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Adjustable Robust Counterpart Optimization Model for Maximum Flow Problems with Box Uncertainty

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ABSTRACT

The maximum flow problem is an optimization problem that aims to find the maximum flow value on a network. This problem can be solved by using Linear Programming. The obstacle that is often faced in determining the maximum flow is the magnitude of the capacity of each side of the network can often be changed due to certain factors. Therefore, we need one of the optimization fields that can calculate the uncertainty factor. The field of optimization carried out to overcome these uncertainties is Robust Optimization. This paper discusses the Optimization model for the maximum flow problem by calculating the uncertainties on parameters and adjustable variables using the Adjustable Robust Counterpart (ARC) Optimization model. In this ARC Optimization model it is assumed that there are indeterminate parameters in the form of side capacity in a network and an uncertain decision variable that is the amount of flow from the destination point (sink) to the source point (source). Calculation results from numerical simulations show that the ARC Optimization model provides the maximum number of flows in a network with a set of box uncertainty. Numerical simulations were obtained with Maple software.

***Keywords*:** Maximum flow problem, Linear Programming, Robust Optimization, Adjustable Robust Counterpart

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1. INTRODUCTION

The maximum flow model is a model that can be used to find out the maximum value of the amount of flow that is passed in a network system. Examples of maximum flow in a network are networks on pipelines, electricity, traffic, communication and others. Several ways to solve the problem of maximum flow, including using linear programming. Furthermore, the problem of maximum flow has also become an important problem in daily life. Maximum flow was first introduced by L.R. Ford and D.R. Fulkerson, in 1956, and continues to be developed to date. However, in reality, there are uncertainty factors that affect the network system. One of the uncertainty factors is the flow capacity on each side in a network that can change. For this reason, optimization techniques are needed that take into account uncertainty in order to obtain an optimal solution that is resistant to data uncertainty.

One area of Optimization that is able to solve various problems related to the problem of uncertainty is Robust Optimization [2]. Robust Optimization has been widely practiced by previous researchers. Referring to [8], the first step towards achieving the Robust Optimization method is carried out by A.L. Soyster, in 1973. To find a solution that is resistant to data uncertainty in linear programming, Soyster enters the worst values of each uncertain parameter into the mathematical programming model. Ben-Tal et al [3] discusses in detail about Robust Optimization.

Gorissen et al [7] given practical guidance on Robust Optimization. Until now Robust Optimization has been growing from year to year. In this framework, the existence of uncertainties in the Robust Counterpart model can produce a feasible solution for all possibilities by using the set of box, ellipsoidal, and polyhedral uncertainty. Referring to [2] and [4], the main challenge of Robust Optimization is to find a set of uncertainties that can be formulated into a computationally tractable optimization problem. Computationally tractable can be analyzed by representing the Robust Counterpart into one of three classes of optimization problems, namely linear programming, conic quadratic, or semidefinit programming.

Robust optimization can be categorized into two, single stage and two stage models. In Robust single stage optimization, all decision variables with "here and now" decisions are considered to be resolved immediately. Meanwhile, in Robust two stage optimization with the "wait and see" decision, the decision variables in the second stage are adjusted to the realization of parameter uncertainty. This two stage Robust optimization is known as Adjustable Robust Counterpart.

This paper discusses the Adjustable Robust Counterpart Optimization model for the problem of maximum current flow with a set of box uncertainty. Uncertainty exists in the parameters of flow capacity on each side and decision variables in the form of the amount of flow from the source point and destination point. In addition, in solving this problem numerical simulations can be searched with Maple software.

2. MATERIALS AND METHODS

To formulate the Adjustable Robust Counterpart Optimization model for the maximum flow problem it must be known in advance about the deterministic model of the maximum flow problem, Robust Optimization, and Adjustable Robust Counterpart Optimization.

2. 1. Maximum Flow Model

Referring to [1], let $G = (V, E)$ is a directed graph, $s, t \in V$ and $k : E \to \mathbb{R}^+$ becomes a function of capacity. Let x is the amount of flow from source node s to destination node t and x_{ij} is the flow from node i to node j over arc (i, j) where $i, j \in V$ and $(i, j) \in E$. In the case of maximum flow, the goal is to find the maximum $s - t$ flow value below k. To find the maximum $s - t$ flow value, add a side from node t to node s with its side capacity value $k_{ts} = \infty$. The maximum flow model can be formulated generally as follows:

$$\begin{aligned}
&\max x_{ts} \\
&\text{s.t} \sum_{j\in V} x_{sj} - \sum_{k\in V} x_{ks} - x_{ts} = 0 \\
&\sum_{j\in V} x_{tj} - \sum_{k\in V} x_{kt} + x_{ts} = 0 \\
&\sum_{j\in V} x_{ij} - \sum_{k\in V} x_{ki} = 0 \qquad \forall\ i \in V - \{s,t\} \\
&0 \le x_{ij} \le k_{ij} \quad \forall\ (i, j) \in E
\end{aligned} \tag{1}$$

Other forms of maximum flow models are as follows:

$$\begin{aligned}
&\max x_{ts} \\
&\text{s.t } Ax = 0 \\
&0 \le x_{ij} \le k_{ij},\ \forall (i, j) \in E
\end{aligned} \tag{2}$$

2. 2. Robust Optimization

Referring to [3], Robust Optimization is a method to solve Optimization problems with data uncertainty and is only known in a set of uncertainties. The general form of the problem of indefinite linear optimization can be formulated as in equation (3) follows:

$$\begin{aligned}
&\min\ c^T x \\
&\text{s.t } Ax \le b \\
&(c, A, b) \in \mathrm{U}
\end{aligned} \tag{3}$$

where $c \in \mathbb{R}^n, A \in \mathbb{R}^{mxn}, b \in \mathbb{R}^n$, the three decision variables are indefinite coefficients. U is a notation of the set of uncertainties.

There are three basic assumptions in Robust Optimization, namely all decision variables state decisions "here and now", decision makers are fully responsible for the consequences of decisions made, if and only if the actual data has been determined in the set of uncertainties U, and constraints on programming problems linear with uncertainty is "hard". In addition, referring to [7], in dealing with Linear Robust Optimization, three things are also assumed. First, the objective function is certainly valuable. If there are uncertainties in the objective function, then the problem can be formulated by replacing the objective function with a single variable function such that uncertainty arises in a constraint function. Second, the right vertex vector b is of course value. If b is not certain, an extra variable x_{n+1} can be introduced. Third, robustness against U can be formulated as a constraint-wise problem and the set of uncertainties U is a closed and convex set.

Assuming that $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ are of certain value, the Robust Counterpart formulation of equation (3) is equivalent to equation (4) below.

$$\begin{aligned} &\min_x \; c^T x \\ &\text{s.t } a^T(\zeta)x \le b \\ &\forall \; \zeta \in Z \end{aligned} \tag{4}$$

Note the uncertain constraint in equation (4) and define the uncertain parameter $a(\zeta) = \bar{a} + P\zeta$ where $\bar{a} \in \mathbb{R}^n$ is a nominal value vector and $P \in \mathbb{R}^{nxL}$ is a confounding matrix. The set U is defined as in equation (5).

$$U = \{a \mid a = \bar{a} + P\zeta, \; \zeta \in Z\} \tag{5}$$

where: $\mathcal{Z} \subset \mathbb{R}^L$ is an uncertain set of primitive factors, so equation (6) is obtained.

$$(\bar{a} + P\zeta)^T x \le b, \; \forall \zeta \in Z \tag{6}$$

The optimal solution from Robust Counterpart is called optimal robust. Furthermore, to reformulate the set of uncertainty U into a computationally tractable problem, the following theorem applies.

Theorem 1:

Referring to [2] and [4], assume the set of uncertainty U is an affine image of the limited set $\mathcal{Z} = \{\zeta\} \subset \mathbb{R}^n$, and $\mathcal{U}$ is:

1. The system of linear inequality constraints

$$P\zeta \le p \tag{7}$$

2. The system of Conic Quadratic inequality

$$\|P_i\zeta - p_i\|_2 \le p_i^T\zeta - r_i, \; i = 1, \ldots, M \tag{8}$$

3. Systems of Linear Matrix Inequality

$$p_0 + \sum_{i=1}^{\dim \zeta} \zeta_i P_i \ge 0 \tag{9}$$

In cases (2) and (3) it is also assumed that the system of the constraints defining U is strictly feasible. Then, the Robust Counterpart of equation (3) is equivalent to:

1) Linear programming problems in the case (1)
2) Conic Quadratic problems in cases (2)
3) Semidefinite problems in cases (3)

Computationally tractable robust counterpart representations of uncertain linear programming for different sets of uncertainties can be seen in Table 1.

Table 1. Tractable formulations for constraints with sets of uncertainties.

Uncertainty Set	$\mathcal{Z}$	Robust Counterpart	Tractability
Box	$\|\zeta\|_\infty \leq 1$	$a^T x + \|P^T x\|_1 \leq b$	LP
Ellipsoidal	$\|\zeta\|_2 \leq 1$	$a^T x + \|P^T x\|_2 \leq b$	CQP
Polyhedral	$D\zeta + q \geq 0$	$\begin{cases} a^T x + q^T w \leq b \\ D^T w = -P^T x \\ w \geq 0 \end{cases}$	LP

2. 3. Adjustable Robust Counterpart Optimization

Referring to [3], on Multistage Optimization, the basic paradigm of Robust Optimization, namely the "here and now" decision, can be relaxed. Some decision variables can be adjusted at a later time according to decision rules, which are a function of (some or all parts of) uncertain data. Adjustable Robust Counterpart (ARC) is given as in equation (10).

$$\min_{x,y(.)} \left\{ c^T x : A(\zeta)x + By(\zeta) \leq b, \forall \zeta \in Z \right\} \tag{10}$$

where: $x \in \mathbb{R}^n$ is the first stage decision "here and now" made before $\varsigma \in \mathbb{R}^L$ is realized, $y \in \mathbb{R}^k$ denotes a "wait and see" decision and $B \in \mathbb{R}^{mxk}$ which shows a certain matrix coefficient.

In practice, $y(\varsigma)$ is often through an approach with affine or linear decision rules $y(\zeta) = y^0 + Q\zeta$ with $y^0 \in \mathbb{R}^k$ and $Q \in \mathbb{R}^{kxL}$ is the coefficient in the decision rule, which is to be optimized.

Thus, the reformulation of equation (10) is as follows:

$$\min_{x,y^0,Q} \left\{ c^T x : A(\zeta)x + By^0 + BQ\zeta \leq b \ , \ \ \forall \zeta \in Z \right\} \tag{11}$$

3. RESULT AND DISCUSSION

This paper discusses about the formulation of the Adjustable Robust Counterpart Optimization (ARC) model for the maximum flow problem, the formulation of the Adjustable Robust Counterpart Optimization model for the maximum flow problem with the set of box uncertainty, and the results of numerical simulations in case studies.

3. 1. ARC Optimization Model Formulation for Maximum Flow Problems

Previously, review the model of the maximum flow problem as in equation (2). The first thing to do is determine the parameters of uncertainty. In the case of maximum flow, the flow capacity of each side in the network is an uncertain factor. Therefore, the uncertainty parameter in this maximum flow model is the side capacity of the network or k_{ij} so that it can be assumed that the $k_{ij} \in U$. Furthermore, k_{ij} uncertainty parameter can be written into the following equation:

$$k_{ij} = \bar{k}_{ij} + P_{ij}\zeta, \ \forall \zeta \in \mathrm{Z} \tag{12}$$

where: $\bar{k}_{ij} \in \mathbb{R}^n$ is a nominal value vector of side capacity, $P_{ij} \in \mathbb{R}^{nxL}$ is a confounding matrix, and $\varsigma \in \mathbb{R}^L$ is a primitive uncertainty vector. k_{ij} uncertainty parameters are found in the constraints of the maximum flow model. k_{ij} parameter is a vector of the right hand side of the constraint, so the assumption that the right hand side of the constraint must be of certain valuable can be met by adding an extra variable $\omega_{ij} \ = \ 1$ so that the k_{ij} parameter becomes the coefficient of the ω_{ij} variable as in equation (13) follows:

$$x_{ij} - k_{ij}\omega_{ij} \le 0, \ \forall i, j \ ; \ \omega_{ij} = 1 \tag{13}$$

Substitute equation (12) to equation (13) to get the model for the maximum flow problem with uncertainty in the following parameters:

$$\begin{aligned} & \max x_{ts} \\ & \text{s.t } Ax = 0 \\ & \quad x_{ij} - (\bar{k}_{ij} + P_{ij}\zeta)\omega_{ij} \le 0, \ \forall i, j \\ & \quad \omega_{ij} = 1 \\ & \quad x_{ij} \ge 0, \ \forall i, j \end{aligned} \tag{14}$$

The next step is to determine the adjustable decision variable from the maximum flow model in the form of the number of flows from node t to node s (x_{ts}). The x_{ts} variable can adjust to $x_{ts}(\varsigma)$ decision rules that depend on ζ and can be defined as follows:

$$x_{ts}(\zeta) = \bar{x}_{ts} + Q\zeta \tag{15}$$

where: $\bar{x}_{ts} \in \mathbb{R}^n$ is the nominal vector of the amount of flow from node t to node s, $Q \in \mathbb{R}^{nxL}$ is a confounding matrix, and $\varsigma \in \mathbb{R}^L$ is a primitive uncertainty factor. Note that the adjustable variable x_{ts} is found in the objective function of the maximum flow model, so the assumption that the objective function must be of value can be fulfilled by replacing the objective function with a single variable function t and there are uncertainties in a constraint function where $t \leq x_{ts}$ and $t \in \mathbb{R}^n$. Substitute equation (15) to equation (14) to obtain an Adjustable Robust Counterpart model for the maximum flow problem as in equation (16) follows:

$$\begin{aligned}
&\max t \\
&\text{s.t } \overline{x}_{ts} + Q\zeta - t \geq 0 \\
&\quad Ax = 0 \\
&\quad x_{ij} - (\overline{k}_{ij} + P_{ij}\zeta)\omega_{ij} \leq 0, \ \forall i, j \\
&\quad \omega_{ij} = 1 \\
&\quad x_{ij} \geq 0, \ \forall i, j \\
&\quad \zeta \in \mathcal{Z}
\end{aligned} \tag{16}$$

3. 2. ARC Optimization Model for Maximum Flow Problems with Box Uncertainty Sets

Assume that the uncertain parameters and decision variables in the Adjustable Robust Counterpart model for the maximum flow problem are in the set of box uncertainty. Define the set of box uncertainty as follows:

$$Z = \{\zeta : \|\zeta\|_\infty \leq 1\} \tag{17}$$

Robust Counterpart formulation for constraints with the set of box uncertainty as in equation (18) follows:

$$\begin{aligned}
&(\overline{a} + P\zeta)^T x \leq b, \ \forall \zeta : \|\zeta\|_\infty \leq 1 \\
&\equiv \overline{a}^T x + \max_{\zeta:\|\zeta\|_\infty \leq 1} (P^T x)^T \zeta \leq b \\
&= \overline{a}^T x + \|P^T x\|_1
\end{aligned} \tag{18}$$

Assuming the uncertainty is in the set of box uncertainty, the third obstacle in equation (16) is equivalent to the following equation:

$$\begin{aligned}
&x_{ij} - (\overline{k}_{ij} + P_{ij}\zeta)\omega_{ij} \leq 0, \ \forall i, j \\
&\equiv x_{ij} - \overline{k}_{ij}\omega_{ij} - P_{ij}\omega_{ij}\zeta \quad \forall i, j \\
&= x_{ij} - \overline{k}_{ij}\omega_{ij} - \max_{\zeta:\|\zeta\|_\infty \leq 1} (P_{ij}\omega_{ij}\zeta) \quad \forall i, j
\end{aligned} \tag{19}$$

Referring to the norm definition, norm-ℓ_∞ is the search for the maximum value of each absolute value of the entry, then the following equation (20) is obtained:

$$\begin{aligned} & x_{ij} - \bar{k}_{ij}\omega_{ij} - \max_{\zeta:\|\zeta\|_\infty \le 1} (P_{ij}\omega_{ij}\zeta), \quad \forall i, j \\ & = x_{ij} - \bar{k}_{ij}\omega_{ij} - \max_{\zeta:|\zeta_i|\le 1} \sum_i \sum_j (P_{ij}\omega_{ij}\zeta), \quad \forall i, j \\ & = x_{ij} - \bar{k}_{ij}\omega_{ij} - \sum_i \sum_j |P_{ij}\omega_{ij}|, \quad \forall i, j \\ & = x_{ij} - \bar{k}_{ij}\omega_{ij} - \|P_{ij}\omega_{ij}\|_1 , \qquad \forall i, j \end{aligned} \tag{20}$$

In the same way, the first obstacle to equation (16) with the set of uncertainty boxes is equivalent to equation (21) follows:

$$\begin{aligned} & \bar{x}_{ts} + Q\zeta - t \ge 0 \\ & \equiv \bar{x}_{ts} + \left(\max_{\zeta:\|\zeta\|_\infty \le 1} Q\zeta \right) - t \\ & = \bar{x}_{ts} + \left(\max_{\zeta:|\zeta_i|\le 1} \sum_i Q_i\zeta_i \right) - t \\ & = \bar{x}_{ts} + \sum_i |Q_i| - t, \ \forall i \\ & = \bar{x}_{ts} + \|Q\|_1 - t \end{aligned} \tag{21}$$

Next, substituting equations (20) and (21) into the Optimization model (16), we obtain the Adjustable Robust Counterpart Optimization model with the set of box uncertainty for the maximum flow problem as in equation (22) below:

$$\begin{aligned} & \max t \\ & \text{s.t } \bar{x}_{ts} + \sum_i |Q_i| - t \ge 0, \ \forall i \\ & \quad Ax = 0 \\ & \quad x_{ij} - \bar{k}_{ij}\omega_{ij} - \sum_i \sum_j |P_{ij}\omega_{ij}| \le 0, \ \forall i, j \\ & \quad \omega_{ij} = 1 \\ & \quad x_{ij} \ge 0, \ \forall i, j \end{aligned} \tag{22}$$

3. 3. Numerical Simulation

The data used for the case study is secondary data that already exists referring to [5] regarding the network on the issue of Energy-Saving Generation Dispatch (ESGD). The ESGD problem used has the aim to minimize carbon gas emissions resulting from the use of coal fuel by minimizing the cost function and optimizing the electric current in the distribution system.

In this electric power distribution system, electricity will be sent from a power source (generator) through several components such as the transformer and capacitors to the consumer demand. Basically the electrical distribution system can be described as a network in which there are components that play a role in the delivery of electricity. Referring to [5] a power distribution system has been converted into a network. Here is a picture of the network that has been converted as in Figure 3. The main focus in this case study is to find the maximum electric current that can flow on the network such as Figure 3 and the maximum current flow with an uncertainty in the electric power capacity.

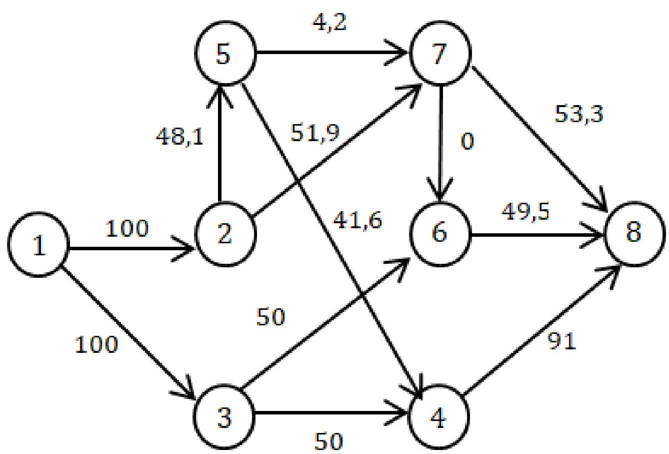

Figure 3. Network problems with ESGD

Next, the Optimization model for the maximum flow problem in the ESGD network case study as in equation (23) follows:

$$
\begin{aligned}
&\max x_{81} \\
&\text{s.t } x_{12} + x_{13} - x_{81} = 0 \\
&\quad -x_{12} + x_{25} + x_{27} = 0 \\
&\quad -x_{25} + x_{54} + x_{57} = 0 \\
&\quad -x_{13} + x_{36} + x_{34} = 0 \\
&\quad -x_{27} - x_{57} + x_{76} + x_{78} = 0 \\
&\quad -x_{34} - x_{54} + x_{48} = 0 \\
&\quad -x_{36} - x_{76} + x_{68} = 0 \\
&\quad -x_{78} - x_{68} - x_{48} + x_{81} = 0 \\
&\quad x_{ij} \le k_{ij}, \ \forall (i, j) \in E \\
&\quad x_{ij} \ge 0
\end{aligned}
\tag{23}
$$

Numerical simulations are carried out using Maple 18. The maximum amount of electric current that can flow on the network is 193.2 Amperes. The following is the amount of electric current on each side as in Table 2.

As for the Adjustable Robust Counterpart Optimization model with the set of box uncertainty for maximum flow problems on the ESGD network as follows:

$$\begin{aligned} &\max t \\ &\text{s.t } x_{81} + Q - t \geq 0 \\ &\quad x_{12} + x_{13} - x_{81} - Q = 0 \\ &\quad -x_{12} + x_{25} + x_{27} = 0 \\ &\quad -x_{25} + x_{54} + x_{57} = 0 \\ &\quad -x_{13} + x_{36} + x_{34} = 0 \\ &\quad -x_{27} - x_{57} + x_{76} + x_{78} = 0 \\ &\quad -x_{34} - x_{54} + x_{48} = 0 \\ &\quad -x_{36} - x_{76} + x_{68} = 0 \\ &\quad -x_{78} - x_{68} - x_{48} + x_{81} + Q = 0 \\ &\quad x_{ij} - k_{ij} - P_{ij} \leq 0, \ \forall (i, j) \in E \\ &\quad x_{ij} \geq 0 \end{aligned} \tag{24}$$

Table 2. The amount of electric current for each side of the network.

Variable	Amount of Electric Current (Ampere)
x_{12}	93,7
x_{13}	99,5
x_{25}	41,8
x_{27}	51,9
x_{34}	50
x_{36}	49,5
x_{48}	91
x_{54}	41
x_{57}	0,79

x_{68}	49,5
x_{76}	0
x_{78}	52,7
x_{81}	193,2

Let the value in the variable P is a random number obtained through Maple software. Based on equation (24) and the value of the variable P, using Maple 18 software the maximum amount of electric current that can flow on the network is 219,1948 Amperes. The following is the amount of electric current on each side with the set of uncertainty boxes as in Table 3 below:

Table 3. The amount of electric current for each side with a set of box uncertainty.

Variable	Amount of Electric Current (Ampere)
x_{12}	104,1759
x_{13}	115,0188
x_{25}	42,8248
x_{27}	61,3511
x_{34}	58,4124
x_{36}	56,6064
x_{48}	95,4278
x_{54}	37,0154
x_{57}	5,8093
x_{68}	56,6064
x_{76}	0
x_{78}	67,1605
x_{81}	0
Q	219,1948
t	219,1948

4. CONCLUSIONS

After researching and discussing the above description, it can be concluded that the model for maximum flow problems can be formulated into the Adjustable Robust Counterpart Optimization model for maximum flow problems with a set of box uncertainty. The Adjustable Robust Counterpart Optimization Model for the maximum flow problem is obtained as a computationally tractable Optimization model. The results of numerical simulations that have been carried out are optimal robust solutions for maximum flow problems. In numerical simulations it is also obtained that the robust results with the set of uncertainty boxes are resistant to interference with the uncertainty of parameters and adjustable variables.

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