Available online at www.worldscientificnews.com

World Scientific News

An International Scientific Journal

WSN 140 (2020) 139-155 EISSN 2392-2192

Nonlinear Analysis of a Large-Amplitude Forced Harmonic Oscillation System using Differential Transformation Method-Padé Approximant Technique

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ABSTRACT

This work presents the nonlinear analysis of forced harmonic oscillation system using differential transformation method-Padé approximant techniques. Without any series expansion of the included sine and cosine of the angular displacement in the nonlinear model of the system, an improved analytical solution of the dynamic model is presented. The high level of accuracy and validity of the analytical solutions obtained by the differential transformation method are shown through comparison of the results of the solution with the corresponding numerical solutions obtained by fourth-fifth-order Runge-Kutta method, homotopy perturbation method and energy balance methods. Also, with the aid of the analytical solutions, parametric studies are carried to study the impacts of the model parameters on the dynamic behavior of the large-amplitude nonlinear oscillation system. The method avoids any numerical complexity and it is very simple, suitable and useful as a mathematical tool for dealing the nonlinear problems.

***Keywords*:** Large amplitude; Oscillation system; Strong nonlinearity; Forced vibration; Differential transformation method-Padé approximant techniques

(Received 28 December 2019; Accepted 18 January 2020; Date of Publication 19 January 2020)

1. INTRODUCTION

Oscillation systems have been widely applied in various systems such as mass on moving belt, simple pendulum, pendulum in a rotating plane, chaotic pendulum, elastic beams supported by two springs, vibration of a milling machine [1–3]. These wide areas of applications have made them to be extensively investigated. Also, many practical physics and engineering components consist of vibrating systems that can be modeled using the oscillator systems [4–6]. In order to simply describe the motion of the oscillation systems and provide physical insights into the dynamics, the nonlinear equation governing the pendulum motion has been linearized and solved [7-9].

The applications of such linearized model are found in the oscillatory motion of a simple harmonic motion, where the oscillation amplitude is small and the restoring force is proportional to the angular displacement and the period is constant. However, when the oscillation amplitude is large, the pendulum's oscillatory motion is nonlinear and its period is not constant, but a function of the maximum angular displacement. Consequently, much attentions have been devoted to develop various analytical and numerical solutions to the nonlinear vibration of large-amplitude oscillation system. In one of the works, Beléndez et al. [10] developed an analytical solution to conservative nonlinear oscillators using cubication method while Gimeno and Beléndez [11] adopted rational-harmonic balancing approach to the nonlinear pendulum-like differential equations.

Lima [12] submitted a trigonometric approximation for the tension in the string of a simple pendulum. Amore and Aranda [13] utilized improved Lindstedt–Poincaré method to provide approximate analytical solution of the nonlinear problems. Momeni et al. [14] applied He's energy balance method to Duffing harmonic oscillators. With the aid of the He's energy balance method, Ganji et al. [15] provided a periodic solution for a strongly nonlinear vibration system. The He's energy balance method as well as He's Variational Approach was used by Askari et al. [16] for the nonlinear oscillation systems. The numerical method was also adopted by Babazadeh et al. [17] for the analysis of strongly nonlinear oscillation systems. Biazar and Mohammadi [18] explored multistep differential transform method to describe the nonlinear bahaviour of oscillatory systems while Fan [19] and Zhang [20] applied He's frequency–amplitude formulation for the Duffing harmonic oscillator.

The above reviewed works are based on approximate solutions for free vibration of oscillation systems. In order to gain better physical insight into the nonlinear problems, Beléndez et al. [21] presented the exact solution for the nonlinear free vibration of pendulum using Jacobi elliptic functions. However, an exact analytic solution, as well as an exact expression for the period, both in terms of elliptic integrals and functions, is known only for the oscillatory regime ($0 < \theta_0 < \pi$). In this regime, the system oscillates between symmetric limits. Consequently, the quest for analytical and numerical solutions for the large-amplitude nonlinear oscillation systems continues. Therefore, with the aid of a modified variational iterative method, Herisanu and Marinca [22] described the behavior of the strongly nonlinear oscillators.

Kaya et al. [23] applied He's variational approach to analyze the dynamic behaviour of a multiple coupled nonlinear oscillators while Khan and Mirzabeigy [24] submitted an improved accuracy of He's energy balance method for the analysis of conservative nonlinear oscillator. A modified homotopy perturbation method was utlized by Khan et al. [25] to solve the nonlinear oscillators. In another work, a coupling of homotopy and the variational approach was used by Khan et al. [26] to provide a physical insight into the vibration of a conservative oscillator with strong odd-nonlinearity. Wu et al. [27] presented analytical approximation to a large amplitude oscillation of a nonlinear conservative system while homotopy perturbation method was used by He [28] to provide an approximation analytical solution for nonlinear oscillators with discontinuous.

Belato et al. [29] analyzed a non-ideal system, consisting of a pendulum whose support point is vibrated along a horizontal guide. Few years later, Cai et al. [30] used methods of multiple scales and Krylov-Bogoliubov-Mitropolsky to developed the approximate analytical solutions to a class of nonlinear problems with slowly varying parameters. In some years after, Eissa and Sayed [31] examined the vibration reduction of a three-degree-of-freedom spring-pendulum system, subjected to harmonic excitation. Meanwhile, Amore and Aranda [32] utilized Linear Delta Expansion to the Linstedt–Poincaré method and developed improved approximate solutions for nonlinear problems. The chaotic behavior and the dynamic control of oscillation systems has been a subject of major concern as presented in the literature. Therefore, with the aid of a shooting method, Idowu et al. [33] studied chaotic solutions of a parametrically undamped pendulum. Also, Amer and Bek [34] applied multiple scales method to study the chaotic response of a harmonically excited spring pendulum moving in a circular path. In the previous year, Anh et al. [35] examined the vibration reduction for a stable inverted pendulum with a passive mass-spring-pendulum-type dynamic vibration absorber. Ovseyevich [36] analyzed the stability of the upper equilibrium position of a pendulum in cases where the suspension point makes rapid random oscillation with small amplitude.

The above reviewed past works were focused on free vibration of the system which were studied numerically and analytically. However, the relatively new approximate analytical methods such as differential transformation method, Adomian decomposition method, homotopy analysis method, homotopy perturbation method, optimal homotopy asymptotic method, optimal homotopy perturbation method, variational iteration method, variation of parameter method, double decomposition method, Daftardar-Gejiji and Jafari method, Temini and Ansari method etc. have not been widely applied to analyze these problem. Although, these methods have been proven to be very efficient in handling nonlinear linear problems as they are not limited by the presence of small perturbation parameters as in the standard asymptotic and traditional perturbation methods such as regular perturbation methods, Poincare-Linstedt method, the method of multiple time scale, and the method of matched asymptotic expansions which are limited to the study of small amplitude responses to small disturbances.

Among the relatively new approximate analytical methods, differential transformation method (DTM) as introduced by Zhou [37] has proven to be more effective than most of the other approximate analytical methods as it does not require many computations. It solves nonlinear problems without linearization, discretization, restrictive assumptions, perturbation and discretization or round-off error. It is capable of greatly reducing the size of computational work while still accurately providing the series solution with fast convergence rate. Also, it reduces the complexity of expansion of derivatives and the computational difficulties of the other traditional and approximation analytical methods [38].

Ghafoori et al. [39] displayed the efficiency of the differential transformation method over homotopy perturbation method (HPM) and variation iteration method (VIM). In their work, it was shown that the results obtained by DTM are much more accurate in comparison with VIM and HPM; where for $t > 0$ the values achieved by HPM and VIM are not reliable. On the other hand, DTM solution presents very accurate results even for times much greater than zero [39]. The advantage of using DTM is the possibility for dealing with wide range of excitation frequency.

Therefore, in this work, differential transformation method-Padé approximant techniques is applied to analyze the nonlinear behaviour of forced harmonic oscillation system. Parametric studies were carried to study the impacts of the model parameters on the nonlinear dynamic behavior of the large-amplitude nonlinear oscillation system.

2. MODEL FORMULATION FOR THE LARGE AMPLITUDE NONLINEAR FORCED OSCILLATION SYSTEMS

Consider an oscillating system under a harmonic force as shown in Fig. 1. The system is displaced through a large amplitude. Using Newton's law for the rotational system, the differential equation for the large model forced vibration of the system is given in Eq. (1) as

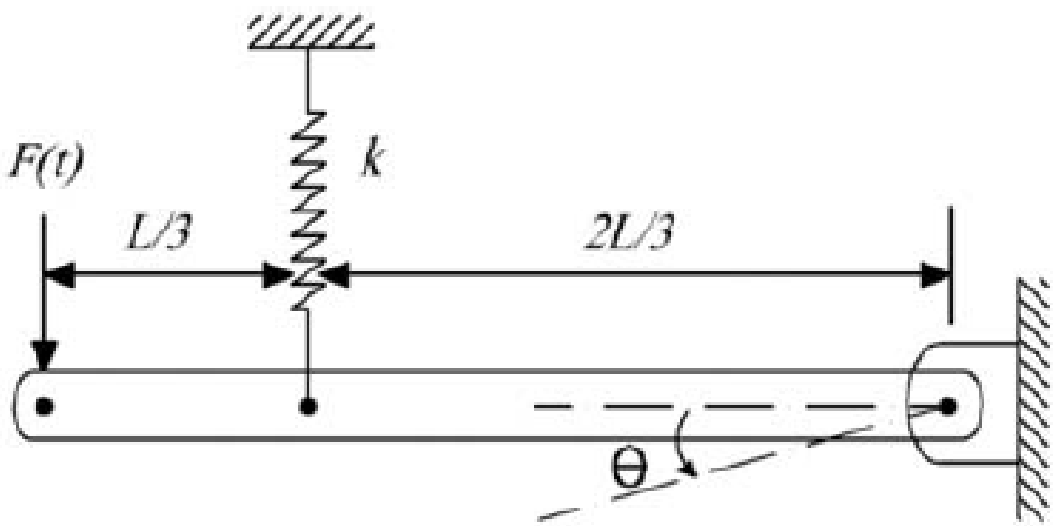

Fig. 1. The physical model of the oscillatory system

$$\frac{d^2\theta}{dt^2} + \frac{4k}{3m} sin\theta = \frac{3F_0}{ml} sin\omega_0 t$$
(1)

the initial conditions are

$$\theta(0) = \theta_0, \quad \dot{\theta}(0) = 0$$
(2)

θ_0 is the initial angular displacement or the amplitude of the oscillations.

3. METHOD OF SOLUTION: DIFFERENTIAL TRANSFORM METHOD

The nonlinear equation (1) is solved using differential transformation method as introduced by Zhou [37]. The basic definitions and the operational properties of the method are as follows:

If $u(t)$ is analytic in the domain T, then the function $u(t)$ will be differentiated continuously with respect to time *t*.

$$\frac{d^{p}u(t)}{dt^{p}} = \varphi(t, p) \quad \text{for} \quad \text{all } t \in T \tag{3}$$

for $t = t_i$, then $\varphi(t, p) = \varphi(t_i, p)$, where *p* belongs to the set of non-negative integers, denoted as the *p*-domain. We can therefore write Eq. (3) as

$$U(p) = \varphi(t_i, p) = \left[\frac{d^{p}u(t)}{dt^{p}}\right]_{t=t_i} \tag{4}$$

where U_p is called the spectrum of $u(t)$ at $t = t_i$

Expressing $u(t)$ in Taylor's series as:

$$u(t) = \sum_{p}^{\infty}\left[\frac{(t-t_i)^{p}}{p!}\right] U(p) \tag{5}$$

where Equ. (3) is the inverse of $U(k)$ us symbol 'D' denoting the differential transformation process and combining (4) and (5), we have

$$u(t) = \sum_{p=0}^{\infty}\left[\frac{(t-t_i)^{p}}{p!}\right] U(p) = D^{-1}U(p) \tag{6}$$

3. 1. Differential transformation method to the nonlinear simple pendulum

The application of differential transform method to the nonlinear simple pendulum is demonstrated in this section.

Eq. (1) can be written as:

$$\ddot{\theta} + \frac{4k}{3m} sin\theta = \frac{3F_0}{ml} sin\omega_0 t \tag{7}$$

In order to apply differential transformation method without series expansion of the included sine and cosine of the angular displacement in the nonlinear simple pendulum:

let $h = \sin\theta$ (8)

then

$$\frac{dh}{d\theta} = \cos\theta \text{, i.e.} \qquad \frac{dh}{d\theta} = \frac{dh}{dt}\frac{dt}{d\theta} = \cos\theta \tag{9}$$

which gives

$$\frac{dh}{dt} = \cos\theta \frac{d\theta}{dt} \tag{10}$$

Eq. (10) can be written as

$$\dot{h} = \dot{\theta}\cos\theta \tag{11}$$

From Eq. (11), we have

$$\cos\theta = \frac{\dot{h}}{\dot{\theta}} \tag{12}$$

The application of chain rule to Eq. (10) gives

$$\frac{d^2h}{dt^2} = \cos\theta \frac{d^2\theta}{dt^2} - \sin\theta\left(\frac{d\theta}{dt}\right)^2 \tag{13}$$

which can be written as

$$\ddot{h} = \ddot{\theta}cos\theta - h\left(\dot{\theta}\right)^2 \tag{14}$$

Substitution of Eq. (12) into Eq. (7) yields

$$\ddot{\theta} + \frac{4k}{3m}h = \frac{3F_0}{ml}sin\omega_0 t \tag{15}$$

and when Eq. (12) is substituted into Eq. (14), we have

$$\dot{\theta}\ddot{h} - \dot{h}\ddot{\theta} + \left(\dot{\theta}\right)^3 h = 0 \tag{16}$$

The obtained Eqs. (15) and (16) are the desired conserved forms of Eq. (7) that are to be solved analytically using differential transform method (DTM).

Using the DTM Table in our previous publication [38], the DTM recursive relations for the fully coupled Eqs, (15) and (16) are

$$(k+1)(k+2)\theta_{k+2}+\frac{4k}{3m}H_k=\frac{3F_0\omega_0^k sin\left(\frac{\pi k}{2}\right)}{mlk!} \tag{17}$$

$$\sum_{l=0}^{k}(l+1)(l+2)H_{l+2}(k-l+1)\theta_{k-l+1}-\sum_{l=0}^{k}(l+1)(l+2)\theta_{l+2}(k-l+1)H_{k-l+1}$$
$$+\sum_{n=0}^{k}\left(\sum_{m=0}^{n}\left(\sum_{l=0}^{m}(l+1)\theta_{l+1}(m-l+1)\theta_{m-l+1}(n-m+1)\theta_{n-m+1}H_{k-n}\right)\right)=0$$
(18)

subject to

$$\theta[0]=a,\ \theta[1]=c,\ H[0]=b,\ H[1]=d \tag{19}$$

Using Eq. (20) in Eqs. (17) and (18), the term by term solution becomes;

$$\theta_2=-\frac{\alpha b}{2}$$

$$\theta_3=-\frac{(\alpha d-\beta\omega)}{6}$$

$$\theta_4=\frac{\alpha b\left(c^3+\alpha d\right)}{24c}$$

$$\theta_5=-\frac{3\alpha^2b^2c^2-\alpha c^3d+\beta c\omega^3-\alpha^2d^2+\alpha\beta d\omega}{120c}$$

$$\theta_6=\frac{\alpha b\left(3\alpha^2b^2c^2-c^6-11\alpha c^3d+4\beta c^3\omega-\alpha^2d^2\right)}{720c^2}$$

$$\theta_7=\frac{\begin{pmatrix}15\alpha^2b^2c^5+33\alpha^3b^2c^2d-10\alpha^2b^2\beta c^2\omega-\alpha c^6d+\beta c^2\omega^5\\-11\alpha^2c^3d^2+10\alpha\beta c^3d\omega+\alpha\beta cd\omega^3-\alpha^3d^3+\alpha^2\beta d^2\omega\end{pmatrix}}{5040c^2}$$

$$\theta_8 = -\frac{\left(\alpha b \begin{pmatrix} 78\alpha^2 b^2 c^5 - c^9 + 33\alpha^3 b^2 c^2 d - 57\alpha c^6 d + 20\beta c^6 \omega + 6\beta c^4 \omega^3 \\ -102\alpha^2 c^3 d^2 + 90\alpha\beta c^3 d\omega - 10\beta^2 c^3 \omega^2 - \alpha^3 d^3 \end{pmatrix}\right)}{40320 c^3}$$

Applying the principle of DTM inversion, the desired solution becomes;

$\theta(t) = \sum_{\xi=0}^{N} \theta_\xi t^\xi$, which in expanded form becomes;

$$\theta(t) = \left(\begin{array}{l} a + ct - \dfrac{\alpha b t^2}{2} - \dfrac{(\alpha d - \beta\omega)t^3}{6} + \dfrac{\alpha b\left(c^3 + \alpha d\right)}{24c} t^4 - \dfrac{\left(3\alpha^2 b^2 c^2 - \alpha c^3 d + \beta c \omega^3 - \alpha^2 d^2 + \alpha\beta d\omega\right)}{120c} t^5 \\ + \dfrac{\alpha b\left(3\alpha^2 b^2 c^2 - c^6 - 11\alpha c^3 d + 4\beta c^3 \omega - \alpha^2 d^2\right)}{720c^2} t^6 + \dfrac{\begin{pmatrix} 15\alpha^2 b^2 c^5 + 33\alpha^3 b^2 c^2 d - 10\alpha^2 b^2 \beta c^2 \omega \\ -\alpha c^6 d + \beta c^2 \omega^5 - 11\alpha^2 c^3 d^2 + 10\alpha\beta c^3 d\omega \\ +\alpha\beta c d\omega^3 - \alpha^3 d^3 + \alpha^2 \beta d^2 \omega \end{pmatrix}}{5040c^2} t^7 - \\ \dfrac{\left(\alpha b \begin{pmatrix} 78\alpha^2 b^2 c^5 - c^9 + 33\alpha^3 b^2 c^2 d - 57\alpha c^6 d + 20\beta c^6 \omega + 6\beta c^4 \omega^3 \\ -102\alpha^2 c^3 d^2 + 90\alpha\beta c^3 d\omega - 10\beta^2 c^3 \omega^2 - \alpha^3 d^3 \end{pmatrix}\right) t^8}{40320c^3} \end{array}\right) + \ldots$$

where

$$\alpha = \frac{4k}{3m}, \qquad \beta = \frac{3F_0 \omega_0^k}{ml}$$

4. THE BASIC CONCEPT AND THE PROCEDURE OF PADÉ APPROXIMANT

The limitation of power series methods to a small domain have been overcome by after-treatment techniques. These techniques increase the radius of convergence and also, accelerate the rate of convergence of a given series. Among the so-called after treatment techniques, Padé approximant technique has been widely applied in developing accurate analytical solutions to nonlinear problems of large or unbounded domain problems of infinite boundary conditions [40]. The Padé-approximant technique manipulates a polynomial approximation into a rational function of polynomials. Such a manipulation gives more information about the mathematical behaviour of the solution. The basic procedures are as follows.

Suppose that a function $f(\eta)$ is represented by a power series:

$$f(\eta) = \sum_{i=0}^{\infty} c_i \eta^i \tag{22}$$

This expression is the fundamental point of any analysis using Padé approximant. The notation $c_i, i = 0,1,2,...$ is reserved for the given set of coefficient and $f(\eta)$ is the associated function. [*L/M*] Padé approximant is a rational function defined as:

$$f(\eta) = \frac{\sum_{i=0}^{L} a_i \eta^i}{\sum_{i=0}^{M} b_i \eta^i} = \frac{a_0 + a_1\eta + a_2\eta^2 + ... + a_L\eta^L}{b_0 + b_1\eta + b_2\eta^2 + ... + b_M\eta^M} \tag{23}$$

which has a Maclaurin expansion, agrees with equation (22) as far as possible. It is noticed that in Eq. (23), there are *L+1* numerator and *M+1* denominator coefficients. So there are *L+1* independent number of numerator coefficients, making *L+M+1* unknown coefficients in all.

This number suggest that normally (*L/M*) out of fit the power series Eq. (22) through the orders $1, \eta, \eta^2, ..., \eta^{L+M}$

In the notation of formal power series

$$\sum_{i=0}^{\infty} c_i \eta^i = \frac{a_0 + a_1\eta + a_2\eta^2 + ... + a_L\eta^L}{b_0 + b_1\eta + b_2\eta^2 + ... + b_M\eta^M} + \mathrm{O}(\eta^{L+M+1}) \tag{24}$$

which gives

$$(b_0 + b_1\eta + b_2\eta^2 + ... + b_M\eta^M)(c_0 + c_1\eta + c_2\eta^2 + ...) = a_0 + a_1\eta + a_2\eta^2 + ... + a_L\eta^L + \mathrm{O}(\eta^{L+M+1}) \tag{25}$$

Expanding the LHS and equating the coefficients of $\eta^{L+1}, \eta^{L+2}, ..., \eta^{L+M}$ we get

$$b_M c_{L-M+1} + b_{M-1} c_{L-M+2} + b_{M-2} c_{L-M+3} + ... + b_2 c_{L-1} + b_1 c_L + b_0 c_{L+1} = 0$$

$$b_M c_{L-M+2} + b_{M-1} c_{L-M+3} + b_{M-2} c_{L-M+4} + ... + b_2 c_L + b_1 c_{L+1} + b_0 c_{L+2} = 0$$

$$b_M c_{L-M+3} + b_{M-1} c_{L-M+4} + b_{M-2} c_{L-M+5} + ... + b_2 c_{L+1} + b_1 c_{L+2} + b_0 c_{L+3} = 0$$

$$\vdots$$

$$b_M c_L + b_{M-1} c_{L+1} + b_{M-2} c_{L+2} + ... + b_2 c_{L+M-2} + b_1 c_{L+M-1} + b_0 c_{L+M} = 0 \tag{26}$$

If $i < 0, c_i = 0$ for consistency. Since $b_0 = 1$, Eqn. (26) becomes a set of *M* linear equations for *M* unknown denominator coefficients

$$\begin{pmatrix} c_{L-M+1} & c_{L-M+2} & c_{L-M+3} & \cdots & c_L \\ c_{L-M+2} & c_{L-M+3} & c_{L-M+4} & \cdots & c_{L+1} \\ c_{L-M+3} & c_{L-M+4} & c_{L-M+4} & \cdots & c_{L+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_L & c_{L+1} & c_{L+2} & \cdots & c_{L+M-1} \end{pmatrix} \begin{pmatrix} b_M \\ b_{M-1} \\ b_{M-2} \\ \vdots \\ b_1 \end{pmatrix} = - \begin{pmatrix} c_{L+1} \\ c_{L+2} \\ c_{L+3} \\ \vdots \\ c_{L+M} \end{pmatrix} \tag{27}$$

From the above Eq. (27), b_i may be found. The numerator coefficient $a_0, a_1, a_2, ..., a_L$ follow immediately from Eq. (25) by equating the coefficient of $1, \eta, \eta^2, ..., \eta^{L+M}$ such that

$$\begin{aligned}
&a_0 = c_0 \\
&a_1 = c_1 + b_1 c_0 \\
&a_2 = c_2 + b_1 c_1 + b_2 c_0 \\
&a_3 = c_3 + b_1 c_2 + b_2 c_1 + b_3 c_0 \\
&a_4 = c_4 + b_1 c_3 + b_2 c_2 + b_3 c_1 + b_4 c_0 \\
&a_5 = c_5 + b_1 c_4 + b_2 c_3 + b_3 c_2 + b_4 c_1 + b_5 c_0 \\
&a_6 = c_6 + b_1 c_5 + b_2 c_4 + b_3 c_3 + b_4 c_2 + b_5 c_1 + b_6 c_0 \\
&\vdots \\
&a_L = c_L + \sum_{i=1}^{min[L/M]} b_i c_{L-i}
\end{aligned} \tag{28}$$

The Eq. (25) and Eq. (26) normally determine the Pade numerator and denominator and are called Padé equations. The [L/M] Padé approximant is constructed which agrees with the equation in the power series through the order η^{L+M}. To obtain a diagonal Padé approximant of order [L/M], the symbolic calculus software Maple is used.

It should be noted as mentioned previously that Δ and Ω in the solutions are unknown constants. In order to compute their values for extended large domains solutions, the power series as presented in Eq. (21) is converted to rational functions using [18/19] Pade approximation through the software Maple and then, the large domains are applied. The resulting simultaneous equations are solved to obtain the values of Δ and Ω for the respective values of the model parameters under considerations.

5. RESULTS AND DISCUSSION

The accuracy of the differential transformation method with Padé approximant is shown in Tables 1 & 2. The Tables depicted the high level of accuracy and agreements of the symbolic solutions of the modified DTM when compared to the numerical solutions of the fourth-fifth-order Runge-Kutta method. From the results in the Tables, it could be stated that the differential transformation method gives highly accurate results and avoids any numerical complexity.

Table 1. Comparison of results when $L = 0.5$, $m = 10$, $k = 1200$, $F_o = 1$, $\omega_o = 2$, $A = \pi/6$

Time	RK4	EBM[41]	HPM[41]	DTM- Padé
0.00	0.5236	0.5326	0.5326	0.5326
0.05	0.4265	0.4257	0.4264	0.4265
0.10	0.1695	0.1689	0.1691	0.1695
0.15	-0.1520	-0.1507	-0.1526	-0.1520
0.20	-0.4150	-0.4140	-0.4154	-0.4150
0.25	-0.5214	-0.5232	-0.5215	-0.5214
0.30	-0.4337	-0.4381	-0.4331	-0.4337
0.35	-0.1825	-0.1906	-0.1814	-0.1825
0.40	0.1394	0.1274	0.1406	0.1394
0.45	0.4089	0.3981	0.4099	0.4089
0.50	0.5259	0.5215	0.5260	0.5259

Table 2. Comparison of results when $L = 1.5$, $m = 5$, $k = 800$, $F_o = 3$, $\omega_o = 2$, $A = \pi/9$

Time	RK4	EBM[41]	HPM[41]	DTM- Padé
0.00	0.3491	0.3491	0.3491	0.3491
0.10	0.0427	0.0432	0.0426	0.0427
0.20	-0.3368	-0.3379	-0.3368	-0.3368
0.30	-0.1204	-0.1319	-0.1203	-0.1204
0.40	-0.3135	-0.3020	-0.3136	-0.3135
0.50	0.2037	0.2155	0.2032	0.2037
0.60	-0.2559	-0.2417	-0.2563	-0.2559
0.70	-0.2560	-0.2830	-0.2557	-0.2560
0.80	0.2041	0.1639	0.2045	0.2041
0.90	0.4089	0.3981	0.4099	0.4089
1.00	-0.1178	-0.0798	-0.1186	-0.1178

Also, the higher accuracy of the differential transformation method over homotopy perturbation method (HPM) and Energy balance method (EBM) is shown. It is shown that the results obtained by EBM are not reliable. It should be stated that the solutions of HPM and EBM obtained by Bayat et al. [41] is based on series expansion of sine and cosine of the angular displacement in the nonlinear model. The expansions were truncated after the fourth terms. Indisputably, such approximated series expansion limits the predictive power of the methods. Without any series expansion of the included sine and cosine of the angular displacement in the nonlinear model of the system, an improved solution with increased predictive power and high level of accuracy of DTM is presented in this work.

Naturally, when a pendulum displaces from its equilibrium position and releases, it oscillates freely with a natural frequency to and fro about its mean position and eventually, its motion dies out due to the opposing forces in the medium or the ever present damping forces in the surrounding. However, the pendulum can be forced to oscillate continuously when subjected to an external agency such as harmonic force. Such oscillations are known as forced or driven oscillations. The system oscillates with a frequency known as the driven frequency and not its natural frequency.

Fig. 2a and 2b shows the further verification of the results of the DTM- Padé approximant techniques and the effects of the initial angular displacement or the amplitude of the oscillations on the nonlinear vibration of the simple pendulum on the time response of the system. As expected, the results show that by increasing the initial angular displacement of the pendulum, the system frequency decreases as shown in Fig. 3. For high amplitudes, the periodic motion exhibited by a simple pendulum is practically harmonic but its oscillations are not isochronous i.e. the period is a function of the amplitude of oscillations.

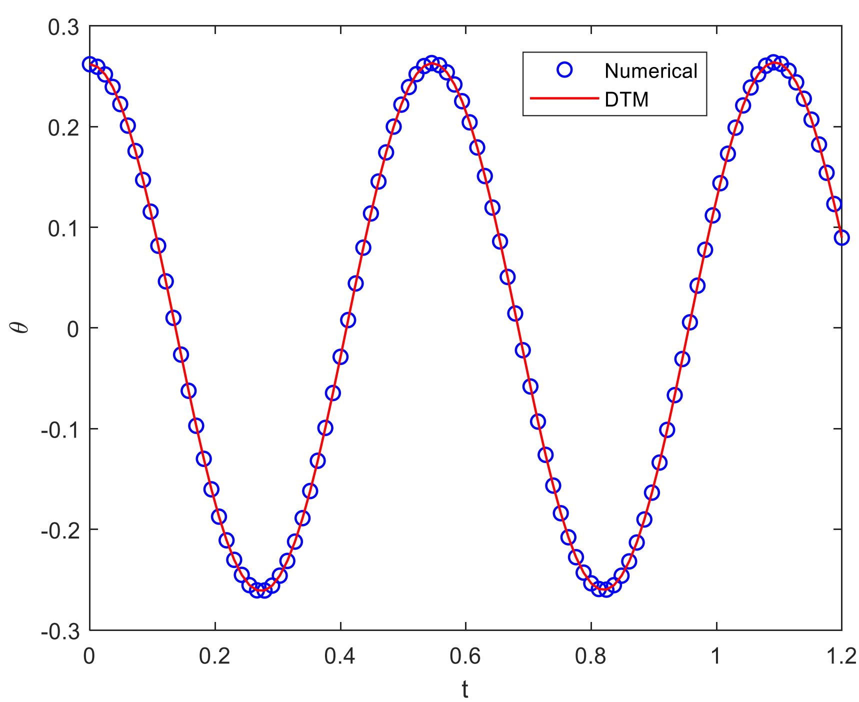

Fig. 2a. Time response of the system when $A = \pi / 12$

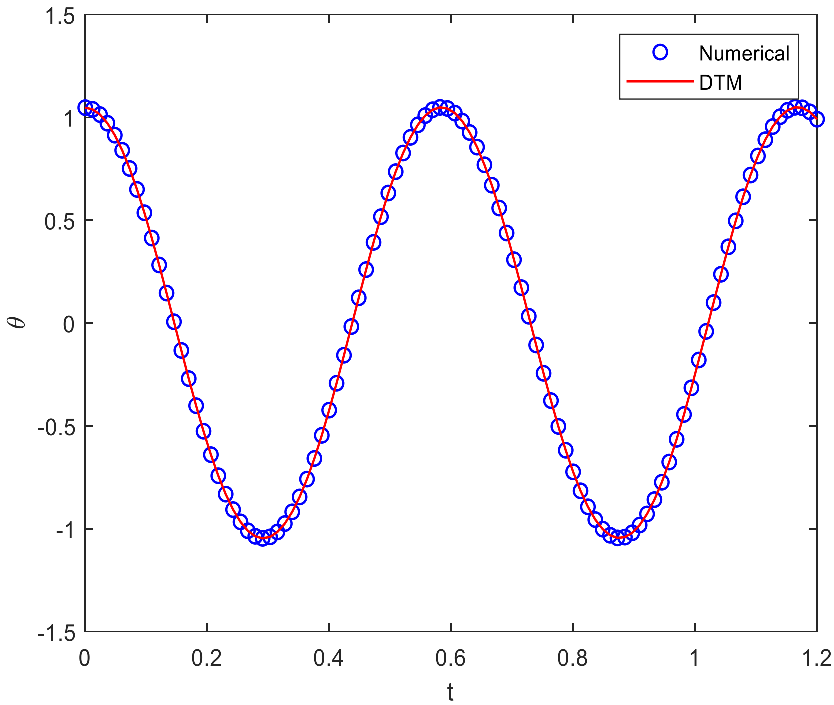

Fig. 2b. Time response of the system when $A = \pi / 3$

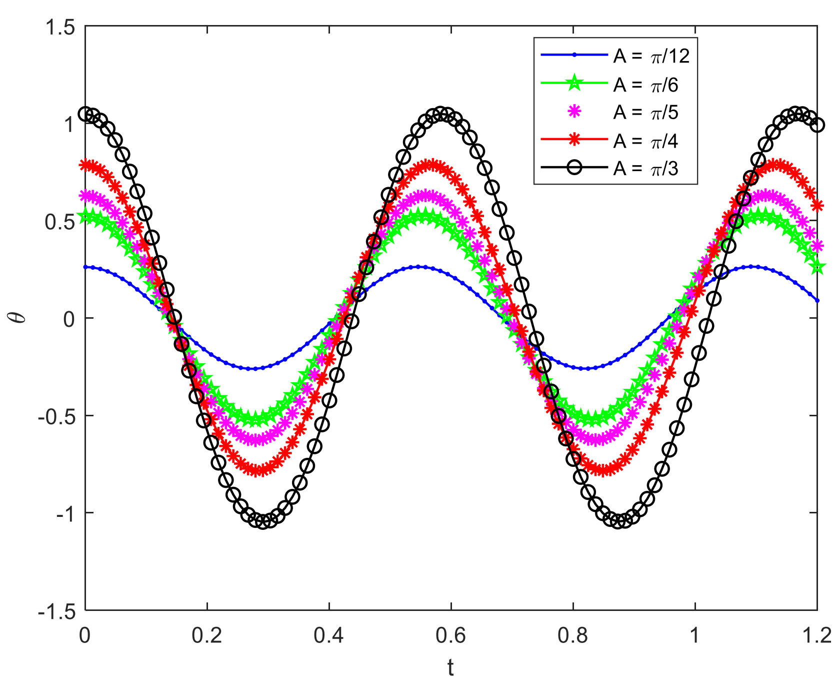

Fig. 3. Effect of amplitude on time responses of the system

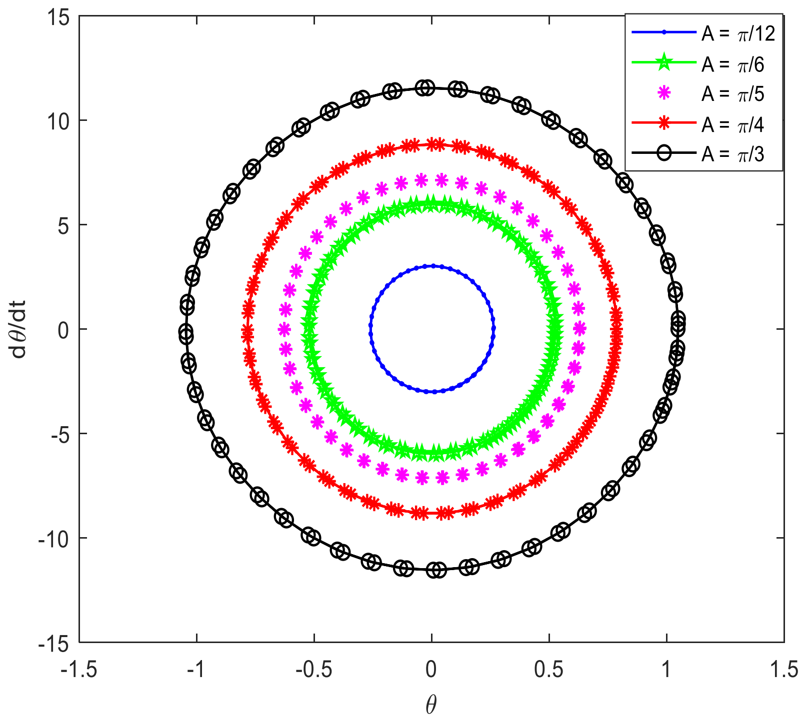

Fig. 4. Effect of amplitude on phase plots of the system

Figs. 4 show the effects of initial displacement or amplitude on the phase plots. The circular curves around (0,0) in figures show that the pendulum goes into a stable limit cycle. The effect of nonlinearity shows that as the resonance causes the amplitude of the motion to increase, the relation between period and amplitude (which is a characteristic effect of nonlinearity) causes the resonance to detune, decreasing its tendency to produce large motions. It is evident that when β approaches zero, the nonlinear frequency approaches the linear frequency of the simple harmonic motion.

6. CONCLUSIONS

In this work, the effectiveness and convenience of differential transformation method to the nonlinear analysis of forced harmonic oscillation system have been shown. Without any series expansion of the included sine and cosine of the angular displacement in the nonlinear model of the system, an improved analytical solution of the dynamic model has been presented. The high level of accuracy and validity of the analytical solutions obtained by the differential transformation method-Padé approximant technique are shown through comparison of the results of the solution with the corresponding numerical solutions obtained by fourth-fifth-order Runge-Kutta method, homotopy perturbation method and energy balance methods. Also, parametric studies were carried to study the effects of the model parameters on the dynamic behavior of the large-amplitude nonlinear oscillation system. The method is very efficient, simple, suitable and useful as a mathematical tool for dealing the nonlinear problems.

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