closed form solutions for the homogenous equations and for the nonhomogeneous equation, when the matrix $\lambda A + B$ is one-to-one, and when it is onto. Also, we give closed form solutions of $A\,x_{n+1} + Bx_n = f_n$ for the case A is onto, and for the case B is onto.

Throughout this paper, $\mathbb{C}^{m\times r}$ is the set of $m \times r$ complex matrices. The range of A is denoted by $R(A)$, the null space of A is denoted by $\mathcal{N}(A)$. A^* is the transposed conjugate complex of A. The conjugate of λ is denoted by $\bar{\lambda}$. The Trace of A is denoted by $Tr(A)$. I is an identity matrix, O is a zero matrix such that $O^0 = I$.

2. PRELIMINARIES

In this section, we introduce some of basic concepts that are important to understand our proofs in the next section.

Definition 2.1. Let $A \in \mathbb{C}^{m\times m}$. The Drazin inverse of A is the unique matrix $A^D \in \mathbb{C}^{m\times m}$ satisfies the following conditions:

$$(i)\quad A^D A A^D = A^D,$$

$$(ii)\quad AA^D = A^D A\;,$$

$$(iii)\quad A^{k+1}A^D = A^k,$$

where $k = Ind(A)$ is called the index of A, it is the smallest non-negative integer such that

$$rank(A^k) = rank(A^{k+1}).$$

Note that A^D always exists, and $A^D = A^{-1}$ for $Ind(A) = 0$. Properties, algorithms for computing A^D can be found in $[9, 10]$.

Definitions 2.2. If $A \in \mathbb{C}^{m\times r}$, then $A^\dagger \in \mathbb{C}^{r\times m}$ is unique, and it is called the Moore-Penrose generalized inverse of A if it satisfies the following condition

$$(1) \qquad AA^{\dagger}A = A,$$

$$(2) \qquad A^{\dagger}AA^{\dagger} = A^{\dagger},$$

$$(3) \qquad (AA^{\dagger})^{*} = AA^{\dagger},$$

$$(4) \qquad (A^{\dagger}A)^{*} = A^{\dagger}A.$$

Properties, algorithms for computing $A^{\dagger}$ can be found in [5].

Definition 2.3. A vector $c \in \mathbb{C}^{r}$ is called a consistent initial condition for the difference equation $A\,x_{n+1} + B\,x_n = f_n$, if the initial value problem $A\,x_{n+1} + B\,x_n = f_n$, $x_0 = c$, $n = 1, 2, \dots$, possesses at least one solution for x_n.

Proposition 2.1. Suppose that $A \in \mathbb{C}^{m\times r}$.
Then

$$R(A) = N(A^{*})^{\perp}.$$

Lemma 2.1. [2] The homogeneous difference equation $A\,x_{n+1} + B\,x_n = \mathbf{0}$ is tractable if and only if there exist a scalar $\lambda \in \mathbb{C}$ such that $(\lambda A + B)^{-1}$ exists.

Theorem 2.1. [2] Suppose that $A\,x_{n+1} + B\,x_n = \mathbf{0}$, with $rank(A) = 1$ and $Tr(A) \neq 0$, is tractable.
Then the general solution is given by

$$x_n = \begin{cases} -(I+K)q, & if\ n = 0 \\ (\hat{A}^{D}\hat{B})^{n}q\,, & if\ n = 1, 2, 3, \dots \end{cases}$$

where $q \in \mathbb{C}^{m}$,

$$K = \begin{bmatrix} \frac{-(Tr\,(A) - a_{11})}{Tr(A)} & \frac{a_{12}}{Tr(A)} & \dots & \frac{a_{1m}}{Tr(A)} \\ \frac{a_{21}}{Tr(A)} & \frac{-(Tr(A) - a_{22})}{Tr(A)} & \dots & \frac{a_{2m}}{Tr(A)} \\ \dots & & \dots & \dots \\ \frac{a_{m1}}{Tr(A)} & \dots & \dots & \frac{-(Tr(A) - a_{mm})}{Tr(A)} \end{bmatrix} \in \mathbb{C}^{m\times m},$$

$$\hat{A}_{\lambda} = (\lambda A + B)^{-1}A = \begin{bmatrix} \hat{a}_{11} & \hat{a}_{12} & \dots & \hat{a}_{1m} \\ \hat{a}_{21} & \hat{a}_{22} & \dots & \hat{a}_{2m} \\ \dots & \dots & \dots & \dots \\ \hat{a}_{m1} & \hat{a}_{m2} & \dots & \hat{a}_{mm} \end{bmatrix},$$

$$\hat{B}_{\lambda} = (\lambda A + B)^{-1}B.$$

3. THE MAIN RESULTS

In this section, we extend results of Campbell [1] on linear systems of differential equations $A\,\dot{x} + Bx = f_n$ in the case A and B are rectangular matrices, and results of Kanan [2] to the linear systems of difference equations $A\,x_{n+1} + Bx_n = f_n$, $n \geq 0$, when A and B are rectangular.

In this section, let $A, B \in \mathbb{C}^{m\times r}$, so $\lambda A + B$ is singular. Let λ is a scalar in $\mathbb{C}$.

We define

$$P = (\lambda A + B)(\lambda A + B)^{\dagger},$$

when $\lambda A + B$ is one-to-one, and

$$P = (\lambda A + B)^{\dagger}(\lambda A + B),$$

when $\lambda A + B$ is onto, where $(\lambda A + B)^{\dagger}$ is the Moore-Penrose generalized inverse of the matrix $\lambda A + B$.

Theorem 3.1. Suppose that $A\,x_{n+1} + Bx_n = \mathbf{0}$, $n \geq 0$, where $A, B \in \mathbb{C}^{m\times r}$.
Then $A\,x_{n+1} + Bx_n = \mathbf{0}$, $n \geq 0$, is tractable if and only if $\lambda A + B$ is one-to-one for some a scalar $\lambda \in \mathbb{C}$.

Proof. The first direct is proven in Theorem 3.4 in [2].
Suppose now that $\lambda A + B$ is one-to-one for some a $\lambda \in \mathbb{C}$. We know that every solution of $A\,x_{n+1} + Bx_n = \mathbf{0}$ is a solution of $\hat{A}x_{n+1} + \hat{B}x_n = \mathbf{0}$. Since $\lambda\hat{A} + \hat{B} = I$, the latter equation is tractable, therefor $A\,x_{n+1} + Bx_n = \mathbf{0}$ is tractable.

Theorem 3.2. Suppose $\lambda A + B$ is one-to-one for some a scalar $\lambda \in \mathbb{C}$. Then all solutions of $A\,x_{n+1} + Bx_n = \mathbf{0}$ where $A, B \in \mathbb{C}^{m\times r}$, are given by

$$x_n = \left(-\hat{A}^D\hat{B}\right)^n q, \qquad n \geq 0, \tag{1}$$

where

$$\hat{A} = (\lambda A + B)^{\dagger}A, \qquad \hat{B} = (\lambda A + B)^{\dagger}B,$$

and q must satisfy

$$q = \hat{A}\hat{A}^D q \text{ and } [I - (\lambda A + B)(\lambda A + B)^{\dagger}]\,A\,\hat{A}^D\left(-\hat{A}^D\hat{B}\right)^n q = \mathbf{0},\ n \geq 0. \tag{2}$$

Proof. Suppose that $\lambda A + B$ is one-to-one. So by Theorem 3.1 $A\,x_{n+1} + Bx_n = \mathbf{0}$ is tractable, then $\hat{A}x_{n+1} + \hat{B}x_n = \mathbf{0}$ is tractable. If x_n is a solution of $A\,x_{n+1} + Bx_n = \mathbf{0}$ then x_n is a solution of $\hat{A}x_{n+1} + \hat{B}x_n = \mathbf{0}$. Since $\hat{A}\hat{B} = \hat{B}\hat{A}$ and $\lambda\hat{A} + \hat{B} = I$. Hence by Theorem 2.1 (replacing $(I + K)$ by $\widehat{A}\hat{A}^D$) we have

$$x_n = \begin{cases} \hat{A}\hat{A}^D q, & if\ n = 0 \\ (-\hat{A}^D\hat{B})^n q\ , & if\ n = 1, 2, 3, \dots \end{cases} ,$$

or

$$x_n = \left(-\hat{A}^D\hat{B}\right)^n q, \text{ where } q = \hat{A}\hat{A}^D q.$$

By substitute from x_n in $A\ x_{n+1} + \ Bx_n = \mathbf{0}$ we get

$$A\left(-\hat{A}^D\hat{B}\right)^{n+1} q + B\left(-\hat{A}^D\hat{B}\right)^n \hat{A}\hat{A}^D q = \mathbf{0},$$

or

$$(-A\hat{A}^D\hat{B}\hat{A}\hat{A}^D + B\hat{A}\hat{A}^D)\left(-\hat{A}^D\hat{B}\right)^n q = \mathbf{0}.$$

Thus

$$(-A\hat{A}^D\hat{B} + B\hat{A}\hat{A}^D)\left(-\hat{A}^D\hat{B}\right)^n q = \mathbf{0}$$

or

$$\left(B\hat{A} + A\hat{B}\right)\hat{A}^D\left(-\hat{A}^D\hat{B}\right)^n q = \mathbf{0}.$$

But

$$A\hat{B} = A(\lambda A + B)^\dagger B$$

$$= A(\lambda A + B)^\dagger\ (\lambda A + B) - A\ (\lambda A + B)^\dagger \lambda A$$

$$= A - \lambda A\ (\lambda A + B)^\dagger A$$

$$= A - (\lambda A + B)(\lambda A + B)^\dagger\ A + B\ (\lambda A + B)^\dagger A$$

$$= [I - (\lambda A + B)(\lambda A + B)^\dagger\ ]A + B\ \hat{A}.$$

Hence

$$\left(B\hat{A} + A\hat{B}\right)\hat{A}^D\left(-\hat{A}^D\hat{B}\right)^n q$$

$$= \left(B\hat{A} - [I - (\lambda A + B)(\lambda A + B)^\dagger\ ]A - B\ \hat{A}\right)\hat{A}^D\left(-\hat{A}^D\hat{B}\right)^n q$$

$$= -[I - (\lambda A + B)(\lambda A + B)^\dagger\ ]A\hat{A}^D\left(-\hat{A}^D\hat{B}\right)^n q = \mathbf{0}.$$

Corollary 1. If $\lambda A+B$ is one-to-one, and $\mathcal{N}\left(\bar{\lambda}A^{*}+B^{*}\right)=\mathcal{N}(A^{*})\cap\mathcal{N}(B^{*})$, then all solutions of $A\,x_{n+1}+\,Bx_n=\mathbf{0}$ are given by

$$x_n=\left(-\hat{A}^D\hat{B}\right)^n A\hat{A}^D q.$$

where q is an arbitrary vector.

Proof.

$$R(\lambda A+B)^{\perp}=\mathcal{N}(\bar{\lambda}A^{*}+B^{*})$$

$$=\mathcal{N}(A^{*})\cap\mathcal{N}(B^{*}).$$

But from Proposition 2.1

$$R(A)\perp\mathcal{N}(A^{*})^{\perp},$$

so that

$$R(A)\subseteq R(\lambda A+B)^{\perp}.$$

Thus

$$[I-(\lambda A+B)(\lambda A+B)^{\dagger}\,]\,A\hat{A}^D\left(-\hat{A}^D\hat{B}\right)^n q=0,\ \ n\geq 0,$$

for all $q\in R(AA^D)$.

Theorem 3.3. Suppose $\lambda A+B$ is one-to-one, and $A\,x_{n+1}+\,Bx_n=f_n$ is consistent. Then all solutions of such equation are given by

$$x_n=\left(-\hat{A}^D\hat{B}\right)^n A\hat{A}^D q+A^D\sum_{i=0}^{n-1}\left(-\hat{A}^D\hat{B}\right)^{n-i-1}\hat{f}_i\ +\left(I-\hat{A}\hat{A}^D\right)\sum_{i=0}^{k-1}\left(-\hat{A}\hat{B}^D\right)^i\hat{B}^D\,\hat{f}_{n+i}\ \ ,\quad (3)$$

where

$$\hat{A}=(\lambda A+B)^{\dagger}A,\ \ \hat{B}=(\lambda A+B)^{\dagger}B,\ \ k=Ind(\hat{A}),\text{ and }\widehat{f_n}=(\lambda A+B)^{\dagger}f_n\,.$$

Proof. Let

$$P=(\lambda A+B)(\lambda A+B)^{\dagger}.$$

then $A\,x_{n+1}+\,Bx_n=f_n$ is equivalent to

$$PA\,x_{n+1}+\,PBx_n=f_n,\qquad (4)$$

$$(I-P)(A\,x_{n+1}+\,Bx_n)=(I-P)f_n.\qquad (5)$$

But (4) is equivalent to

$$\hat{A}x_{n+1} + \hat{B}x_n = \hat{f}_n, \tag{6}$$

since $(\lambda A + B)^{\dagger}P = (\lambda A + B)^{\dagger}$. The equation (6) is consistent by Theorem 2.1 and uniquely determines x_n. Thus $(I - P)f_n$ is determined by (5).
Note that $A\,x_{n+1} + Bx_n = f_n$ is consistent if and only if (5) satisfied, where x_n given by (3). We can use (5) by several ways. For example, in Lemma 2.1 f_n was given as zero and (5) was used to determine the consistent initial conditions. On the other hand view x_0 as given and use (5) to determine the consistent f_n, as in the following corollary.

Corollary 2. If $\lambda A + B$ is one-to-one, and $A, B \in \mathbb{C}^{m\times r}$, then there exists f_n for which $A\,x_{n+1} + Bx_n = f_n$ is inconsistent.
In the following, we discuss the special cases, when A is one-to-one, and when B is one-to-one. The next theorem gives the necessary and sufficient conditions for the solutions x_n of the initial value problem to be unique.

Theorem 3.4. The equation

$$A\,x_{n+1} + Bx_n = \mathbf{0},\ \ x_0 = \ c,\ \ n = 1, 2, \dots\ ,$$

has unique solutions for all x_0 if and only if A is one-to-one $(A^{\dagger}A = I)$ and $R(B) \subseteq R(A)$, $(AA^{\dagger}B = B)$.

Proof. Assume that $A\,x_{n+1} + Bx_n = \mathbf{0}$ has unique solutions for all x_0. Then by (2), $\hat{A}^D\hat{B}\ x_0 = x_0$ for all x_0. From this we get

$$\hat{A} = (\lambda A + B)^{\dagger}A,$$

hence A is one-to-one. New, to prove that $AA^{\dagger}B = B$. If we multiply $A\,x_{n+1} + Bx_n = \mathbf{0}$ by $AA^{\dagger}$ we get

$$AA^{\dagger}A\ \ x_{n+1} + AA^{\dagger}Bx_n = \mathbf{0},$$

or

$$A\ \ x_{n+1} + A\,A^{\dagger}Bx_n = \mathbf{0}.$$

So, every solution of $Ax_{n+1} + Bx_n = \mathbf{0}$ also satisfies $A\,x_{n+1} + AA^{\dagger}Bx_n = \mathbf{0}$. Thus $A^{\dagger}A\,Bx_n = Bx_n$ for all solutions x_n. Hence $A^{\dagger}A\,B = B$.
Conversely, assume A is one-to-one. This means that $A^{\dagger}A = I$. Also, assume $A^{\dagger}A\,B = B$. Then x_n is a solution of $Ax_{n+1} + Bx_n = \mathbf{0}$ if and only if x is a solution of $A\,x_{n+1} + A^{\dagger}Bx_n = \mathbf{0}$. However $A\,x_{n+1} + A^{\dagger}Bx_n = \mathbf{0}$ has unique solutions for all x_0.

Theorem 3.5. Suppose $\lambda A + B$ is one-to-one, where A and B are rectangular, and

$$\mathcal{N}\left(\bar{\lambda}A^* + B^*\right) = \mathcal{N}(A^*) \cap \mathcal{N}(B^*).$$

Then $A\,x_{n+1} + Bx_n = f_n$ is consistent if and only if

$$[I - (\lambda A + B)(\lambda A + B)^\dagger\,]\,f_n = \mathbf{0}.$$

Proof. Suppose that $\lambda A + B$ is one-to-one and $\mathcal{N}\left(\bar{\lambda}A^* + B^*\right) = \mathcal{N}(A^*) \cap \mathcal{N}(B^*)$. We can see that $(\lambda A + B)(\lambda A + B)^\dagger$ is the identity on $R(\lambda A + B)$. But

$$R(\lambda A + B) = \mathcal{N}\left(\bar{\lambda}A^* + B^*\right)^\perp$$

$$= (\mathcal{N}(A^*) \cap \mathcal{N}(B^*))^\perp \supseteq R(A) \cup R(B).$$

Thus

$$(\lambda A + B)(\lambda A + B)^\dagger A = A,$$

and

$$(\lambda A + B)(\lambda A + B)^\dagger B = B.$$

Hence for any x_n, if we take $f_n = A\,x_{n+1} + Bx_n$ we get

$$(\lambda A + B)(\lambda A + B)^\dagger f_n = f_n.$$

We can note that, if $(\lambda A + B)(\lambda A + B)^\dagger f_n = f_n$ then $A\,x_{n+1} + Bx_n = f_n$ is equivalent to $\hat{A}x_{n+1} + \hat{B}x_n = \hat{f}_n$. Since $\hat{A}x_{n+1} + \hat{B}x_n = \hat{f}_n$ is consistent, so $A\,x_{n+1} + Bx_n = f_n$ is consistent.
New, we discuss the case when $\lambda A + B$ is onto. Thus, note that

$$P = (\lambda A + B)^\dagger(\lambda A + B),$$

so $Ax_{n+1} + Bx_n = f_n$ becomes

$$APx_{n+1} + BPx_n = f_n - A(I - P)x_{n+1} - B(I - P)x_n,$$

or equivalently

$$A(\lambda A + B)^\dagger[(\lambda A + B)x_n] + B(\lambda A + B)^\dagger[(\lambda A + B)x_n] = f_n - A(I - P)x_{n+1} - B(I - P)x_n\,. \tag{7}$$

But

$$A(\lambda A + B)^\dagger[(\lambda A + B)x_n] + [B(\lambda A + B)^\dagger] = I,$$

so, (7) in terms of $(\lambda A + B)x_n$ is a differential equation, that it has a solution.

Theorem 3.6. Suppose A is one-to-one. Then $A\,x_{n+1} + Bx_n = f_n$ is consistent if and only if f_n is of the form

$$f_n = AA^{\dagger}h_n \oplus (I - AA^{\dagger})Bg_n \tag{8}$$

where h_n is an arbitrary function, and

$$g_n = (-A^{\dagger}B)^n + A^{\dagger}\sum_{i=0}^{n-1}(-A^{\dagger}B)^{n-i-1}h_i \tag{9}$$

q an arbitrary consistent. Conversely, if f_n has the form (8), then g_n given in (9) is general solution of $A\,x_{n+1} + Bx_n = f_n$.

Proof. Assume that A is one-to-one. Then multiplying $A\,x_{n+1} + Bx_n = f_n$ by $A^{\dagger}$, then by $I - AA^{\dagger}$ we get

$$x_{n+1} + A^{\dagger}Bx_n = A^{\dagger}f_n, \tag{10}$$

$$(I - AA^{\dagger})Bx_n = (I - AA^{\dagger})f_n, \tag{11}$$

so, $A\,x_{n+1} + Bx_n = f_n$ is equivalent to the last pair of equations. We can choose $AA^{\dagger}f_n$ to be arbitrary, we say $AA^{\dagger}h_n$. From (10) x_n is determined uniquely in terms of $A^{\dagger}f_n$. Then $(I - AA^{\dagger})f_n$ results from substituting x_n into (11).
We can get a similar result if B is one-to-one.

Theorem 3.7. Suppose B is one-to-one. Then $A\,x_{n+1} + Bx_n = f_n$ is consistent if and only if f is of the form

$$f_n = BB^{\dagger}h_n + (I - BB^{\dagger})Ag_{n+1} \tag{12}$$

where h_n is arbitrary and

$$g_n = [-(B^{\dagger}A)^D]^n(B^{\dagger}A)^D(B^{\dagger}A)q + (B^{\dagger}A)^D\sum_{i=0}^{n-1}[-(B^{\dagger}A)^D]^{n-i-1}B^{\dagger}h_n$$
$$+ [I - (B^{\dagger}A)^D\,(B^{\dagger}A)]\sum_{i=0}^{k-1}(-B^{\dagger}A)^iB^{\dagger}h_{n+i}, \tag{13}$$

where $k = Ind(B^{\dagger}A)$, q an arbitrary constant. Conversely, if f_n has the form (12) then g_n in (13) is the general solution.

Proof. Suppose B is one-to-one. By the similar way in Theorem 3.6 then $A\,x_{n+1} + Bx_n = f_n$ is equivalent to the pair

$$B^{\dagger}Ax_{n+1} + x_n = B^{\dagger}f_n, \tag{14}$$

and

$$(I - BB^{\dagger})Ax_{n+1} = (I - BB^{\dagger})f_n, \tag{15}$$

$BB^{\dagger}f_n$ can be chosen arbitrary, we say $BB^{\dagger}h_n$. Then (14) uniquely determines x_n given (12). Substituting x_n into (15) gives $(I - BB^{\dagger})f_n$.
The next theorem gives closed form solutions when $\lambda A + B$ is onto, under some conditions.

Theorem 3.8. Suppose $\lambda A + B$ is onto, and f_n is n-times differentiable. Let

$$g_n = f_n - A[I - (\lambda A + B)^{\dagger}(\lambda A + B)]h_{n+1} - B[I - (\lambda A + B)^{\dagger}(\lambda A + B)]h_n,$$

where h_n is an arbitrary and $(n+1)$-time differentiable vector valued function. Then all solutions of $A\, x_{n+1} + Bx_n = f_n$ are given by

$$x_n = (\lambda A + B)^{\dagger}\left[\left(-\hat{A}^D\hat{B}\right)^n \hat{A}\hat{A}^D q + \hat{A}^D \sum_{i=0}^{n-1}\left(-\hat{A}^D\hat{B}\right)^{n-i-1} g_n + \left(I - \hat{A}^D\hat{A}\right)\sum_{i=0}^{k-1}\left(-\hat{A}\hat{B}^D\right)^i \hat{B}^D\, \hat{g}_{n+i}\right] + [I - (\lambda A + B)^{\dagger}\,(\lambda A + B)]h_n$$

where q an arbitrary constant vector, $k = Ind(\hat{A})$, and

$$\hat{A} = A(\lambda A + B)^{\dagger}\,, \hat{B} = B(\lambda A + B)^{\dagger}\,.$$

The next theorem comes from Theorem 3.8 by set $\lambda = 0$ and noting that $\hat{B} = I$, and $\hat{A} = AB^{\dagger}$.

Theorem 3.9. Suppose B is onto, then all solutions of $A\, x_{n+1} + Bx_n = f_n$ are given by

$$x_n = B^{\dagger}[\,(-(AB^{\dagger})^D)^n(AB^{\dagger})(AB^{\dagger})^D q + (AB^{\dagger})^D \sum_{i=0}^{n-1}(-(AB^{\dagger})^D)^{n-i-1}\, g_n] + [I - (AB^{\dagger})^D(AB^{\dagger})]\sum_{i=0}^{k-1}(-AB^{\dagger})^i\, \hat{g}_{n+i} + [I - B^{\dagger}B\,]h_n.$$

Theorem 3.10. Suppose A is onto, then all solutions of $A\, x_{n+1} + Bx_n = f_n$ are given by

$$x_n = A^{\dagger}[\,(-BA^{\dagger})^n q + \sum_{i=0}^{n-1}(-BA^{\dagger})^{n-i-1}\, g_i] + [I - A^{\dagger}A]h_n,$$

where h_n is an arbitrary function, and $g_n = f_n - B[I - A^{\dagger}A]h_n$.

Proof. Assume A is onto. We can rewrite $A\,x_{n+1} + Bx_n = f_n$ as

$$(A\,x_{n+1}) + BA^{\dagger}(Ax_n) = f_n - B[I - A^{\dagger}A]x_n\,. \tag{16}$$

Setting $[I - A^{\dagger}A]x_n$ arbitrary, we can solve uniquely for Ax_n, $A^{\dagger}Ax_n = x_n$ to get (16).

4. CONCLUSION

Our study showed the importance of the matrix $\lambda A + B$ in explicating the solutions of the linear system of difference equations $A\,x_{n+1} + Bx_n = f_n$, where A and B are rectangular. All solutions of such system was given for the cases: $\lambda A + B$ is one-to-one, $\lambda A + B$ is onto, A is onto and B is onto, we hope that this study can be applied to linear systems with nonconstant coefficients.

References

[1] S. L. Campbell, Linear systems of differential equations with singular coefficients. *SIAM Journal on Mathematical Analysis* 8 (1977) 1057-1066

[2] A. M. Kanan, Solutions of a class of singular linear systems of difference equations. Part 1. *World Scientific News*, 143 (2020) 53-66

[3] R. Penrose, A generalized inverse for matrices. *Proc. Camb. Phil. Soc.* 51 (1955) 406-413

[4] A. Ben-Israel, Generalized Inverses of Matrices and Their Applications. Springer, (1980) 154-186.

[5] A. M. Kanan, A. A. Elbeleze, and A. Abubaker, Applications of the Moore-Penrose Generalized Inverse to Linear Systems of Algebraic Equations. *American Journal of Applied Mathematics* 7 (2019) 152-156

[6] J. Lopez-Bonilla, R. Lopez-vazquez, S. Vidal-Beltran, Moore-Penrose's inverse and solutions of linear systems. *World Scientific News* 101 (2018) 246-252

[7] T. N. E. Greville, Some Applications of the Pseudoinverse of a Matrix. *SIAM Review*, 2(1) (1960) 15-22

[8] S. L. Campbell, The Drazin inverses of an infinite matrix. *SIAM J. Appl. Math.* 31 (1976) 492-503

[9] A. M. Kanan, K. Hassan, Solution of linear systems of differential equations with singular constant coefficients by the Drazin inverse of matrices. *World Scientific News* 137 (2019) 229-236

[10] S. L. Campbell, C. D. Meyer, Jr., and N. J. Rose, Applications of the Drazin inverse to Linear Systems of differential equations with Singular Constant Coeffficients. *SIAM J. Appl. Math.* 31 (1976) 411-425

[11] I K. Dassios. On non-homogeneous linear generalized linear discrete time systems. *Circuits Systems and Signal Processing* 31 (2012) 1699-1712

[12] I. Dassios, G. Kalogeropoulos. On a non-homogeneous singular linear discrete time system with a singular matrix pencil. *Circuits Syst Signal Process* 32 (2013) 1615-1635

[13] I. K. Dassios. On a boundary value problem of a class of generalized linear discrete-time systems. *Advances in Difference Equations*, 2011: 51 (2011).

[14] A. A. Soliman, Stability analysis of difference systems via cone valued Liapunov`s function method. *J. Math. Kyoto Univ.* 46 (2006) 65-74

[15] S. Zhang, Stability analysis of delay difference systems. *Comput. Math. Appl.* 33 (1997) 41-52

[16] P. K. Anh, L. C. Loi, On multipoint boundary-value problems for linear implicit non-autonomous systems of difference equations, *Vietnam of Mathematics*, 29 (2001) 281-286

[17] C. Peng, Z. Cheng-Hui, Indefinite linear quadratic optimal control problem for singular linear discrete-time system: Krein space method. *Acta Automatica Sinica* 33 (2007) 635-639

[18] I. K. Dassios. On stability and state feedback stabilization of singular linear matrix difference equations. *Advances in Differential Equations,* 2012: 75 (2012).

[19] S. L. Campbell, Limit Behavior of Solutions of Difference Equations. *Linear Algebra and its Applications*, 23 (1979) 167-178

[20] P. K. Anh, N. H. Du, and L. C. Loi, Connections between Implicit Difference Equations and Differential -Algebraic Equations. *Acta Mathematica Vietnamica* 29 (2004) 23-39

[21] P. K. Anh, D. S. Hoang, Stability of a Class of Singular Difference Equations, *International Journal of Difference Equations,* 1 (2006) 181-193

[22] P. K. Anh, N. H. Du, and L. C. Loi, Singular Difference Equations: An Overview. *Vietnam Journal of Mathematics,* 35 (2007) 339-372