On the nonlinear analysis of isotropic circular plate resting on viscoelastic foundation

Saheed Afolabi Salawu¹,*, Gbeminiyi Musibau Sobamowo²

¹Department of Civil and Environmental Engineering, University of Lagos, Akoka, Nigeria
²Department of Mechanical Engineering, University of Lagos, Akoka, Lagos, Nigeria
E-mail address: safolu@outlook.com

ABSTRACT

Nonlinear analysis of isotropic circular plate resting on viscoelastic foundation is investigated. The dynamic analogue of Von Kármán equations is considered in establishing the governing equation also, consideration is given to symmetric and axisymmetric mode. Thereafter, the coupled nonlinear partial differential equations are transformed to Duffing equation using Galerkin method and analysed using Optimal Homotopy Asymptotic Method (OHAM). Subsequently, the analytical solutions are used to investigate the influence of various parameters on the dynamic response of the plate. It is observed that, nonlinear frequency ratio increases with increase linear Winkler, Pasternak foundation and tensile force. Nevertheless, it is established that the nonlinear frequency ratio of the plate decreases as nonlinear Winkler foundation and compressive force increase. Also, the results revealed that both clamped and simply supported edge condition results in softening nonlinearity behaviour. Conversely, axisymmetric case of vibration gives lower nonlinear frequency ratio compared to asymmetric case. Furthermore, maximum deflection occurs when excitation force is zero, likewise the presence of viscoelastic foundation results in attenuation of deflection for the circular plate. It is expected that findings from the study will add values to the existing knowledge of classical vibration.

Keywords: nonlinear vibration, circular plate, Winkler and Pasternak, Optimal Homotopy Asymptotic Methods, Duffing equations
1. INTRODUCTION

Plates resting on viscoelastic foundation are used extensively in railway engineering, aerospace, mechanical engineering and telecommunication. There are numerous studies on vibration analysis of circular plates resting on foundation. The topic has attracted interest of so many researchers. In the study, consideration is given to the structural response of plates subject to excitation forces. Investigation of plates resting on viscoelastic foundation is paramount so as to understand the vibration control of the system and avoid breakdown of the system. Thin circular plate can exhibit huge flexural vibrations of the order of the plate thickness. In that case, a linear model cannot sufficiently predict the dynamic behaviour of the circular plate.

In this study, Von Kármán equations is used to include the geometric nonlinearity in the model. Vibration analysis can be used as a guide in the designing of resonance free system. In view of this, Chien and Chen investigated nonlinear frequency vibrations of plates placed on a nonlinear foundation [1–3]. The authors realized that, aside the vibration amplitude, foundation stiffness, initial stresses and modulus ratio influence the frequency responses of the plates. The governing differential equation was obtained using Galerkin method and solved numerically using Runge Kutta method. In another work, Ping et al. [4] analysed the dynamic behavior of circular plates resting on nonlinear foundation with Duffing equation.

Also, Younesian et al. [5] focused on investigation of resonance frequency sensitivity with respect to initial amplitude of the plate resting on nonlinear foundation. In the presence of large deformation, most engineering materials exhibit nonlinear behaviour. In relation to that, Jain and Nath [6] studied the hardening and softening nonlinear response of orthotropic shallow spherical shells under clamped edge condition. It was discovered that the maximum deformation decreases with increasing hardening nonlinear foundation. Foundations play a major role in preventing structural system under oscillation from total collapse. There are different foundation models, the simplest is the Winkler foundation [7].

The shortcoming of the model is inability in presenting the viscoelastic behaviour of the materials [8]. Therefore, viscoelastic foundation is adopted in this study. Two-parameter models are introduced to take care of the deficiency of Winkler model. The foundation elements are connected by a horizontal layer that acts as linkage to the vertical elements. Pasternak model has attracted the most attention of researchers since the introduction of two-parameter foundations model, there is a general assumption that it is the most generalized two-parameter foundations [9–12]. Wang and Stephens studied the shear layer effect of a beam resting on the Winkler and Pasternak foundation by comparing the natural frequencies and realized that the natural frequencies increase as a result of the shear layer [13]. Subsequently, they investigated the buckling and post buckling behaviour of orthotropic plates. In another work, Shen [14] studied the buckling and post buckling phenomenon on a Pasternak foundation. They submitted that post-buckling is strongly affected by the foundation parameters.

Analytical method presents the exact and direct solutions of the model. However, the application of the method to nonlinear models are often difficult thereby lose their functionality. In view of this, numerical methods are adopted [15–17]. The assumptions and constraints involved in numerical methods make them adaptable for finding solutions to nonlinear and complicated problems. Boundary element methods and Finite element (BEM and FEM) are mostly applied to problems with complicated boundary conditions and geometry. Several researchers had used FEM in solving vibrations of different structures resting on Winkler, nonlinear and two-parameter foundations [18–22]. However, numerical method is associated
with limitations of huge volume of calculations, computational cost, stability and convergence study. Approximate analytical methods have successfully provided merit of numerical method to certain extent. Some of the approximate analytical solutions applied to vibration problems include, Variational iteration method (VIM), Perturbation techniques, Homotopy analysis method (HAM), Adomian decomposition method (ADM) and Differential transformation method (DTM). Dynamic response of beam on elastic foundation was done using VIM by Ozturk [23]. In further works, VIM has been applied by many researchers for system resting on foundation. Younesian et al. [24] used the method to present solutions to free vibration of beam on nonlinear foundation. The method is also associated with problem of searching for Wronskian multiplier.

The search for small parameter associated with perturbation, coefficient ergodicity associated with HAM, rigour of Adomian polynomial determination in ADM and transformation of governing equation into recursive form and small domain limitation of DTM are issues overcome by optimal Homotopy asymptotic method (OHAM). Herisanu et al. [25] introduced the method while investigating nonlinear dynamic response of an electric machine under nonlinear vibration. It was realized that the method is more flexible than HAM based on the in-built convergence conditions [26]. [27-29] studied the solution of nonlinear heat problems with OHAM, compared the solutions to numerical and found good harmony of OHAM solutions. The method is simple, effective precise and easy to use. Therefore, it is justified to apply the method to nonlinear vibration problem

This study has been inspired by the fact that, scattered contributions on the topic are presented in the literature. To the best of authors’ knowledge, nonlinear analysis of axisymmetric and asymmetric isotropic circular plate resting on viscoelastic foundation has not been investigated. Therefore, this study focuses on nonlinear analysis of isotropic circular plate resting on viscoelastic foundation. The analytical solutions are used for parametric investigation.

2. PROBLEM FORMULATION AND MATHEMATICAL ANALYSIS

Considering a circular plate as shown in Fig. 1. This study investigates the circular plate under clamped and simple-support conditions on the assumption of Von- Kármán’s deflection theory [30, 31].

1) The plate is assumed to be thin, i.e., \( h/a \ll 1 \)
2) The Kirchhoff–Love hypotheses are assumed to be satisfied.
3) The in-plane and rotatory inertia terms are neglected.
4) Assuming there is Perfect bonding between the foundation and the plate.

The governing differential equations as reported by [32-34] are

\[
D \nabla^4 w^* (r, \theta, t) + \kappa \nabla^2 w^* (r, \theta, t) - \kappa_p w^{**} (r, \theta, t) + c \frac{\partial w^* (r, \theta, t)}{\partial t} + \rho h \frac{\partial^2 w^* (r, \theta, t)}{\partial t^2} = \frac{h}{r} \frac{\partial}{\partial r} \left( \frac{\partial F(\partial w^*)}{\partial \theta} \right) = q \cos (\omega t),
\]

(1)
\[ \nabla^4 F = -\left( \frac{E}{r} \right) \frac{\partial^2 w^*}{\partial r} \frac{\partial^2 w^*}{\partial r^2}, \]

(2)

where \( t \) is the time, \( r \) is the radial coordinate, \( \rho \) is the material density, \( h \) is the thickness of the plate, flexural rigidity \( D = \frac{Eh^3}{12(1-v^2)} \), Poisson’s ratio is \( v \), \( w \) is the transverse deflection, \( E \) is the Young’s modulus of the plate. The foundation of the circular plate is Visco-Pasternak medium. \( k_p \) is the nonlinear Winkler \( k_w \) is the Winkler foundation, \( k_s \) is the Pasternak foundation, \( q \) is the amplitude of excitation, \( \omega \) is the natural frequency, \( F \) is the Airy stress function and \( c \) is the damper modulus parameters, respectively.

**Fig. 1.** Showing linear varying thickness circular plate resting on four-parameter foundations.

The deflection of the plate is assumed to be

\[ W(R, \theta, t) = \sum_{m=0}^{\infty} W_m(R, t) \cos(m\theta), \]

where \( r \) and \( \theta \) are polar coordinates. \( m \) is the number of nodal diameter.

Using dimensionless parameters

\[
\begin{align*}
\text{w} &= \frac{w}{h}, \quad F = \frac{F}{Eh^3}, \quad R = \frac{r}{a}, \quad D = \frac{Eh^3}{12(1-v^2)}, \\
&= -\frac{1}{12(1-v^2)} \left( \frac{\partial^3 w}{\partial R^2} + \frac{2}{R} \frac{\partial^2 w}{\partial R^2} - \left( \frac{2m^2 + 1}{R^2} \right) \frac{\partial^2 w}{\partial R^2} + \left( \frac{m^2 + 1}{R^3} \right) \frac{\partial w}{\partial R} - \left( \frac{4m^2 - m^4}{R^4} \right) \right) - \frac{1}{R} \frac{\partial}{\partial R} \left( \frac{\partial F}{\partial R} \frac{\partial w}{\partial R} \right) \\
&= -\frac{1}{12(1-v^2)} \frac{\partial^2 w}{\partial \tau^2} - k_s \frac{\partial^2 w}{\partial R^2} + k_w \frac{\partial w}{\partial R} - k_p w + 3 k_q \frac{w^3}{4} - \frac{c}{12(1-v^2)} \frac{\partial w}{\partial \tau} + Q_0 \cos(\omega t),
\end{align*}
\]

(3)

\[
\frac{\partial^2 F}{\partial R^2} + \frac{1}{R} \frac{\partial F}{\partial R} \frac{1}{R^2} F = -\left( \frac{1}{R} \right) \frac{\partial F}{\partial R} \frac{\partial^2 w}{\partial R^2},
\]

(4)

\[ \tau = \left[ \sqrt{\frac{D}{\gamma ha^4}} \right] t, \quad k_w = \frac{k_w a^4}{Eh^3}, \quad k_p = \frac{k_p a^4}{Eh^3}, \quad k_s = \frac{k_s a^2}{Eh^3}, \quad Q_0 = \frac{qa^4}{Eh^3}, \quad C = \frac{ca^4}{Eh^3} \]
2. 1. Boundary Condition

Consideration is given to $R = 0$ and $R = 1$

- **Clamped edge condition:**

$$ W|_{R=1} = 0, \frac{\partial W}{\partial R}_{R=1} = 0, $$

$$ W|_{R=0} = 0, \frac{\partial W}{\partial R}_{R=0} = 0, $$

(5)

- **Simply Supported edge condition:**

$$ W|_{R=1} = 0, \left( \frac{\partial^2 W}{\partial R^2} + \frac{\nu \partial W}{R \partial R} \right)_{R=1} = 0, $$

$$ W|_{R=0} = 0, \left( \frac{\partial^2 W}{\partial R^2} + \frac{\nu \partial W}{R \partial R} \right)_{R=0} = 0, $$

(6)

- **The initial conditions:**

$$ W|_{t=0} = T_0; \frac{\partial W}{\partial \tau}_{t=0} = 0, $$

(7)

For the Airy stress function conditions, free edge and constrained immovable edge is considered. The boundary conditions in dimensionless form are

$$ R = 1:\ \frac{\partial F}{\partial R} = 0, \ \frac{\partial^2 F}{\partial R^2} - \frac{\nu \partial F}{R \partial R} = 0, $$

(8)

3. METHODS

3. 1. Description of Optimal Homotopy Asymptotic Method

Optimal Homotopy Asymptotic Method (OHAM) presented by Marinca and Herisanu [25], is an effective method of solving strongly nonlinear equations. OHAM uses the combine idea of traditional perturbation method and topology technique, but not limited to small parameters as related to classical perturbation methods. To illustrate the principle of its operation, assume the general nonlinear governing of the form.

$$ L(\phi (t) + g(t) + N(\phi (t))) = 0, \quad t \in \Omega, $$

with the following boundary conditions
where $L$ is the linear operator, $t$ is the independent variable, $\varphi(t)$ is the unknown function, $N$ is the nonlinear operator, $\Omega$ is the domain, $B$ is the boundary operator and $\partial \Omega$ represents the boundary of the domain.

According to OHAM [29]

$$(1 - p)[L(\phi(t, p)) + g(t)] = H(p)[L(\phi(t, p)) + g(t) + N(\phi(t, p))],$$

$$B\left(\phi(t, p), \frac{\partial \phi(t, p)}{\partial t}\right) = 0,$$

where $p \in [0,1]$ is an embedding operator, $H(p)$ is a nonzero auxiliary function for $p \neq 0$ and $\phi(t, p)$ is an unknown function, obviously, when $p = 0$ and $p = 1$, it holds $\phi(t, 0) = \varphi_0(t)$ and $\phi(t, 1) = \varphi(t)$. Therefore, $p$ varies from 0 to 1 while $\varphi_0(t)$ is obtained from Eq. (11) for $p = 0$

**Zero-order deformation**

$$L(\varphi_0(t) + g(t)) = 0, \quad B\left(\varphi_0, \frac{d \varphi_0}{dt}\right) = 0$$

The auxiliary function $H(p)$ in the form

$$H(p) = pC_1 + p^2C_2 + \cdots$$

where $C_1$ and $C_2$ are constants

By expanding the Taylor series of $\phi(t, p, C)$ about $p$, we obtain

$$\phi(t, p, C) = \varphi_0(t) + \sum_{k=1}^{\infty} \varphi_k(t, C_1, C_2, \ldots, C_k) p^k$$

Now substituting Eq. (15) in Eq. (11) and equating the equal powers of $p$, we obtain the following linear equations:

**The first and second order problems are as follows:**

$$L(\varphi_1(t) + g(t)) = C_1N_0(\varphi_0(t)), \quad B\left(\varphi_1, \frac{d \varphi_1}{dt}\right) = 0$$

$$L(\varphi_2(t) - L(\varphi_1(t)) = C_2N_0(\varphi_0(t)) + C_1[L(\varphi_1(t)) + N_1(\varphi_0(t)\varphi_1(t))], \quad B\left(\varphi_2, \frac{d \varphi_2}{dt}\right) = 0$$
The general governing equations for $\varphi_k(t)$ are given by

$$L(\varphi_k(t)) - L(\varphi_{k-1}(t)) = C_k N_0(\varphi_0(t)) + \sum_{j=1}^{k-1} C_j \left[ L(\varphi_{k-j}(t)) + N_{k-j}(\varphi_0(t), \varphi_1(t), \ldots, \varphi_{k-j}(t)) \right]$$

$$B\left(\varphi_k, \frac{d\varphi_k}{dt}\right) = 0,$$

where $k = 2, 3, \ldots$ and $N_m(\varphi_0(t), \varphi_1(t), \ldots, \varphi_{k-j}(t))$ is the coefficient of $p^m$ in the expansion of $N(\phi(t, p; C_i))$ about the embedding parameter $p$.

$$N(\phi(t, p; C_i)) = N_0(\varphi_0(t)) + \sum_{m=1}^{\infty} N_m(\varphi_0, \varphi_1, \varphi_2, \ldots, \varphi_m) p^m$$

It is observed that the convergence of Eq. (15) depends upon the auxiliary constants $C_1, C_2, \ldots$ at $p = 1$, one has

$$\tilde{\varphi}(t, C_1, C_2, \ldots, C_m) = \varphi_0(t) + \sum_{j=1}^{m} \varphi_j(t, C_1, C_2, \ldots, C_m)$$

Substitute Eq. (20) into Eq. (9) will give

$$R(t, C_1, C_2, \ldots, C_m) = L(\tilde{\varphi}(t, C_1, C_2, \ldots, C_m)) + g(t) + N(\tilde{\varphi}(t, C_1, C_2, \ldots, C_m))$$

If $R = 0$ then $\tilde{\varphi}$ will be the exact solution. Generally, it does not happen, especially in nonlinear problems. For the determinations of auxiliary constants $i = 1, 2, 3, \ldots, m$, we choose $a$ and $b$ in a manner which leads to the optimum values of $C_i, s$ for the convergent solution of the desired problem. There are many methods of residuals to find the optimal value of $C_i, i = 1, 2, 3, \ldots, m$. We apply the method of least squares

$$J(C_1, C_2, \ldots, C_m) = \int_a^b R^2(t, C_1, C_2, \ldots, C_m) dt$$

where $\text{R}$ is residual and

$$R = L(\tilde{\varphi}) + g(t) + N(\tilde{\varphi})$$

Also

$$\frac{\partial J}{\partial C_1} = \frac{\partial J}{\partial C_2} = \ldots = \frac{\partial J}{\partial C_m} = 0,$$
where \( a \) and \( b \) are properly chosen numbers to locate the desired \( C_i, i = 1, 2, 3, \ldots m \).

### 3.2. Transformation of coupled PDE to ODE

An approximate solution is obtained by assuming the nonlinear free vibrations to have the same spatial shape, i.e.,

\[
w(r, \tau) = (c_1 r^2 + c_2 r^2 + 1) \varphi(t) \Rightarrow w \varphi(t),
\]  

(25)

Substitute Eq. (25) into Eq. (4) and solve the ODE, we have

\[
F(r, t) = \frac{c_1}{r} - c_1 r + \frac{(\varphi(t))^2 r^3 \left(2 c_1^2 r^4 + 4 c_2 r^2 + 3 c_2^2\right)}{6},
\]

(26)

The value of \( F \) is accordingly found to be finite at the origin \( c_1 = 0 \). Additionally, \( c_3 \) is the constant of integration to be determined from in plane boundary conditions.

The Substitution of the expression \( w \) for \( w \) and \( F \) given by Eqs. (25) and (26) respectively and \( k_b = 5 \) and \( \nu = 0.3 \) into Eq. (3) with the application of the Galerkin procedure in the nonlinear time differential equation obtained in the form

\[
M \ddot{q}_1(t) + G \dot{q}_2(t) + K \ddot{q}_2(t) + V \dot{q}_3(t) = \ddot{Q} \cos \omega t,
\]

\[
\dot{q}_1(t) + \zeta \ddot{q}_1(t) + \omega^2 q_1(t) + \mu \ddot{q}_3(t) = Q \cos \omega t,
\]

where

\[
M = \frac{25 c_1^2}{3822} \left(\frac{5}{273} + \frac{25 c_2^2}{1638}\right) c_4 + \frac{25}{1638} + \frac{25 c_2^2}{1092} + \frac{5 c_2^2}{546},
\]

(29)

\[
K = \left(-\frac{25}{546} + \frac{k_\infty}{10} - \frac{c_2}{2} - \frac{k_\infty}{2}\right) c_4^2 + \left(\frac{25}{52} + \frac{k_\infty}{6} - 2 c_3 - 2 k_\infty\right) c_4 + \frac{25}{182} + \frac{k_\infty}{4} - \frac{2}{3 c_3} - \frac{2}{3 k_\infty}\right) c_2
\]

\[
+ \left(\frac{75}{182} - \frac{k_\infty}{4} - \frac{4}{3 c_3} - \frac{4}{3 k_\infty}\right) c_4^2 + \left(\frac{25}{39} + \frac{k_\infty}{5} - 2 c_3 - 2 k_\infty\right) c_4 + \frac{k_\infty}{6} - \frac{25}{182},
\]

(30)
\[ V = \left( \frac{2}{5} - \frac{3k_p}{56} \right) c_2^4 + \left( 2 - \frac{3}{16}k_p \right) c_4 + \frac{1}{2} - \frac{k_p}{4} \right) c_2^4 + \left( \frac{10}{3} - \frac{k_p}{4} \right) c_4 + \frac{2}{9} - \frac{k_p}{14} \right) c_4 + \frac{9k_p}{20} \right) c_2^2 \\
+ \left( \frac{5}{2} - \frac{3k_p}{20} \right) c_4 + \left( 20 - \frac{9k_p}{16} \right) c_2^4 \right) \right) c_4 + \left( \frac{20}{27} - \frac{3k_p}{88} \right) c_4 + \left( \frac{20}{21} - \frac{k_p}{6} \right) c_4^4 \right) \right) \frac{9c_2^4k_p}{28} + \frac{3}{10} c_4k_p - \frac{k_p}{8} \right), \]
\[ G = \frac{5Cc_2^2}{546} + \left( \frac{25C}{1092} + \frac{25Cc_4}{1638} \right) c_2 + \frac{25C}{1638} + \frac{5Cc_4}{273} + \frac{25Cc_4^2}{3822}, \]
\[ \bar{Q} = \left( \frac{-25Q_0}{1638} - \frac{5Q_0c_4}{546} - \frac{25Q_0c_2}{2184} \right) \cos(\omega t), \]
\[ z = G, \quad \omega_0^2 = \frac{K}{M}, \quad \mu = \frac{V}{M}, \quad Q = \frac{\bar{Q}}{M}. \]

The initial and boundary conditions are
\[ \varphi(0, r) = a, \quad \varphi(0, r) = 0, \quad (34) \]

For a plate with an elastically restrained outer edge, with rotational and in-plane stiffnesses \( k_b \) and \( k_i \), subjected to an applied in-plane radial force resultant \( N^* \) at the outer edge, the boundary conditions are:
\[ r = a: \quad M_r = k_b \frac{\partial w^*}{\partial r}, \quad N_r = N^* - k_i u^*, \quad (35) \]

where \( u^* \) is the radial displacement at mid-plane. Introduce dimensionless parameters \( k_b, k_i \) and \( N \)
\[ k_b = \frac{12k^*_a}{Eh}, \quad N = \left( \frac{N}{Eh} \right) \left( \frac{a}{h} \right)^2, \quad k_i = \frac{k^*_a}{Eh}, \quad (36) \]

The dimensionless boundary conditions are
\[ R = 1: \quad W = 0, \quad (37) \]
\[ \left[ (1 - \nu^2)k_b + \nu \right] \frac{\partial W}{\partial R} + \frac{\partial^2 W}{\partial R^2} = 0, \quad (38) \]
\[ k_i \left( \frac{\partial^2 F}{\partial R^2} - \nu \frac{\partial F}{\partial R} \right) + \frac{\partial F}{\partial R} = N, \quad (39) \]
Eq. (37) and (38) are used to find constants $c_2$ and $c_4$ while the constant of integration $c_3$ is obtained using Eq. (39)

\[
c_2 = \frac{2(k_b \nu^2 - k_b - \nu - 3)}{k_b \nu^2 - k_b - \nu - 5}, \quad c_4 = \frac{k_b \nu^2 - k_b - \nu - 1}{k_b \nu^2 - k_b - \nu - 5},
\]

\[
c_3 = -\frac{3(Nk_b^2 \nu^4 - 5k_b^2 \nu^5 + 10k_b^2 \nu^4 - 6Nk_b^2 \nu^2 - 6Nk_b v^3 + 10k_b^2 \nu^3 + 10k_b \nu^4 - 30Nk_b \nu^2 - 20k_b \nu^2)}{22k_b \nu^3 + 3Nk_b^2 \nu^3 + 3N \nu^2 - 5k_b \nu^2 - 94k_b \nu^2 - 5 \nu^3 + 30Nk_b \nu + 30N \nu + 10k_b \nu^2 - 22k_b \nu}
\]

\[
c_3 = -\frac{3(k_b \nu^2 - k_b - \nu - 5)^2 (-2 + \nu)}{3},
\]

While for Clamped edge condition are

\[
c_2 = -2, \quad c_4 = 1, \quad c_3 = \frac{3N + 10 - 5 \nu}{6 + 3 \nu},
\]

(41)

3.3. Application of the OHAM

Considering the following Duffing equation:

\[
\ddot{\phi}_s(t) + \zeta \dot{\phi}_s(t) + \omega_0^2 \phi_s(t) + \mu \dot{\phi}_s(t) = Q \cos \Omega t, \quad \phi_s(0) = a, \quad \dot{\phi}_s(0) = 0,
\]

(42)

We can construct a Homotopy in the following form

\[
(1 - p)\ddot{\phi}_s(t) + \omega_0^2 \phi_s(t) - H(p) \left( Q \cos \Omega t - \zeta \dot{\phi}_s(t) - \mu \dot{\phi}_s(t) \right) = 0, \quad p \in [0, 1].
\]

(43)

where $\zeta$ is the damping coefficient, $\mu$ is the cubic nonlinearity, $\Omega$ is excited frequency and $\omega_0^2 = K / M$ which is the natural frequency of the underlying linear system. In the case of primary resonance, the excited frequency $\Omega$ is assumed to be close to $\omega_0$. $H(p) = pC_1 + p^2C_2 + \cdots$ is non-zero auxiliary function for $p \neq 0$

The solution can be expanded in a series of

\[
\phi(t, p, C_i) = \phi_0(t, p) + \sum_{k=1}^{\infty} \phi_k(t, C_i) p^k, \quad i = 1, 2,
\]

(44)

Generally, one iteration cannot be done due to the complicated nonlinear equation. Therefore, an additional iteration method is needed. In that case, the perturbed natural frequency $\omega$ is useful. Using the parameter $p$ as the expanding parameter the following equation is obtained.
\[ \Omega^2 = \omega_0^2(1 + p\sigma_1 + p^2\sigma_2 + p^3\sigma_3, \ldots), \]  
(45)

where, \( \Omega \) is approximate nonlinear frequency \( \omega_0 \) is the linear natural frequency and \( \sigma \), are unknowns arriving from by removing the secularity conditions due to inhomogeneity in the equation. [35]

\[ \omega_0^2 = \lim_{p \to 0} \Omega^2 = \omega_0^2(1 + \sigma_1 + \sigma_2 + \sigma_3, \ldots), \]  
(46)

The expansion is like Lindstedt-Poincaré method, we introduce a new variable \( \tau \)

\[ \tau = \Omega t, \]  
(47)

where \( \omega \) is the natural frequency. Then Eq. (42) can be rewritten as

\[ \Omega^2 \phi(t) + \omega_0^2 \phi(t) = p \left( Q \cos \Omega t - \zeta \phi_1(t) - \mu \phi_3(t) \right), \]  
(48)

Then substituting Eq. (44) and Eq. (45) into (43), and equating the terms with the identical powers of \( p \), yields the following equations:

Zeroth Order of \( P^0 \):

\[ \frac{d^2 \phi_0}{d\tau^2} + \phi_0 = 0, \]  
(49)

\[ \phi(0) = A, \quad \phi(0) = 0, \]

First Order of \( P^1 \):

\[ \frac{d^2 \phi_1}{d\tau^2} + \phi_1 = -c_1 \left( \frac{d^2 \phi_0}{d\tau^2} + \phi_0 + \zeta \frac{d \phi_0}{d\tau} + \mu \frac{d^3 \phi_0}{d\tau^3} - \frac{Q}{\omega_0^2} \cos(\tau) \right) - \sigma_1 \frac{d^2 \phi_0}{d\tau^2} - \frac{d^2 \phi_0}{d\tau^2} + \phi_0, \]  
(50)

\[ \phi(0) = 0, \quad \phi(0) = 0, \]

Second Order of \( P^2 \):

\[ \frac{d^2 \phi_2}{d\tau^2} + \phi_2 = -c_2 \left( \frac{d^2 \phi_0}{d\tau^2} + \phi_0 + \zeta \frac{d \phi_0}{d\tau} + \mu \frac{d^3 \phi_0}{d\tau^3} - \frac{Q}{\omega_0^2} \cos(\tau) \right) - \sigma_1 \frac{d^2 \phi_1}{d\tau^2} - \sigma_2 \frac{d^2 \phi_0}{d\tau^2} + \phi_1 \]

\[ - c_1 \left( \frac{d^2 \phi_1}{d\tau^2} + \sigma_1 \frac{d^2 \phi_0}{d\tau^2} + \phi_1 + \zeta \frac{d \phi_0}{d\tau} + \sigma_1 \zeta \frac{d \phi_0}{d\tau} + 3 \mu \frac{d^3 \phi_0}{d\tau^3} \right) + \frac{d^2 \phi_0}{d\tau^2} + \sigma_1 \frac{d^2 \phi_0}{d\tau^2}, \]  
(51)

\[ \phi(0) = 0, \quad \phi(0) = 0, \]

Solving Eq. (49) along with the conditions, we have

\[ \phi_0 = A \cos(\tau), \]  
(52)
Substitute Eq. (52) into Eq. (50), we have

\[
\frac{d^2 \varphi}{d \tau^2} + \varphi = A c_i \left( \frac{\sigma_i - \frac{3A^2 \mu}{4 \omega_0^2} + \frac{Q}{A \omega_0^2}}{c_i} \right) \cos(\tau) + c_i \zeta A \sin(\tau) - \frac{A^3 c_i \mu}{4 \omega_0^3} \cos 3\tau, \quad (53)
\]

To eliminate the secular terms, the following should be satisfied

\[c_i \zeta A = 0,\]

\[A c_i \left( \frac{\sigma_i - \frac{3A^2 \mu}{4 \omega_0^2} + \frac{Q}{A \omega_0^2}}{c_i} \right) = 0, \quad (54)\]

Therefore

\[
\frac{\sigma_1}{c_1} = \frac{3A^2 \mu}{4 \omega_0^2} - \frac{Q}{A \omega_0^2}, \quad (55)
\]

Particular solution of Eq. (53) is

\[
\frac{d^2 \varphi_1}{d \tau^2} + \varphi_1 = -\frac{1}{4} \frac{c_i \mu A^3}{\omega_0^5} \cos 3\tau,
\]

\[
\varphi_1 = \frac{1}{32} \frac{c_i \mu A^3}{\omega_0^5} \cos 3\tau, \quad (56)
\]

Substituting \(\sigma_i, \varphi_0\) and \(\varphi_1\) into Eq. (52) one gets further iterations.

\[
\frac{d^2 \varphi_2}{d \tau^2} + \varphi_2 = A \left( \sigma_2 - \sigma_1 - \frac{3A^2 \mu}{4 \omega_0^2} + \frac{c_2 Q}{A \omega_0^2} + c_2 c_i \right) \cos(\tau) + A \left( c_2 \zeta + c_i \sigma_1 \zeta \right) \sin(\tau)
\]

\[
- \left( \frac{9\sigma_1 c_i \mu A^3}{32 \omega} - \frac{1}{4} \frac{c_2 \mu A^3}{\omega} - \frac{1}{4} \frac{c_2 \mu A^3}{\omega} + \frac{1}{4} \frac{c_i^2 \mu A^3}{\omega} + 3 \frac{\mu^2 A^5 \left( \cos (2\eta) + 1 \right) c_i}{64 \omega} \right) \right) \cos(3\tau)
\]

\[
+ \frac{3c_i^2 \zeta \mu A^3}{32 \omega} \sin(3\tau), \quad (57)
\]

To eliminate the secular terms, the following should be satisfied

\[A \left( c_2 \zeta + c_i \sigma_1 \zeta \right) = 0,\]

\[A \left( \sigma_2 - \sigma_1 - \frac{3A^2 \mu}{4 \omega_0^2} + \frac{c_2 Q}{A \omega_0^2} + c_2 c_i \right) = 0, \quad (58)\]
Therefore

\[ \sigma_z = \sigma_1 + \frac{3A^2}{4} \frac{c_2 \mu}{\omega_0^2} - \frac{c_3 Q}{A \omega_0^2} - c_1 \sigma_1, \]

(59)

Particular solution of Eq. (57) is

\[
\begin{align*}
\frac{d^2 \varphi_z}{dt^2} + \varphi_z &= - \left( \frac{9 \sigma_1 c_1 \mu A^3}{32 \omega} - \frac{3 \mu A^3 (\cos(2\eta) + 1)c_1}{64 \omega} \cos(3 \tau) + \frac{3c_1^2 \zeta A^3}{32 \omega} \sin(3 \tau) \right) \\
\varphi_z &= \frac{9c_1^2 \zeta A^3 \sin(\tau)}{256 \omega_0^2} - \frac{4}{3} \mu A^2 c_1 (\cos(\tau))^5 + \frac{32c_1^2}{3} - \frac{1}{3} \mu A^2 + 12\sigma_1 - \frac{32}{3} c_1 - \frac{32c_2}{3} (\cos(\tau))^3 + \\
&\quad + \left( -8c_1^2 + \frac{19}{12} \mu A^2 - 9\sigma_1 + 8 \right) c_1 + 8c_2 \right) \cos(\tau) + \sin(\tau) c_1 \left( A^2 \eta \mu - \zeta c_1 \right)
\end{align*}
\]

(60)

Accordingly, the first order approximate solution of Eq. (44) is

\[
\begin{align*}
\varphi &= \lim_{\rho \to 1} \left( \varphi_0 + \rho \varphi_1 \right) = \varphi_0 + \varphi_1 + \varphi_2, \\
\varphi &= \cos(\tau) + \frac{1}{32} \frac{c_1 \mu A^3}{\omega_0^2} - \frac{9c_1^2 \zeta A^3 \sin(\tau)}{256 \omega_0^2} - \frac{\cos(\tau) \left( \mu A^2 c_1 - 32c_1^2 - 36\sigma_1 c_1 + 32c_1 + 32c_2 \right) A^3 \mu}{1024 \omega_0^2} \\
&\quad + \left( -\frac{4}{3} \mu A^2 c_1 (\cos(\tau))^5 + \frac{32c_1^2}{3} - \frac{1}{3} \mu A^2 - 12\sigma_1 - \frac{32}{3} c_1 - \frac{32c_2}{3} (\cos(\tau))^3 + \\
&\quad + \left( -8c_1^2 + \frac{19}{12} \mu A^2 - 9\sigma_1 + 8 \right) c_1 + 8c_2 \right) \cos(\tau) + \sin(\tau) c_1 \left( A^2 \eta \mu - \zeta c_1 \right)
\end{align*}
\]

(61)

Also, Applying Eq. (46), we have the first order nonlinear frequency ratio

\[
\begin{align*}
\frac{\omega_N^2}{\omega_0^2} &= \lim_{\rho \to 1} \Omega^2 = (1 + \sigma_1 + \sigma_2 + \sigma_3 \cdots), \quad \text{also} \quad c_1 \to 1 \\
\frac{\omega_N}{\omega_0} &= \sqrt{1 + \left( \frac{3A^2}{4} \frac{\mu}{\omega_0^2} - \frac{Q}{A \omega_0^2} \right) + \zeta A},
\end{align*}
\]

(62)

-36-
Substituting Eq. (61) into Eq. (42), we get the residual

\[ R(\tau; C_1, C_2) = \frac{d^2 \varphi}{d\tau^2} + \omega_0^2 \varphi + \zeta \frac{d\varphi}{d\tau} + \mu \varphi^3 - \bar{Q} \cos(\tau), \]  

(63)

For \( C_1 \) and \( C_2 \) we minimize the function as follows

\[ J(C_1, C_2) = \int_0^1 (R(\tau, C_1, C_2))^2 d\tau \]  

(64)

The unknown constant \( C_1 \) and \( C_2 \) can be identified by using the following conditions

\[ \frac{\partial J(C_1)}{\partial C_1} = \frac{\partial J(C_2)}{\partial C_2} = 0, \]  

(65)

We obtained \( C_1 = \frac{990539693}{20000000000} \) and \( C_2 = \frac{-660503801}{500} \).

Thus, we obtained the first order deflection series as

\[ \varphi(\tau) = A \cos(\tau) + \frac{154771827 \mu A^1 \cos(3\tau)}{1.0 \times 10^{10} A_0^6} + \frac{862354641 \zeta \mu A^1 \sin(\tau)}{1.0 \times 10^{10} A_0^6} - \frac{6603597953 \mu A^2 \cos(\tau)}{1.0 \times 10^{10}} + \frac{704537363}{50} - \frac{51590609 \mu A^2}{3.12 \times 10^8} + \frac{2.20 \times 10^{10} \mu A^2}{1.0 \times 10^{10} A_0^6} + \frac{88812659}{5.0 \times 10^{10}} \bigg(\cos(\tau)^3 \bigg) \]  

\[ - \frac{3 \mu A^3}{256 A_0^6} + \frac{2452922209 \zeta \sin(\tau) \cos(\tau)^2}{2.5 \times 10^{10}} + \frac{264201511}{25} + \frac{7841772569 \mu A^3}{1.0 \times 10^{10}} - \frac{4.13 \times 10^{10} \mu A^3}{2.5 \times 10^9 A_0^6} + \frac{2168277}{4.8 \times 10^{10} A_0^6} + \frac{990539693 \sin(\tau)}{2.0 \times 10^{10}} \bigg( A^2 \eta \mu - \frac{990539693 \zeta}{2.0 \times 10^{10}} \bigg) \]  

(66)

4. RESULTS

Considering thin isotropic circular plate under simply supported and clamped edge boundary condition resting on viscoelastic foundation and made of thickness \( h = 0.003 m \), mass density \( \rho = 7850 kg/m^3 \), Young’s modulus \( E = 210 GPa \), Poisson’s ratio \( v = 0.3 \). Validation of the proposed analytical is presented in Table 1. A comparison study was done for isotropic circular
plate, the investigation revealed that the proposed results are in good harmony with values presented by [36], maximum of 0.51% discrepancy is observed.

Table 1. Frequency ratio for an isotropic circular plate under clamped edge boundary condition for different values of non-dimensional vibration amplitudes.

<table>
<thead>
<tr>
<th>W&lt;sub&gt;max&lt;/sub&gt;/h</th>
<th>Haterbouch and Benamar [37]</th>
<th>Present study</th>
<th>% Variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>1.0072</td>
<td>1.007</td>
<td>0.02</td>
</tr>
<tr>
<td>0.4</td>
<td>1.0284</td>
<td>1.0238</td>
<td>0.46</td>
</tr>
<tr>
<td>0.5</td>
<td>1.0439</td>
<td>1.0414</td>
<td>0.25</td>
</tr>
<tr>
<td>0.6</td>
<td>1.0623</td>
<td>1.0597</td>
<td>0.26</td>
</tr>
<tr>
<td>0.8</td>
<td>1.1073</td>
<td>1.1049</td>
<td>0.24</td>
</tr>
<tr>
<td>1</td>
<td>1.1615</td>
<td>1.1564</td>
<td>0.51</td>
</tr>
</tbody>
</table>

4.1. Effect of foundation parameters

The influence of Viscoelastic Winkler, Pasternak foundations are studied on nondimensional amplitude with nonlinear frequency ratio of the isotropic circular plate as presented in Fig. 2-4. The simulation is done varying the values of the foundation stiffness.

It is observed in Fig. 2 that; the amplitude frequency ratio decreases as nonlinear Winkler parameters increase in the two boundary conditions considered and for both asymmetric and axisymmetric cases are considered. This is a case of softening nonlinearity properties. This may be attributed to the fact that; the bending stiffness of the foundation decreases as the value of the nonlinear Winkler foundation increases and in consequent the deflection of the circular plate decreases.

Fig. 2. Influence of nonlinear foundation stiffness variation on amplitude of vibration for axisymmetric case

Fig. 3. Influence of Pasternak foundation variation on amplitude of vibration for axisymmetric case
Fig. 4. Influence of Winkler foundation stiffness variation on amplitude of vibration for axisymmetric case

Figs. 3 and 4 illustrate the nonlinear frequency against the amplitude curve of the isotropic circular plate for viscoelastic foundation. From the results, it is observed that, nonlinear frequency of the circular plate increases with the increase in elastic linear foundation parameters. This is because, increasing values of the linear foundation parameters leads to increase in the circular plate shear stiffness thereby resulting to vibrate at higher frequency. Pronounced effect on the nonlinear frequency amplitude curve of the circular plate is observed with Pasternak foundation. The parameters can be used to control the nonlinearity of structures.

4.2 Effect of radial force

In order to illustrate the effect of radial force on the dynamic behaviour of the circular plate resting on viscoelastic foundation, the analytical results obtained are presented in
graphical forms as shown Figs. 5 and 6. Fig. 5 shows the results obtained for tensile force while Fig. 6 depicts that of compressive force. It is clearly shown from the figures that; as the value of N, the tensile force increases, the nonlinear frequency ratio increases. However, the nonlinear frequency decreases as the compressive force increases. More pronounced effect is observed in compressive force. This is attributed to the fact variation of radial force N may affect the stiffness of the isotropic circular plate.

4.3. Effect of viscoelastic foundation

The influence of the viscoelastic foundation is presented in Fig. 7 and 8. As the viscoelastic parameter increases, the deflection on the plate decreases. Attenuation of the deflection of the circular plate observed means that damages in the system as a result of vibration can be reduced with the presence of viscoelastic foundation.

Fig. 7. Influence of viscoelastic foundation on vibration of the circular plate for Simply supported boundary condition

Fig. 8. Influence of viscoelastic foundation on vibration of the circular plate for clamped edge boundary condition

Fig. 9. Comparison of midpoint deflection time history for the linear and nonlinear analysis

Fig. 10. Midpoint deflection time history for the nonlinear analysis of isotropic circular plate
Figs. 9 and 10 show the comparison of linear with nonlinear vibration of the isotropic circular plate, it is observed that the difference is more significant with an increase in value of maximum vibration of the structure while Fig. 8 shows the midpoint deflection time history of the isotropic circular plate.

4.4 Comparison of asymmetric and antisymmetric case

To illustrate the comparative study of asymmetric and axisymmetric cases considered, results obtained from the analysis are shown graphically in Figs. 11-13. From the results, axisymmetric case is shown to possess better result than the asymmetric case. The frequency ratio is lower for axisymmetric case compared to symmetric case due to higher stiffness possessed by axisymmetric case. This shows that spatial property of the circular plate has an impact on vibration of the circular plate.

Fig. 11. Influence of axisymmetric and asymmetric parameter m on the nonlinear amplitude-frequency response curves of the isotropic circular plate

Fig. 12. Influence of asymmetric and symmetric parameter m on the deflection time history for nonlinear analysis of the isotropic circular plate simply supported edge

Fig. 13. Influence of asymmetric and symmetric parameter m on the deflection time history for nonlinear analysis of the isotropic circular plate clamped edge
4. 5. Primary resonance response

To further study the primary resonance of the circular plate, the partial differential governing equations are transformed to nonlinear ordinary equation using Galerkin method. Subsequently, the Duffing equation obtained is analyzed using OHAM and results presented in Fig. 14. The primary resonance response of the isotropic circular plate is shown in Fig. 14. It is apparent from the presented result that, vibration amplitude is lower than the thickness of the plate which validate the reliability of the results. The results further show that, the nonlinear frequency is a function of amplitude. Increasing the forcing term increase the amplitude of vibration

![Fig. 14. Influence of varying vibration amplitude on the nonlinear amplitude-frequency response curves of the isotropic circular plate](image)

4. 6. Effect of boundary condition

![Fig. 15. Influence of simply supported boundary conditions on the nonlinear amplitude-frequency response curves of the isotropic circular plate](image)

![Fig. 16. Influence of clamped edge boundary conditions on the nonlinear amplitude-frequency response curves of the isotropic circular plate](image)
Fig. 17 Midpoint deflection time history for the Clamped and simply supported condition of nonlinear analysis of isotropic circular plate.

Figs. 15-17 depict the effect of boundary conditions on the nonlinear amplitude frequency response curve of the isotropic circular plate. From the results, case of softening nonlinearity is observed. Moreover, Fig. 17 shows that clamped edge boundary condition attenuate the deflection more than the simply supported edge based on the fact that, clamped edge possessed higher stiffness than simply supported edge condition.

5. DISCUSSION

This research investigates the nonlinear analysis of isotropic circular plates resting on viscoelastic foundation. This article can be used by various researchers from many disciplines as technical source of understanding the potential vibration behaviour of system. So far, practical applications have been suggested for the research, deflection and frequency ratio obtained, but there is a lack of research on the investigation of intelligent materials on foundations. Meanwhile, future work is focused on this study.

6. CONCLUSIONS

In this study, nonlinear analysis of isotropic circular plates resting on viscoelastic foundation is investigated. The governing coupled partial differential equation is transformed into nonlinear ordinary differential equation using Galerkin method of separation. The ODE is solved analytically using OHAM technique. Asymmetric and Axisymmetric cases are considered. The results obtained are verified with results published in cited literature and good harmony is observed. The developed analytical solutions are used to investigate the effect of
elastic foundations, radial force, damper and varying amplitude on the dynamic behaviour of the isotropic circular plate. Based on the parametric studies, the following were observed

i. Nonlinear Winkler foundation parameter results into reduction in the frequency ratio. While, nonlinear frequency ratio increases with increase linear elastic foundation. Pasternak parameter has a pronounced effect on the nonlinear frequency.

ii. Nonlinear frequency ratio increases for tensile force where contrariwise is observed for compressive force. The effect of compressive force is more pronounced than that of tensile force.

iii. The deflection of the circular plate decreases as the viscoelastic foundation parameters increases.

iv. For primary resonance obtained, vibration amplitude is lower than the thickness of the plate and maximum amplitude occurs at \( Q = 0 \).

v. Axisymmetric case of vibration gives lower frequency ratio compared to symmetric case.

vi. Same softening nonlinearity is observed for both simply supported and clamped edge condition.

From the present investigation, it can be concluded that OHAM is found to be very reliable, simple and powerful when applied to nonlinear vibration problem. One of the distinct advantages of the method is overcoming the small domain limitation. It is hope that the present study will improve understanding in theory of vibration of circular plate.

Abbreviations

- r: Radius of the plate
- C: Clamped edge plate
- E: Young’s modulus
- F: Free edge support
- S: Simply supported edge
- \( \Omega \): natural frequency
- \( \frac{d}{dr} \): Differential operator
- f: Dynamic deflection
- h: plate thickness
- \( \rho \): Mass density
- D: Modulus of elasticity

References


