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Existence and Uniqueness of Time Periodic Solution to the Viscous Modified Degasperis-Procesi Equation

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ABSTRACT

In this paper, we establish the existence and uniqueness criteria of time periodic solution to the viscous modified Degasperis-Procesi (vmDP for short) equation with periodic boundary value conditions. The analysis of this study is based on Galerkin's method and Leray-Schauder fixed point theorem. Using Galerkin's method some uniform priori estimates of approximate solution to the corresponding equation of vmDP has been constructed. Furthermore, the efficient and straightforward existence and uniqueness criteria of time periodic solution to the vmDP with periodic boundary value conditions has been obtained.

Keywords**:** viscous modified Degasperis-Procesi equation, time periodic solution, Galerkin's method, Leray-Schauder fixed point theorem

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1. INTRODUCTION

The nonlinear partial differential equations have proved to be valuable tools to the modeling of many physical, chemical and biological phenomena. The study of the solitary wave

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solutions for nonlinear partial differential equationsplay an important role in various fields, such as, quantum mechanics, electricity, plasma physics, chemical kinematics, optical fibers, biological model, electromagnetic field, viscoelasticity, electrochemistry, physics, control theory, fluid mechanics and population model [1-4], etc. In most of the cases, it is difficult to obtain the exact solution of these nonlinear partial differential equations. As a result, in the last few years different analytical methods have been developed, such as, Adomian decomposition method [5], the homotopy analysis method [6], the variational iteration method [7-9], the homotopy perturbation method [10-12], and variational homotopy perturbation method [13-14], modified decomposition method [15], etc.

In 2006, Wazwaz [16] studies a family of important physical equations which is known as modified k - equation and he provides the following form of that modified k - equation:

$$u_t - u_{xxt} + (k+1)u^2 u_x - k u_x u_{xx} - u u_{xxx} = 0, \tag{1}$$

where, k is a positive integer. By taking $k=3$, Wazwaz [16] reduces Eq. (1) to the following modified Degasperis-Procesi (mDP) equation:

$$u_t - u_{xxt} + 4u^2 u_x - 3u_x u_{xx} - u u_{xxx} = 0. \tag{2}$$

Wazwaz [17], proved that the Eq. (2) has the following solitary wave solution:

$$u(x,t) = -2\sec h^2\left(\frac{x}{2} - \frac{5}{4}t\right). \tag{3}$$

After Wazwaz [17], Eq. (2) has been also investigated by many researchers, see for instance [18-20] and references therein. Yousif et al. [20] studied the Eq. (2) using variational homotopy perturbation method and obtained the following solitary wave solution:

$$u(x,t) = -\frac{15}{8}\sec h^2\left(\frac{x}{2}\right) - \frac{225}{16} t \sec h^4\left(\frac{1}{2}x\right)\tanh\left(\frac{1}{2}x\right). \tag{4}$$

In this paper, we consider the following one-dimensional viscous version of mDP equation given by Eq. (2):

$$u_t - u_{xxt} - \mu\left(u_{xx} - u_{xxxx}\right) + 4u^2 u_x - 3u_x u_{xx} - u u_{xxx} = f(x,t),\ t>0,\ x\in\mathbb{R}, \tag{5}$$

with the following periodic boundary conditions:

$$u(x+L,t) = u(x,t),\ t>0,\ x\in\mathbb{R}, \tag{6}$$

$$u(x,t+\omega) = u(x,t),\ t>0,\ x\in\mathbb{R}, \tag{7}$$

where $\mu > 0$ is a viscosity constant and the forcing term f is L- periodic in spatial x and ω-periodic in time t. Without loss of generality, we assume that $\int_{\Omega} f(x,t)\,dx = 0$, where $\Omega = [0,\ L]$ Using different techniques, Foias et. al. [21], Gao and Shen [22] and Gao et. al. [23] have been studied the viscous version of modified Camassa-Holm equation which is obtained by setting $k = 2$ in Eq. (1). Motivated by the works of Foias et. al. [21], Gao and Shen [22], Gao et. al. [23] and Fu and Guo [26] here we consider the Eq. (5) which we named as vmDP equation.

Now a days, many researchers devoted themselves in the study of existence of time-periodic solution to various nonlinear evolution equations, see for instance [23-26] and references therein. If a system is periodically dependent on time t, then there arises a natural question about the existence of time-periodic solution with same period for that system. Recently, E.E.

Obinwanne and U. Collins [27] applied the Leray-Schauder fixed point theorem [24, 29] to obtain solution of Duffing's equation. Moreover, to the best of our knowledge, there is no any work considering the existence and uniqueness of time-periodic solution to the Eqs. (5)-(7), using Galerkin's method [24, 28] and Leray-Schauder fixed point theorem [24, 29]. Therefore, the main purpose of this paper is to establish the existence and uniqueness criteria of time-periodic solution to the system given by Eqs. (5)-(7) using Galerkin's method and Leray-Schauder fixed point theorem.

The rest of this work is furnished as follows:

In Section 2, we provide some basic definitions, inequalities and introduce Galerkin's method and Leray-Schauder fixed point theorem. Section 3 is used to formulate some uniform priori estimates for the existence of approximate solution of the vmDP equation given Eqs. (5)-(7), which will be apply in next section. Section 4 is devoted to state and prove the existence and uniqueness criteria of time-periodic solutions to the vmDP equation given by Eqs.(5)-(7). Finally, we give a conclusion.

2. MATERIALS AND METHODS

In this section, we introduce some necessary definitions and preliminary facts which will be used throughout this paper.

Definition 2.1. ([30]). Let B be a Banach space. For $1 \le p \le \infty$, the space $L^p(B;\omega)$ is defined as the set of ω-periodic B-measurable functions on $\mathbb{R}$ such that

$$\|u\|_{L^p(B;\omega)} = \begin{cases} \left(\int_0^{\omega} \|u\|_B^p\, ds\right)^{1/p} < \infty, & 1 \le p < \infty \\ \sup\limits_{0 \le t \le \omega} \|u\|_B < \infty, & p = \infty. \end{cases}$$

Definition 2.2. ([30]).The space $W^{h,p}(B;\omega)$denote the set of functions which belong to $L^p(B;\omega)$ together with their derivatives up to order h, and if B is a Hilbert space, then we write $W^{h,2}(B;\omega)=H^h(B;\omega)$.

According to definitions 2.1 and 2.2, the following inequalities hold (see [30]):

$$\|u\|_\infty \le h_1\|u\|_{H^1}, \tag{8}$$

$$\left\|D^i u\right\|_p \le h_2\|u\|_{H^m}^{\theta}\|u\|^{1-\theta}, \tag{9}$$

where, $D^i u=\dfrac{\partial^i u}{\partial x^i}$, $\dfrac{1}{p}=i+\theta\left(\dfrac{1}{2}-m\right)+(1-\theta)\dfrac{1}{2}$, as $0\le i<m,\ i/m\le\theta\le 1.$

$$\|u\|\le h_3\|u_x\|,\ \int_\Omega u(x)\,dx=0. \tag{10}$$

Now, we state Leray-Schauder fixed point theorem which will be used as the tools to establish the main results.

Theorem 2.1. (Leray-Schauder fixed point theorem [28]). Let B be a Banach space and $T:B\to B$ be a completely continuous (continuous and compact) operator with the following property: there exists $R>0$ such that the statement ($u=rTu\ with\ r\in[0,1)$) implies $\|u\|_B<R$. Then T has a fixed point u^* such that $\|u^*\|\le R$.

Now we give a brief discussion on the Galerkin's method.

The Galerkin's method is a very strong andgeneral method. The main idea of this method is as follows. To tackle a problem posed in an infinite dimensional space, start with a studying its approximation on a nested sequence of finite dimensional sub-spaces. Solving the approximate problem is generally simpler than solving the infinite dimensional one. Passing to the limit, we construct a solution of the original problem. Here we also introduce the Galerkin's method with an abstract problem posed as a weak formulation on a Hilbert space H, namely,

find $u\in H$ such that for all $v\in H,\ a(u,v)=f(v)$.

where $a(.,.)$ is a bilinear form and $f(v)$ is a bounded linear functional on H.

Choose a subspace $H_n\subset H$ of dimension n and solve the projected problem:

$$\text{find } u_n\in H_n \text{ such that for all } v_n\in H_n,\ a(u_n,v_n)=f(v_n). \tag{11}$$

The Eq. (11) is known as the Galerkin equation. Notice that the Eq. (11) has remained unchanged and only the spaces have changed. Reducing the problem to a finite-dimensional vector subspace allows us to numerically compute u_n as a finite linear combination of the basis vectors in H_n. The key property of the Galerkin approach is that the error is orthogonal to the chosen sub-spaces. Since $H_n \subset H$, we can use v_n as a test vector in the original equation.

Subtracting the two, we get the Galerkin orthogonality relation for the error $e_n = u - u_n$, which is the error between the solution of the original problem, u and the solution of the Galerkin equation, u_n

$$a(e_n, v_n) = a(u, v_n) - a(u_n, v_n) = f(v_n) - f(v_n) = 0.$$

Since the aim of Galerkin's method is the production of a linear system of equations. Hence, we build its matrix form, which can be used to compute the solution algorithmically.

Let $b_1, b_2, b_3, \cdots, b_n$ be a basis for H_n .Then, it is sufficient to use these in turn for testing the Galerkin equation Eq. (11), that is: find $u_n \in H_n$ such that $a(u_n, b_i) = f(b_i),\ i = 1, 2, 3, \cdots, n$

We expand u_n with respect to this basis, $u_n = \sum_{j=1}^{n} u_j b_j$ and inserting it into the above equation we obtain that

$$a\left(\sum_{j=1}^{n} u_j b_j, b_i\right) = \sum_{j=1}^{n} u_j a(b_j, b_i) = f(b_i),\ i = 1, 2, 3, \cdots, n. \tag{12}$$

The Eq. (12) is actually a linear system of equations of the form $A_{ij} u_j = f_i$, where $A_{ij} u_j = \sum_{j=1}^{n} u_j a(b_j, b_i)$, and $f_i = f(b_i)$.

3. UNIFORM PRIORI ESTIMATES FOR THE EXISTENCE OF APPROXIMATE SOLUTION

In this section, we formulate some uniform priori estimates for the existence of approximate solution to the vmDP equation by applying Galerkin's method and theorem 2.1. If we denote the unbounded linear operator by $Au = -u_{xx}$ on

$$B = L^2 \cap \left\{u : u(x+L) = u(x), \int_{\Omega} u dx = 0\right\},$$

then we obtain a set of all linearly independent eigenvectors $\{\omega_j\}_{j=0}^{\infty}$ of A, i.e., $A\omega_j = \lambda_j \omega_j$, with $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \to \infty$, which form an orthogonal basis of $L^2(\Omega)$. Now by Galerkin's method, for any n and a sequence of functions $\{a_{jn}(t)\}_{j=1}^{n}$, where $a_{jn}(t) \in C^1(R;\omega)$,

$(j=1,2,3,\cdots,n)$, we can say that the function $u_n=\sum_{j=1}^{n}a_{jn}(t)\omega_j\in C^1(H^n;\,\omega)$is an approximate solution to the Eqs. (5) - (7) if itsatisfies the following equation:

$$\left(u_{nt}-u_{nxxt}-\mu\left(u_{nxx}-u_{nxxxx}\right),\omega_j\right)=\left(Nu_n+f,\,\omega_j\right),\ j=1,2,3,\cdots,n, \tag{13}$$

where, $Nu_n=-4u_n^{\;2}u_{nx}+3u_{nx}u_{nxx}+u_nu_{nxxx}$ and $S_n=span\{\omega_1,\omega_2,\omega_3,\cdots,\omega_n\}$. By classical theory of ordinary differential equations, we can say that for any fixed $v_n(t)=\sum_{j=1}^{n}b_{jn}(t)\omega_j\in C^1(H^n;\,\omega)$, the equation

$$\left(u_{nt}-u_{nxxt}-\mu\left(u_{nxx}-u_{nxxxx}\right),\omega_j\right)=\left(Nv_n+f,\,\omega_j\right),\ j=1,2,3,\cdots,n, \tag{14}$$

has a unique ω -periodic solution u_n and the mapping $T:v_n\to u_n$ is continuous and compact in $C^1(S_n;\,\omega)$. Hence for proving the existence of the time periodic solution of Eq. (13) by applying theorem 2.1, it is enough to shown that the inequality $\sup_{0\le t\le\omega}\|u_n\|^2\le c$ holds for all possible solutions of Eq. (13) and the nonlinear term Nu_n is replaced by λNu_n, $(0\le\lambda\le1)$, where c is a constant only depending on $L,\omega,\mu,h,h_1,h_2,h_3,f$.

Now, we establish some lemmas which convey the required uniform priori estimators for the existence of approximate time periodic solution of (13).

Lemma 3.1. *If* $f\in C^1(H^{-1}(\Omega);\omega)$, *then* $\sup_{0\le t\le\omega}\left(\|u_n\|^2+\|u_{nx}\|^2\right)\le c_1$, *where* c_1 *is a constant only depending on* $L,\omega,\mu,\lambda_1,f,\varepsilon,M=\sup_{0\le t\le\omega}\left\{\|f(x,t)\|^2_{H^{-1}(\Omega)}\right\}$ *and* $d_1=\min\{2\mu\lambda_1,2\mu-\varepsilon\}>0$.

Proof. Multiplying both sides of Eq. (13) by $a_{jn}(t)$and summing up over j from 1 to n, we yield$\left(u_{nt}-u_{nxxt}-\mu\left(u_{nxx}-u_{nxxxx}\right),u_n\right)=\left(Nu_n+f,\,u_n\right)$, which gives the following equation:

$$\frac{1}{2}\frac{d}{dt}\left(\|u_n\|^2+\|u_{nx}\|^2\right)+\mu\left(\|u_{nx}\|^2+\|u_{nxx}\|^2\right)=\left(Nu_n+f,\,u_n\right). \tag{15}$$

It is clear that

$$-4\int_\Omega u_n^3u_{nx}dx=0,3\int_\Omega u_nu_{nx}u_{nxx}dx+\int_\Omega u_n^2u_{nxxx}dx=0 \tag{16}$$

and

$$\|u_{nxx}\| = \int_{\Omega}\left|\left(\sum_{j=1}^{n} a_{jn}(t)\omega_j\right)_{xx}\right|^2 dx = \int_{\Omega}\left|\sum_{j=1}^{n} \lambda_j a_{jn}(t)\omega_j\right|^2 dx \geq \lambda_1\|u_n\|^2. \tag{17}$$

From the Young's inequality, we have

$$\int_{\Omega} f u_n dx \leq \frac{\varepsilon}{2}\|u_{nx}\|^2 + \frac{M}{2\varepsilon}, \tag{18}$$

where $\varepsilon > 0$ is a constant.

Combining Eqs. (15), (16) and inequalities (17) and (18), we obtain that

$$\frac{d}{dt}\left(\|u_n\|^2 + \|u_{nx}\|^2\right) + d_1\left(\|u_n\|^2 + \|u_{nx}\|^2\right) \leq \frac{M}{\varepsilon}, \tag{19}$$

where $d_1 = \min\{2\mu\lambda_1, 2\mu - \varepsilon\} > 0.$

Applying the time periodicity of u_n and integrating the inequality (19) over the closed interval $[0,\omega]$, we get $d_1\int_0^{\omega}\left(\|u_n\|^2 + \|u_{nx}\|^2\right)dt \leq \frac{M\omega}{\varepsilon}$. Thus there exists a $t^* \in [0,\omega]$ such that

$$\|u_n(t^*)\|^2 + \|u_{nx}(t^*)\|^2 \leq \frac{M}{d_1\varepsilon}. \tag{20}$$

Hence from inequalities (19) and (20), we get

$$\frac{d}{dt}\left(\|u_n\|^2 + \|u_{nx}\|^2\right) \leq \frac{M}{\varepsilon}. \tag{21}$$

Integrating the inequality (21) with respect to t from t^* to $t \in [t^*, t^* + \omega]$, we have

$$\|u_n(t)\|^2 + \|u_{nx}(t)\|^2 - \left(\|u_n(t^*)\|^2 + \|u_{nx}(t^*)\|^2\right) \leq \frac{M\omega}{\varepsilon}$$

$$\Rightarrow \|u_n(t)\|^2 + \|u_{nx}(t)\|^2 \leq \frac{M\omega}{\varepsilon} + \|u_n(t^*)\|^2 + \|u_{nx}(t^*)\|^2 \leq \frac{M\omega}{\varepsilon} + \frac{M}{d_1\varepsilon} = \frac{M\omega}{\varepsilon}\left(\omega + \frac{1}{d_1}\right).$$

Therefore, we deduce that

$$\sup_{0\leq t\leq\omega}\left(\|u_n\|^2 + \|u_{nx}\|^2\right) \leq \frac{M\omega}{\varepsilon}\left(\omega + \frac{1}{d_1}\right) \triangleq c_1. \tag{22}$$

This completes the proof.

Remark 3.1. From theorem 2.1 and lemma 3.1, we can conclude that the sequence $\{u_n\}_{n=1}^{\infty}$ represents the sequence of approximate solutions of Eq. (13) and hence the sequence $\{u_n\}_{n=1}^{\infty}$ also represents the sequence of approximate solutions to system given by Eqs. (5) - (7).

Now, we find the convergence of the sequence $\{u_n\}_{n=1}^{\infty}$ of approximate solutions, and for this we need to establish a priori estimates for the high order derivers of that sequence $\{u_n\}_{n=1}^{\infty}$.

Lemma 3.2. If $f \in C^1\left(H^{-1}(\Omega);\omega\right)$, then $\sup\limits_{0\le t\le\omega}\left(\|u_{nx}\|^2 + \|u_{nxx}\|^2\right) \le c_2$, where c_2 is a constant only depending on $L, \omega, \mu, \lambda_1, f, \varepsilon, h_1, h_2$ *and* c_1.

Proof. Multiplying both sides of Eq. (13) by $-\lambda_j a_{jn}(t)$ and summing up over j from 1 to n, we yield $\left(u_{nt} - u_{nxxt} - \mu\left(u_{nxx} - u_{nxxxx}\right), u_{nxx}\right) = \left(Nu_n + f, u_{nxx}\right)$, which gives the following equation:

$$-\frac{1}{2}\frac{d}{dt}\left(\|u_{nx}\|^2 + \|u_{nxx}\|^2\right) - \mu\left(\|u_{nxx}\|^2 + \|u_{nxxx}\|^2\right) = \left(Nu_n + f, u_{nxx}\right). \tag{23}$$

From the Young's inequality, we have

$$\left|\int_\Omega f u_{nxx} dx\right| \le \frac{\varepsilon}{2}\|u_{nxxx}\|^2 + \frac{M}{2\varepsilon}, \tag{24}$$

where $\varepsilon > 0$ is a constant and $M = \sup\limits_{0\le t\le\omega}\left\{\|f(x,t)\|^2_{H^{-1}(\Omega)}\right\}$.

Combining inequalities (8), (22) and Young's inequality, we obtain that

$$\begin{aligned}\left|\int_\Omega u_n^2 u_{nx} u_{nxx} dx\right| &\le \|u_n\|_\infty \|u_n\|_\infty \int_\Omega |u_{nx} u_{nxx}| dx \le h_1^2 \|u_n\|^2_{H^1} \int_\Omega |u_{nx} u_{nxx}| dx \\ &\le h_1^2 c_1\left(\frac{\varepsilon}{2h_1^2 c_1}\|u_{nxx}\|^2 + \frac{h_1^2 c_1}{2\varepsilon}\|u_{nx}\|^2\right) \le \frac{\varepsilon}{2}\|u_{nxx}\|^2 + \frac{h_1^4 c_1^3}{2\varepsilon}.\end{aligned} \tag{25}$$

Combining inequalities (9), (22), Cauchy-Schwarz inequality, Young's inequality and lemma 3.1, we have

$$\begin{aligned}\left|\int_\Omega u_n u_{nxx} u_{nxxx} dx\right| &= \left|-\frac{1}{2}\int_\Omega u_{nx} u_{nxx}^2 dx\right| \le \frac{1}{2}\|u_{nx}\|\|u_{nxx}\|_4^2 \le \frac{1}{2} c_1^{1/2} h_2^2 \|u_n\|^{1/2} \|u_n\|^{3/2}_{H^3} \\ &\le \frac{3}{4}\varepsilon\|u_n\|^2_{H^3} + \frac{c_1^3 h_2^8}{64\varepsilon^3} \le \frac{3}{4}\varepsilon c_1 + \frac{3}{4}\varepsilon\|u_{nxx}\|^2 + \frac{3}{4}\varepsilon\|u_{nxxx}\|^2 + \frac{c_1^3 h_2^8}{64\varepsilon^3}.\end{aligned} \tag{26}$$

Choosing ε small enough such that $\frac{15}{4}\varepsilon<\frac{\mu}{2}$ and using inequalities (24) - (26) in Eq. (23), we obtain

$$\frac{d}{dt}\left(\|u_{nx}\|^2+\|u_{nxx}\|^2\right)+\mu\left(\|u_{nxx}\|^2+\|u_{nxxxx}\|^2\right)\le\frac{M}{\varepsilon}+\frac{3h_1^4c_1^2}{\varepsilon}+\frac{9\varepsilon c_1}{2}+\frac{3h_2^8c_1^3}{32\varepsilon^3}. \tag{27}$$

Applying the time periodicity of u_n and integrating the inequality (27) with respect to t over the closed interval $[0,\omega]$, we get $\mu\int_0^\omega\left(\|u_{nxx}\|^2+\|u_{nxxx}\|^2\right)dt\le\left(\frac{M}{\varepsilon}+\frac{3h_1^4c_1^2}{\varepsilon}+\frac{9\varepsilon c_1}{2}+\frac{3h_2^8c_1^3}{32\varepsilon^3}\right)\omega.$

Thus there exists a $t^*\in[0,\omega]$ such that

$$\|u_{nxx}(t^*)\|^2+\|u_{nxxx}(t^*)\|^2\le\frac{1}{\mu}\left(\frac{M}{\varepsilon}+\frac{3h_1^4c_1^2}{\varepsilon}+\frac{9\varepsilon c_1}{2}+\frac{3h_2^8c_1^3}{32\varepsilon^3}\right). \tag{28}$$

Hence from inequalities (27) and (28), we get

$$\frac{d}{dt}\left(\|u_{nx}\|^2+\|u_{nxx}\|^2\right)\le\frac{M}{\varepsilon}+\frac{3h_1^4c_1^2}{\varepsilon}+\frac{9\varepsilon c_1}{2}+\frac{3h_2^8c_1^3}{32\varepsilon^3}. \tag{29}$$

Integrating the inequality (29) with respect to t from t^* to $t\in[t^*,t^*+\omega]$, we have

$$\|u_{nx}(t)\|^2+\|u_{nxx}(t)\|^2-\left(\|u_{nx}(t^*)\|^2+\|u_{nxx}(t^*)\|^2\right)\le\left(\frac{M}{\varepsilon}+\frac{3h_1^4c_1^2}{\varepsilon}+\frac{9\varepsilon c_1}{2}+\frac{3h_2^8c_1^3}{32\varepsilon^3}\right)\omega$$

$$\begin{aligned}\Rightarrow\|u_{nx}(t)\|^2+\|u_{nxx}(t)\|^2&\le\left(\frac{M}{\varepsilon}+\frac{3h_1^4c_1^2}{\varepsilon}+\frac{9\varepsilon c_1}{2}+\frac{3h_2^8c_1^3}{32\varepsilon^3}\right)\omega+\|u_{nx}(t^*)\|^2+\|u_{nxx}(t^*)\|^2\\&\le\left(\frac{M}{\varepsilon}+\frac{3h_1^4c_1^2}{\varepsilon}+\frac{9\varepsilon c_1}{2}+\frac{3h_2^8c_1^3}{32\varepsilon^3}\right)\omega+\left(\frac{M}{\varepsilon}+\frac{3h_1^4c_1^2}{\varepsilon}+\frac{9\varepsilon c_1}{2}+\frac{3h_2^8c_1^3}{32\varepsilon^3}\right)\frac{1}{\mu}\\&=\left(\frac{M}{\varepsilon}+\frac{3h_1^4c_1^2}{\varepsilon}+\frac{9\varepsilon c_1}{2}+\frac{3h_2^8c_1^3}{32\varepsilon^3}\right)\left(\omega+\frac{1}{\mu}\right).\end{aligned}$$

Therefore, we deduce that

$$\sup_{0\le t\le\omega}\left(\|u_{nx}\|^2+\|u_{nxx}\|^2\right)\le\left(\frac{M}{\varepsilon}+\frac{3h_1^4c_1^2}{\varepsilon}+\frac{9\varepsilon c_1}{2}+\frac{3h_2^8c_1^3}{32\varepsilon^3}\right)\left(\omega+\frac{1}{\mu}\right)\triangleq c_2. \tag{30}$$

This completes the proof.

In the following lemma, we continue the formation of priori estimates for the high order derivers of the approximation solution sequence $\{u_n\}_{n=1}^{\infty}$ by an inductive argument.

Lemma 3.3. For any $r \geq 0$, if $f \in C^1\left(H^{r-1}(\Omega);\omega\right)$, then $\sup\limits_{0\leq t\leq\omega}\left(\left\|D^{r+1}u_n\right\|^2 + \left\|D^{r+2}u_n\right\|^2\right) \leq b,$ where b is a constant only depending on $L, \omega, \mu, r, h_1, h_2, h_3, f, \varepsilon.$

Proof. If we consider $r = 0$, then lemma 3.3 is obviously hold from lemma 3.2. Assume that for $0 < r \leq r_1 - 1$, where $r_1 \geq 2$, lemma 3.3 holds. By induction method to complete the proof of this lemma, we have to prove that the lemma is also hold for $r = r_1$.

Multiplying both sides of Eq. (13) by $(-1)^{r_1+1}\lambda_j^{r_1+1}a_{jn}(t)$ and summing up over j from 1 to n, we yield

$$(-1)^{r_1+1}\frac{1}{2}\frac{d}{dt}\left(\left\|D^{r_1+1}u_n\right\|^2 + \left\|D^{r_1+2}u_n\right\|^2\right) + (-1)^{r_1+1}\mu\left(\left\|D^{r_1+2}u_n\right\|^2 + \left\|D^{r_1+3}u_n\right\|^2\right) = \left(Nu_n + f,\, D^{2(r_1+1)}u_n\right). \tag{31}$$

From the Young's inequality, we have

$$\left|\int_\Omega f\, D^{2(r_1+1)}u_n dx\right| = \left|\int_\Omega D^{r_1-1}f\, D^{r_1+3}u_n dx\right| \leq \frac{\varepsilon}{2}\left\|D^{r_1+3}u_n\right\|^2 + \frac{1}{2\varepsilon}\left\|D^{r_1-1}f\right\|^2. \tag{32}$$

where $\varepsilon > 0$ is a constant.

Since the lemma is hold for $0 < r \leq r_1 - 1$, then from inequalities (8), (10) and Young's inequality, we obtain that

$$\begin{aligned}
&\left|\int_\Omega u_n^2 u_{nx} D^{2(r_1+1)}u_n dx\right| \leq \left\|u_n\right\|_\infty \left|\int_\Omega\left(\sum_{i=0}^{r_1+1} C_{r_1+1}^i D^i u_n D^{r_1+1-i}u_{nx}\right) D^{r_1+1}u_n dx\right| \\
&\leq \left\|u_n\right\|_\infty \int_\Omega \left|u_n D^{r_1+2}u_n D^{r_1+1}u_n\right| dx + \left\|u_n\right\|_\infty \int_\Omega \left|\left(\sum_{i=0}^{r_1+1} C_{r_1+1}^i D^i u_n D^{r_1+1-i}u_{nx}\right) D^{r_1+1}u_n\right| dx \\
&\leq \varepsilon\left\|D^{r_1+2}u_n\right\|^2 + \frac{\left[h_1^2 h_3^2 b\right]^2}{4\varepsilon}\left\|D^{r_1+1}u_n\right\|^2 + h_1\left\|u_n\right\|_{H^1}\int_\Omega\left|\left(\sum_{i=0}^{r_1+1} C_{r_1+1}^i D^i u_n D^{r_1+1-i}u_{nx}\right) D^{r_1+1}u_n\right| dx \\
&\leq \varepsilon\left\|D^{r_1+2}u_n\right\|^2 + b\left(\varepsilon, h_1, h_3, r_1\right).
\end{aligned} \tag{33}$$

and

$$\begin{aligned}
&\left|\int_\Omega u_{nx}u_{nxx}D^{2(r_1+1)}u_n dx\right| \\
&\quad \le \|u_{nx}\|_\infty \int_\Omega \left|D^{r_1+1}u_n D^{r_1+3}u_n\right| dx + \sum_{i=0}^{r_1-1} C_{r_1+1}^i \left\|D^{i+1}u_n\right\|_\infty \left\|D^{r_1+1-i}u_n\right\|_\infty \int_\Omega \left|D^{r_1+3}u_n\right| dx \\
&\quad \le 2\varepsilon \left\|D^{r_1+3}u_n\right\|^2 + b(r_1,h_1,\varepsilon,L).
\end{aligned} \tag{34}$$

Similarly, we also obtain that

$$\begin{aligned}
&\left|\int_\Omega u_n u_{nxxx}D^{2(r_1+1)}u_n dx\right| \\
&\le \int_\Omega \left|u_n D^{r_1+3}u_n D^{r_1+2}u_n\right| dx + \int_\Omega \left|C_{r_1}^1 Du_n \left(D^{r_1+2}u_n\right)^2\right| dx \\
&\quad + \int_\Omega \left|C_{r_1}^2 D^2u_n D^{r_1+1}u_n D^{r_1+2}u_n\right| dx + \int_\Omega \left|\left(\sum_{i=0}^{r_1-1} C_{r_1}^i D^i u_n D^{r_1+3-i}u_n\right) D^{r_1+2}u_n\right| dx \\
&\le b(r_1,h_1,h_3)\left(\int_\Omega \left|D^{r_1+3}u_n D^{r_1+2}u_n\right| dx + \int_\Omega \left|D^{r_1+2}u_n\right|^2 dx + \int_\Omega \left|D^{r_1+1}u_n D^{r_1+2}u_n\right| dx + \int_\Omega \left|D^{r_1+2}u_n\right| dx\right).
\end{aligned} \tag{35}$$

Again, since the lemma is hold for $0 < r \le r_1 - 1$, then from inequalities (9) and Young's inequality, we get

$$\begin{aligned}
&b(r_1,h_1,h_3)\int_\Omega \left|D^{r_1+3}u_n D^{r_1+2}u_n\right| dx \\
&\le \varepsilon \left\|D^{r_1+3}u_n\right\|^2 + b(r_1,h_1,\ h_2,h_3,\varepsilon)\|u_n\|_{H^{r_1+3}}^{2(r_1+2)/(r_1+3)} \|u_n\|^{2(1-r_1+2/(r_1+3))} \\
&\le \varepsilon \left\|D^{r_1+3}u_n\right\|^2 + \varepsilon \|u_n\|_{H^{r_1+3}}^2 + b(r_1,h_1,\ h_2,h_3,\varepsilon)\|u_n\|^2 \\
&\le 2\varepsilon \left\|D^{r_1+3}u_n\right\|^2 + \varepsilon \left\|D^{r_1+2}u_n\right\|^2 + b(r_1,h_1,\ h_2,h_3,\varepsilon),
\end{aligned}$$

$$\begin{aligned}
&b(r_1,h_1,h_3)\int_\Omega \left|D^{r_1+2}u_n\right|^2 dx \le b(r_1,h_1,\ h_2,h_3)\|u_n\|_{H^{r_1+3}}^{2(r_1+2)/(r_1+3)} \|u_n\|^{2(1-r_1+2/(r_1+3))} \\
&\le \varepsilon \left\|D^{r_1+3}u_n\right\|^2 + \varepsilon \left\|D^{r_1+2}u_n\right\|^2 + b(r_1,h_1,\ h_2,h_3,\varepsilon),
\end{aligned}$$

$$\begin{aligned}
&b(r_1,h_1,h_3)\int_\Omega \left|D^{r_1+1}u_n D^{r_1+2}u_n\right| dx \le \varepsilon \left\|D^{r_1+2}u_n\right\|^2 + b(r_1,h_1,h_3,\varepsilon)\left\|D^{r_1+1}u_n\right\|^2 \\
&\quad \le \varepsilon \left\|D^{r_1+2}u_n\right\|^2 + b(r_1,h_1,h_3,\varepsilon),
\end{aligned}$$

and

$$b(r_1,h_1,h_3)\int_\Omega \left|D^{r_1+2}u_n\right| dx \le \varepsilon \left\|D^{r_1+2}u_n\right\|^2 + b(r_1,h_1,h_3,\varepsilon,L).$$

Combining the above inequalities with the inequality (35), we have

$$\left|\int_\Omega u_n u_{nxxx} D^{2(r_1+1)} u_n dx\right| \le 3\varepsilon \left\|D^{r_1+3}u_n\right\|^2 + 4\varepsilon\left\|D^{r_1+2}u_n\right\|^2 + b\left(r_1,h_1,\ h_2,h_3,\varepsilon,L\right) \tag{36}$$

Using inequalities (31)-(34) inthe inequality (36), we get

$$\begin{aligned}&\frac{d}{dt}\left(\left\|D^{r_1+1}u_n\right\|^2+\left\|D^{r_1+2}u_n\right\|^2\right)+2\mu\left(\left\|D^{r_1+2}u_n\right\|^2+\left\|D^{r_1+3}u_n\right\|^2\right)\\ &\quad\le 15\varepsilon\left\|D^{r_1+3}u_n\right\|^2+14\varepsilon\left\|D^{r_1+2}u_n\right\|^2+b\left(r_1,h_1,\ h_2,h_3,\varepsilon,f,L\right).\end{aligned} \tag{37}$$

Choosing ε small enough such that $15\varepsilon<\mu,$ then from (37) we have

$$\frac{d}{dt}\left(\left\|D^{r_1+1}u_n\right\|^2+\left\|D^{r_1+2}u_n\right\|^2\right)+2\mu\left(\left\|D^{r_1+2}u_n\right\|^2+\left\|D^{r_1+3}u_n\right\|^2\right)\le b\left(r_1,h_1,\ h_2,h_3,\varepsilon,f,L\right). \tag{38}$$

Applying the time periodicity of u_n and integrating the inequality (38) with respect to t over the closed interval $[0,\omega]$, we get

$$2\mu\int_0^\omega\left(\left\|D^{r_1+2}u_n\right\|^2+\left\|D^{r_1+3}u_n\right\|^2\right)dt\le b\left(r_1,h_1,\ h_2,h_3,\varepsilon,f,L\right)\omega.$$

Thus there exists a $t^*\in[0,\omega]$ such that

$$\left\|D^{r_1+2}u_n\left(t^*\right)\right\|^2+\left\|D^{r_1+3}u_n\left(t^*\right)\right\|^2\le\frac{b\left(r_1,h_1,\ h_2,h_3,\varepsilon,f,L\right)}{\mu}. \tag{39}$$

Hence from inequalities (38) and (39), we get

$$\frac{d}{dt}\left(\left\|D^{r_1+1}u_n\right\|^2+\left\|D^{r_1+2}u_n\right\|^2\right)\le b\left(r_1,h_1,\ h_2,h_3,\varepsilon,f,L\right). \tag{40}$$

Integrating the inequality (40) with respect to t from t^* to $t\in\left[t^*,t^*+\omega\right]$, we have

$$\begin{aligned}&\left\|D^{r_1+1}u_n(t)\right\|^2+\left\|D^{r_1+2}u_n(t)\right\|^2-\left(\left\|D^{r_1+1}u_n\left(t^*\right)\right\|^2+\left\|D^{r_1+2}u_n\left(t^*\right)\right\|^2\right)\le\omega b\left(r_1,h_1,\ h_2,h_3,\varepsilon,f,L\right)\\ &\Rightarrow\ \left\|D^{r_1+1}u_n(t)\right\|^2+\left\|D^{r_1+2}u_n(t)\right\|^2\le\omega b\left(r_1,h_1,\ h_2,h_3,\varepsilon,f,L\right)+\left\|D^{r_1+1}u_n\left(t^*\right)\right\|^2+\left\|D^{r_1+2}u_n\left(t^*\right)\right\|^2\\ &\qquad=b\left(r_1,h_1,\ h_2,h_3,\varepsilon,f,L\right)\left(\omega+\frac{1}{\mu}\right).\end{aligned}$$

Therefore, we deduce that

$$\sup_{0\le t\le\omega}\left(\left\|D^{r_1+1}u_n\right\|^2+\left\|D^{r_1+2}u_n\right\|^2\right)\le b\left(r_1,h_1,\ h_2,h_3,\varepsilon,f,L\right)\left(\omega+\frac{h_3^2}{\mu}\right)\triangleq b. \tag{41}$$

Thus, the lemma is hold for $r=r_1$. This completes the proof.

Lemma 3.4. For any $r\ge 0$, if $f\in C^1\left(H^{r+1}(\Omega);\omega\right)$, then $\sup_{0\le t\le\omega}\left(\left\|D^r u_{nt}\right\|^2+\left\|D^{r+1}u_{nt}\right\|^2\right)\le b_1$, where b_1 is a constant only depending on $L,\omega,\mu,\varepsilon,\lambda_n,h,h_1,h_2,h_3$ *and* f.

Proof. We first prove that the lemma is hold of $r=0$, that is we prove that $f\in C^1\left(H^1(\Omega);\omega\right)$ implies, $\sup_{0\le t\le\omega}\left(\left\|u_{nt}\right\|^2+\left\|u_{nxt}\right\|^2\right)\le b_1$.

Multiplying both sides of Eq. (13) by $a'_{jn}(t)$ and summing up over j from 1to n, we get

$$\left\|u_{nt}\right\|^2+\left\|u_{nxt}\right\|^2=\left(Nu_n+f+\mu\left(u_{nxx}-u_{nxxxx}\right),u_{nt}\right). \tag{42}$$

Now by lemma 3.3, we have $f\in C^1\left(H^1(\Omega);\omega\right)$ implies, $\left\|u_n\right\|_{H^4}^2\le b_1$. Hence

$$\left|\left(Nu_n+f+\mu\left(u_{nxx}-u_{nxxxx}\right),u_{nt}\right)\right|\le\left\|Nu_n+f+\mu\left(u_{nxx}-u_{nxxxx}\right)\right\|\left\|u_{nt}\right\|\le b_1\left\|u_{nt}\right\|. \tag{43}$$

Combining Eq. (42) and inequality (43), we obtain that

$$\sup_{0\le t\le\omega}\left(\left\|u_{nt}\right\|^2+\left\|u_{nxt}\right\|^2\right)\le b_1.$$

Thus, the lemma holds for $r=0$. Now we assume that the lemma is hold for $0<r\le r_1$, where $r_1\ge 1$. By induction method to complete the proof of this lemma, we have to prove that the lemma is also hold for $r=r_1+1$.

Multiplying both sides of Eq. (13) by $(-1)^{r_1+1}\lambda_j^{r_1+1}a'_{jn}(t)$ and summing up over j from 1 to n, we yield

$$(-1)^{r_1+1}\left(\left\|D^{r_1+1}u_{nt}\right\|^2+\left\|D^{r_1+2}u_{nt}\right\|^2\right)=\left(Nu_n+f+\mu\left(u_{nxx}-u_{nxxxx}\right),\ D^{2(r_1+1)}u_{nt}\right). \tag{44}$$

Now by lemma 3.3, we have $f\in C^1\left(H^{r_1+2}(\Omega);\omega\right)$ implies, $\left\|D^r u_n\right\|\le b_1$, where $r\le r_1+1$. Hence

$$\left|\left(Nu_n + f + \mu\left(u_{nxx} - u_{nxxxx}\right), D^{2(r_1+1)}u_{nt}\right)\right| \le \left\|D^{r_1+1}\left(Nu_n + f + \mu\left(u_{nxx} - u_{nxxxx}\right)\right)\right\|\left\|D^{r_1+1}u_{nt}\right\| \\ \le b_1\left\|D^{r_1+1}u_{nt}\right\|. \tag{45}$$

Combining Eq. (44) and inequality (45), we obtain that

$$\sup_{0\le t\le\omega}\left(\left\|D^{r_1+1}u_{nt}\right\|^2 + \left\|D^{r_1+2}u_{nt}\right\|^2\right) \le b_1.$$

Therefore, the lemma is hold for $r = r_1 + 1$. This completes the proof.

4. TIME-PERIODIC SOLUTION TO THE vmDP EQUATION

This section is devoted to establish the existence and uniqueness criteria of time-periodic solutions to the vmDP equation given by Eqs. (5) - (7).

We first we establish the existence of time periodic solution for vmDP equation given by Eqs. (5) - (7), for which we construct the following theorem.

Theorem 4.1. For any $f \in C^1\left(H^{r+1}(\Omega);\omega\right)$, $r \ge 0$, there exist a time periodic solution of $u(x,t)$ to Eqs. (5) - (7), such that $u(x,t) \in L^\infty\left(H^{r+4}(\Omega);\omega\right) \cap W^{1,\infty}\left(H^r(\Omega);\omega\right)$.

Poof. According to remark 3.1, we have the sequence $\{u_n\}_{n=1}^{\infty}$ is the sequence of approximate solution of Eqs. (5) - (7). So, to complete the proof of this theorem, we have only to prove that the sequence of approximate solution $\{u_n\}_{n=1}^{\infty}$ is converges and the limit of this sequence is $u(x,t) \in L^\infty\left(H^{r+4}(\Omega);\omega\right) \cap W^{1,\infty}\left(H^r(\Omega);\omega\right)$.

Using lemmas 3.1-3.4 and standard compactness arguments, we conclude that there is a subsequence $\{u_{n_k}\}, k = 1,2,3,\cdots,\infty$, such that for any $f \in C^1\left(H^{r+1}(\Omega);\omega\right)$ $r \ge 0$, we get

$$u_{n_k}(x,t) \to u(x,t), \text{ weaklyin } L^\infty\left(H^{r+4}(\Omega);\omega\right);$$

$$u_{n_k}(x,t) \to u(x,t), \text{ strongly in } L^\infty\left(H^{r+3}(\Omega);\omega\right),$$

$$u_{n_k t}(x,t) \to u_t(x,t), \text{ weaklyin } L^\infty\left(H^{r+1}(\Omega);\omega\right);$$

$$u_{n_k t}(x,t) \to u_t(x,t), \text{ strongly in } L^\infty\left(H^{r}(\Omega);\omega\right),$$

From the above discussion, it is clear that the nonlinear terms of Eq. (13) are well defined.

Thus

$$\left\|u_n^2 u_{nx} - u^2 u_x\right\| \le \left\|u_n^2\left(u_{nx}-u_x\right)\right\| + \left\|u_x\left(u_n^2-u^2\right)\right\| \le \left\|u_n\right\|_\infty^2 \left\|u_{nx}-u_x\right\| + \left\|u_x\right\|_\infty \left\|u_n+u\right\|_\infty \left\|u_n-u\right\| \to 0,$$

as $n \to \infty$, uniformly in t.

$$\left\|u_{nx}u_{nxx} - u_x u_{xx}\right\| \le \left\|u_{nx}\left(u_{nxx}-u_{xx}\right)\right\| + \left\|u_{xx}\left(u_{nx}-u_x\right)\right\| \le \left\|u_{nx}\right\|_\infty \left\|u_{nxx}-u_{xx}\right\| + \left\|u_{xx}\right\|_\infty \left\|u_{nx}-u_x\right\|_\infty \to 0,$$

as $n \to \infty$, uniformly in t.

$$\left\|u_n u_{nxxx} - u u_{xxx}\right\| \le \left\|u_n\right\|_\infty \left\|u_{nxxx}-u_{xxx}\right\| + \left\|u_n-u\right\|_\infty \left\|u_{xxx}\right\| \le \left\|u_n\right\|_\infty \left\|u_{nxxx}-u_{xxx}\right\| + h_1 \left\|u_n-u\right\|_{H^1} \left\|u_{xxx}\right\| \to 0,$$

as $n \to \infty$, uniformly in t.

Therefore, it follows that

$$\left(u_t - u_{xxt} - \mu\left(u_{xx}-u_{xxxx}\right), v\right) = \left(Nu + f,\, v\right),\ v \in L_{per}^2.$$

Now applying the priori estimates obtained in the previous section, we can say that $u(x,t)$ satisfies the following vmDP equation

$$u_t - u_{xxt} - \mu\left(u_{xx}-u_{xxxx}\right) = Nu + f,\ \text{almost every where on } \mathbb{R}^1 \times \Omega.$$

Hence $u(x,t) \in L^\infty\left(H^{r+4}(\Omega);\omega\right) \cap W^{1,\infty}\left(H^r(\Omega);\omega\right)$ is the time periodic solution of vmDP equation given by Eqs. (5) - (7), which is also the converging limit of the sequence of approximate solution. This completes the proof.

Now, we establish the uniqueness of time periodic solution of vmDP equation given by Eqs. (5) - (7), which is in following theorem.

Theorem 4.2. Suppose that the hypothesis of theorem 4.1 holds. If $M = \sup\limits_{0\le t\le\omega}\left\{\left\|f(x,t)\right\|_{H^{-1}(\Omega)}^2\right\}$ is sufficiently small, then the time periodic solution of Eqs. (5) - (7) obtained in theorem 4.1 is unique.

Proof. Let $u = u(x,t)$ and $u^* = u^*(x,t)$ betwo distinct time periodic solutions of Eqs. (5) - (7). If we set $v(x,t) = u(x,t) - u^*(x,t)$, then from Eq. (13), we get

$$v_t - v_{xxt} - \mu\left(v_{xx}-v_{xxxx}\right) = Nu - Nu^*. \tag{46}$$

Taking the inner product in both sides of Eq. (46) with v, we have

$$\frac{1}{2}\frac{d}{dt}\left(\|v\|^2 + \|v_x\|^2\right) + \mu\left(\|v_x\|^2 + \|v_{xx}\|^2\right) = \left(Nu - Nu^*,\, v\right). \tag{47}$$

From inequality (10) and Eq. (47), we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|v\|^2+\|v_x\|^2\right)+\frac{\mu}{2h_3^2}\|v\|^2+\frac{\mu}{2}\|v_x\|^2+\mu\|v_{xx}\|^2\le\left(Nu-Nu^*,\,v\right). \tag{48}$$

Since

$$\begin{aligned}&\left|\left(-4u^2u_x+4u^{*2}u_x^*,v\right)\right|\le\left|\left(-4u^2v_x,v\right)\right|+\left|\left(-4\left(u+u^*\right)vu_x^*,v\right)\right|\\&\le 2\|u\|_\infty^2\left(\|v\|^2+\|v_x\|^2\right)+4\|u_x^*\|_\infty\|u+u^*\|_\infty\|v\|^2\le\left(2h_1^2c_1+8h_1^2c_1^{1/2}c_2^{1/2}\right)\|v\|^2+2h_1^2c_1\|v_x\|^2.\end{aligned} \tag{49}$$

$$\begin{aligned}&\left|\left(3u_xu_{xx}-3u_x^*u_{xx}^*,v\right)\right|\le\|u_x\|_\infty\left(\|v\|^2+\|v_{xx}\|^2\right)+3\|u_x^*\|_\infty\|v_x\|^2+\|u_x^*\|_\infty\left(\|v\|^2+\|v_{xx}\|\right)\\&\le 3h_1c_1^{1/2}\|v\|^2+3h_1c_1^{1/2}\|v_x\|^2+3h_1c_1^{1/2}\|v_{xx}\|^2.\end{aligned} \tag{50}$$

$$\begin{aligned}&\left|\left(uu_{xxx}-u^*u_{xxx}^*,v\right)\right|\le\int_\Omega|u_xvv_{xx}|\,dx+\int_\Omega|uv_xv_{xx}|\,dx+2\int_\Omega|u_x^*v_x^2|\,dx+2\int_\Omega|u_x^*vv_{xx}|\,dx\\&\le\frac{1}{2}\|u_x\|_\infty\left(\|v\|^2+\|v_{xx}\|^2\right)+\frac{1}{2}\|u\|_\infty\left(\|v_x\|^2+\|v_{xx}\|^2\right)+2\|u_x^*\|_\infty\|v_x\|^2+\|u_x^*\|_\infty\left(\|v\|^2+\|v_{xx}\|^2\right)\\&\le\frac{3}{2}h_1c_2^{1/2}\|v\|^2+\left(\frac{1}{2}h_1c_1^{1/2}+2h_1c_2^{1/2}\right)\|v_x\|^2+\left(\frac{1}{2}h_1c_1^{1/2}+\frac{3}{2}h_1c_2^{1/2}\right)\|v_{xx}\|^2.\end{aligned} \tag{51}$$

Now, if M is sufficiently small such that

$$2h_1^2c_1+8h_1^2c_1^{1/2}c_2^{1/2}+\frac{9}{2}h_1c_2^{1/2}\le\frac{\mu}{4h_3^2},\;\;2h_1^2c_1+6h_1c_2^{1/2}+\frac{1}{2}h_1c_1^{1/2}\le\frac{\mu}{4},\;\;\frac{1}{2}h_1c_1^{1/2}+\frac{7}{2}h_1c_2^{1/2}\le\frac{\mu}{2}. \tag{52}$$

Combining the inequalities (48)-(52), we get

$$\frac{d}{dt}\left(\|v\|^2+\|v_x\|^2\right)+\delta\left(\|v\|^2+\|v_x\|^2\right)\le 0, \tag{53}$$

where $\delta\ge 0$ is a suitable constant.

Applying Gronwall's inequality [31] in (53), we obtain that

$$\left(\|v(t)\|^2+\|v_x(t)\|^2\right)\le\left(\|v(0)\|^2+\|v_x(0)\|^2\right)e^{-\delta t},\text{ for any }t\ge 0. \tag{54}$$

Since v is ω-periodic in t, then for any positive integer m we have

$$\left(\|v(t)\|^2+\|v_x(t)\|^2\right)=\left(\|v(t+m\omega)\|^2+\|v_x(t+m\omega)\|^2\right). \tag{55}$$

From (54) and (55), we get

$$\left(\|v(t)\|^2+\|v_x(t)\|^2\right)\le\left(\|v(0)\|^2+\|v_x(0)\|^2\right)e^{-\delta(t+m\omega)}.$$

Which gives us

$$v(0)=v_x(0)=0.$$

Hence $u(x,t)=u^*(x,t)$, i.e., the time periodic solution of Eqs. (5) - (7) is unique. This completes the proof.

5. CONCLUSIONS

In this study, we have proven the new existence and uniqueness criteria for time periodic solution to the vmDP equation applying the Galerkin's method and Leray-Schauder fixed point theorem. The Leray-Schauderfixed point theorem helps us to determine the existence ofapproximate solution point withinthe uniform priori estimates, whereas uniform priori estimates of approximate solution of vmDP equation is constructed by using Galerkin's method. Using theorem 4.1, one can easily be checked the existence of time periodic solution of vmDP equation given by Eq. (5)-(7) and theorem4.2 ensure the uniqueness of that time periodic solution. The established results provide an easy and straightforward technique to cheek the existence and uniqueness of time periodic solution to the vmDP equation. Furthermore, the results of this research extend the corresponding results of Foias et. al. [21], Gao and Shen [22] and Gao et. al. [23].

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