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The Drazin inverse of a class of partitioned matrices

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ABSTRACT

In this work, we give the representations of the Drazin inverse for the partitioned matrix $\begin{bmatrix} A_1 & A_2 \\ O & A_3 \end{bmatrix}$ with A_1 and A_3 square and singular under the conditions that $rank(A_1) = rank(A_3) = 1, Trace(A_1) \neq 0$ and $Trace(A_3) \neq 0$, and then we give the representations of the Drazin inverse for the partitioned EP matrix $\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$ with A_1 is square and non-singular under the conditions that $rank(A_1) = rank\left(\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\right)$, and $\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} A_1 [I \quad Q]$ where $P = A_3\, {A_1}^{-1}$ and $Q = {A_1}^{-1} A_2$. Also, we give the representations of the Drazin inverse for the partitioned matrix $\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$ with A_1 square and singular under the conditions that $rank(A_1) = rank\left(\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\right) = 1, Trace(\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}) \neq 0.$

***Keywords*:** Drazin inverse, Partitioned matrix, Rank, Trace of matrix

1. INTRODUCTION

Let H be a complex square matrix. The Drazin inverse [1, 2] of H is the unique matrix H^D satisfies the following conditions:

$$H^D H H^D = H^D, H H^D = H^D H, H^{k+1} H^D = H^k$$

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where $k = Ind(H)$ is called the index of H, it is the smallest non-negative integer such that $rank(H^k) = rank(H^{k+1})$. We know that H^D always exists and $H^D = H^{-1}$ for $Ind(H) = 0$. Properties of the Drazin inverse can be found in [1, 2].

We know that the Drazin inverse of H and the Moore-Penrose generalized inverse of H $(H^{\dagger})$ are the same if H and its conjugate transpose H^* have the same null space, (that means H is EP a matrix [1, 3]).

In 1979, Campbell and Meyer [2] proposed an open problem to find an explicit representation for the Drazin inverse of partitioned matrix $\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$ with A_1 and A_4 square in terms of A_1, A_2, A_3 and A_4. But because of the difficulty of this problem, until now it has not been solved yet even for the case $A_4 = 0$. There are many representations for the Drazin inverse of special 2×2 partitioned matrix but under special conditions, some of them can be found in [4, 5]. For example, in 2010, Bu and Zhang [4] gave explicit representation for the Drazin inverse of $\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$ under the conditions $A_1A_2A_3 = 0$ and either $A_4A_3 = 0$ or $A_2A_4 = 0$.

Then in 2011, Bu, Zhang and Zhao [5] gave representation for the Drazin inverse of $\begin{bmatrix} A_1 & A_2 \\ -I & O \end{bmatrix}$ with A_1 and A_2 square under the condition that $A_1A_2 = A_2A_1$ and the Drazin inverse of $\begin{bmatrix} A_1 & A_2 \\ A_3 & O \end{bmatrix}$ with A_1 square under the condition that $A_1A_2A_3 = A_2A_3A_1$, also they recalled some other representations in their introduction.

In this article, we give the representations of the Drazin inverse for $\begin{bmatrix} A_1 & A_2 \\ O & A_3 \end{bmatrix}$ with A_1 and A_3 square and singular under the conditions that $rank(A_1) = rank(A_3) = 1$, $Trace(A_1) \neq 0$ and $Trace(A_3) \neq 0$, and some corollaries, and then we give the representation of the Drazin inverse for $\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$ with A_1 is square and non-singular under the conditions that $rank(A_1) = rank\left(\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\right)$ and $\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} A_1 [I \quad Q]$ where $P = A_3\, {A_1}^{-1}$ and $Q = {A_1}^{-1}A_2$. Also, we give the representation of the Drazin inverse for the partitioned matrix $\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$ with A_1 square and singular under the conditions that $rank(A_1) = rank\left(\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\right) = 1$, $Trace(\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}) \neq 0$.

Throughout this paper, $\mathbb{C}^{n \times n}$ is the set of $n \times n$ complex matrices. The identity matrix of $\mathbb{C}^{n \times n}$ is denoted by I or I_n. The trace of a matrix H is denoted by $Tr(H)$. The conjugate transpose of H is denoted by H^*.

2. SOME LEMMAS AND THEOREMS

Lemma 2.1. [2] If A_1, A_2, A_3 and A_4 are matrices such that A_1 is square and non-singular and $rank(A_1) = rank\left(\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\right)$, then

$$A_4 = A_3\, {A_1}^{-1} A_2.$$

Furthermore, if $P = A_3 A_1^{-1}$ and $Q = A_1^{-1} A_2$ then

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} A_1 [I \quad Q].$$

Lemma 2.2. [6] If $A \in \mathbb{C}^{n\times n}$ is such that $rank(A) = 1$ and

$$A^D = \frac{1}{[Tr(A)]^2} A$$

where $Tr(A) \neq 0$, then $AA^D - K = I$, where

$$K = \begin{bmatrix} \frac{-(Tr(A)-a_{11})}{Tr(A)} & \frac{a_{12}}{Tr(A)} & \cdots & \frac{a_{1n}}{Tr(A)} \\ \frac{a_{21}}{Tr(A)} & \frac{-(Tr(A)-a_{22})}{Tr(A)} & \cdots & \frac{a_{2n}}{Tr(A)} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{a_{n1}}{Tr(A)} & \cdots & \cdots & \frac{-(Tr(A)-a_{nn})}{Tr(A)} \end{bmatrix},$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

Theorem 2.1. [2] If $A \in \mathbb{C}^{m\times n}$, then there exists $B \in \mathbb{C}^{m\times r}$, $C \in \mathbb{C}^{r\times n}$ such that $A = BC$ and $r = rank(A) = rank(B) = rank(C)$.

3. MAIN RESULS

Theorem 3.1. Let

$$H = \begin{bmatrix} A_1 & A_2 \\ O & A_3 \end{bmatrix} \in \mathbb{C}^{n\times n}$$

where

$$A_1 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \qquad A_3 = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

are square singular with $rank(A_1) = rank(A_3) = 1$, $Tr(A_1) \neq 0$ and $Tr(A_3) \neq 0$, $Ind(A_1) = l_1$ and $Ind(A_3) = l_2$, then

$$H^D = \begin{bmatrix} A_1^D & Y \\ O & A_3^D \end{bmatrix},$$

where

$$A_1^D = \frac{1}{[Tr(A_1)]^2} A_1,\ A_3^D = \frac{1}{[Tr(A_3)]^2} A_3\ ,$$

$$Y = (A_1^D)^2 \left[\sum_{i=0}^{n} [(A_1^D)^i A_2 A_3^i] \right] (-K_2) + (-K_1) \left[\sum_{i=0}^{n} [A_1^i A_2 (A_3^D)^i] \right] (A_3^D)^2 - A_1^D A_2\, A_3^D$$

$$= (A_1^D)^2 \left[\sum_{i=0}^{l_2-1} [(A_1^D)^i A_2 A_3^i] \right] (-K_2) + (-K_1) \left[\sum_{i=0}^{l_1-1} [A_1^i A_2 (A_3^D)^i] \right] (A_3^D)^2 - A_1^D A_2\, A_3^D \qquad (1)$$

where

$$K_1 = \begin{bmatrix} \frac{-(Tr\,(A_1) - a_{11})}{Tr(A_1)} & \frac{a_{12}}{Tr(A_1)} & \cdots & \frac{a_{1n}}{Tr(A_1)} \\ \frac{a_{21}}{Tr(A_1)} & \frac{-(Tr(A_1) - a_{22})}{Tr(A_1)} & \cdots & \frac{a_{2n}}{Tr(A_1)} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{a_{n1}}{Tr(A_1)} & \cdots & \cdots & \frac{-(Tr(A_1) - a_{nn})}{Tr(A_1)} \end{bmatrix}$$

and

$$K_2 = \begin{bmatrix} \frac{-(Tr\,(A_3) - b_{11})}{Tr(A_3)} & \frac{b_{12}}{Tr(A_3)} & \cdots & \frac{b_{1n}}{Tr(A_3)} \\ \frac{b_{21}}{Tr(A_3)} & \frac{-(Tr(A_3) - b_{22})}{Tr(A_3)} & \cdots & \frac{b_{2n}}{Tr(A_3)} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{b_{n1}}{Tr(A_3)} & \cdots & \cdots & \frac{-(Tr(A_3) - b_{nn})}{Tr(A_3)} \end{bmatrix}.$$

Proof. Expand the term $A_1 Y$ as follows.

$$A_1 Y = \left[\sum_{i=0}^{l_2-1} [(A_1^D)^{i+2} A_2 A_3^i] \right] (-K_2) + (-A_1 K_1) \left[\sum_{i=0}^{l_1-1} [A_1^i A_2 (A_3^D)^{i+2}] \right] - A_1 A_1^D A_2\, A_3^D$$

$$= -A_1^D A_2 K_2 + \left[\sum_{i=0}^{l_2-2} [(A_1^D)^{i+2} A_2 A_3^{i+1}] \right] (-K_2) - A_1 K_1 \left[\sum_{i=1}^{l_1-1} [A_1^{i-1} A_2 (A_3^D)^{i+1}] \right]$$

$$-A_1 k_1 A_1^{l_1-1} A_2\ (A_3^D)^{l_1+1} - A_1 A_1^D A_2\ A_3^D$$

$$= -A_1^D A_2 K_2 + \left[\sum_{i=0}^{l_2-2}\left[(A_1^D)^{i+2} A_2 A_3^{i+1}\right]\right](-K_2) - K_1\left[\sum_{i=1}^{l_1-1}\left[A_1^i A_2 (A_3^D)^{i+1}\right]\right] -$$

$$K_1 A_1^{l_1} A_2\ (A_3^D)^{l_1+1} - A_1 A_1^D A_2\ A_3^D.$$

Now expand the term YA_3 as follows

$$YA_3 = \left[\sum_{i=0}^{l_2-1}\left[(A_1^D)^{i+2} A_2 A_3^{i+1}\right]\right](-K_2) + (-K_1)\left[\sum_{i=0}^{l_1-1}\left[A_1^i A_2 (A_3^D)^{i+1}\right]\right] - A_1^D A_2\ A_3^D A_3$$

$$= \left[\sum_{i=0}^{l_2-2}\left[(A_1^D)^{i+2} A_2 A_3^{i+1}\right]\right](-K_2) + (A_1^D)^{l_2+1} A_2 A_3^{l_2}(-K_2) + (-K_1) A_2\ A_3^D +$$

$$(-K_1)\left[\sum_{i=1}^{l_1-1}\left[A_1^i A_2 (A_3^D)^{i+1}\right]\right] - A_1^D A_2\ A_3^D A_3.$$

Now, we can see that

$$A_1 Y - YA_3 = A_1^D A_2(-K_2) + (-K_1) A_1^{l_1} A_2\ (A_3^D)^{l_1+1} - A_1 A_1^D A_2\ A_3^D$$

$$- (A_1^D)^{l_2+1} A_2 A_3^{l_2}(-K_2) - (-K_1) A_2\ A_3^D + A_1^D A_2\ A_3^D A_3.$$

Using the fact that

$$-K_1 = I - A_1 A_1^D,\ \ -K_2 = I - A_3 A_3^D$$

we get

$$A_1 Y - YA_3 = A_1^D\ A_2 - \ A_2 A_3^D,$$

that means that

$$\begin{bmatrix} A_1 & A_2 \\ O & A_3 \end{bmatrix}\begin{bmatrix} A_1^D & Y \\ O & A_3^D \end{bmatrix} = \begin{bmatrix} A_1^D & Y \\ O & A_3^D \end{bmatrix}\begin{bmatrix} A_1 & A_2 \\ O & A_3 \end{bmatrix}.$$

So the second condition of Definition of the Drazin inverse is satisfied.
Now, we will show that the first condition holds. Note that

$$\begin{bmatrix} A_1^D & Y \\ O & A_3^D \end{bmatrix}\begin{bmatrix} A_1 & A_2 \\ O & A_3 \end{bmatrix}\begin{bmatrix} A_1^D & Y \\ O & A_3^D \end{bmatrix} = \begin{bmatrix} A_1^D & \vdots & A_1^D\ A_1 Y + Y\ A_3 A_3^D + A_1^D\ A_2 A_3^D \\ \cdots & \vdots & \cdots \\ O & \vdots & A_3^D \end{bmatrix}$$

to show that the first condition holds it is necessary to show that

$$A_1^D\, A_1 Y + Y\, A_3 A_3^D + A_1^D\, A_2 A_3^D = Y.$$

This is immediate from (1). Thus the first condition of definition of the is satisfied. Finally, we will show that third condition Drazin inverse satisfied. To do that, we will show that

$$\begin{bmatrix} A_1 & A_2 \\ O & A_3 \end{bmatrix}^{n+2} \begin{bmatrix} A_1^D & Y \\ O & A_3^D \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ O & A_3 \end{bmatrix}^{n+1}.$$

Note that, for any $k > 0$,

$$\begin{bmatrix} A_1 & A_2 \\ O & A_3 \end{bmatrix}^{k} = \begin{bmatrix} A_1^k & S(k) \\ O & A_3^k \end{bmatrix},$$

where

$$S(k) = \sum_{i=0}^{k-1} \left[A_1^{k-1-i} A_2 A_3^i\right].$$

Since $n + 2 > l_1$ and $n + 2 > l_2$, then

$$\begin{bmatrix} A_1 & A_2 \\ O & A_3 \end{bmatrix}^{n+2} \begin{bmatrix} A_1^D & Y \\ O & A_3^D \end{bmatrix} = \begin{bmatrix} A_1^{n+1} & A_1^{n+2} Y + S(n+2) A_3^D \\ O & A_3^{n+1} \end{bmatrix}.$$

Thus, it is only necessary to show that

$$A_1^{n+2} Y + S(n+2) A_3^D = S(n+1).$$

Since $l_1 + l_2 < n + 1$, it must be the case that

$$A_1^n (A_1^D)^i = A_1^{n-1} \text{ for } i = 1, 2, \dots, l_2 - 1.$$

Thus,

$$A_1^{n+2} Y = A_1^n \left[\sum_{i=0}^{l_2-1} \left[(A_1^D)^i A_2 A_3^i \right] \right] (-K_2) - A_1^{n+1} A_2 A_3^D$$

$$= \left[\sum_{i=0}^{l_2-1} \left[A_1^{n-i} A_2 A_3^i\right]\right](-K_2) - A_1^{n+1} A_2 A_3^D.$$

and

$$S(n+2) A_3^D = \sum_{i=0}^{n+1} \left[A_1^{n+1-i} A_2 A_3^i A_3^D\right]$$

$$= \sum_{i=0}^{l_2}\left[A_1^{n+1-i}A_2A_3^iA_3^D\right] + \sum_{i=l_2+1}^{n+1}\left[A_1^{n+1-i}A_2A_3^{i-1}\right].$$

Note that

$$\sum_{i=0}^{l_2}\left[A_1^{n+1-i}A_2A_3^iA_3^D\right] = A_1^{n+1}A_2A_3^D + \sum_{i=1}^{l_2}\left[A_1^{n+1-i}A_2A_3^iA_3^D\right]$$

$$= A_1^{n+1}A_2A_3^D + \sum_{i=0}^{l_2-1}\left[A_1^{n-i}A_2A_3^{i+1}A_3^D\right]$$

So

$$S(n+2)A_3^D = A_1^{n+1}A_2A_3^D + \sum_{i=0}^{l_2-1}\left[A_1^{n-i}A_2A_3^{i+1}A_3^D\right] + \sum_{i=l_2+1}^{n+1}\left[A_1^{n+1-i}A_2A_3^{i-1}\right].$$

It is easy to see that

$$A_1^{n+2}Y + S(n+2)A_3^D = \sum_{i=0}^{l_2-1}\left[A_1^{n-i}A_2A_3^i\right](-K_2) + \sum_{i=0}^{l_2-1}\left[A_1^{n-i}A_2A_3^{i+1}A_3^D\right] +$$

$$\sum_{i=l_2+1}^{n+1}\left[A_1^{n+1-i}A_2A_3^{i-1}\right].$$

Using the fact $(-K_2) = I - A_3A_3^D$ we get

$$A_1^{n+2}Y + S(n+2)A_3^D = \sum_{i=0}^{l_2-1}\left[A_1^{n-i}A_2A_3^i\right] + \sum_{i=l_2+1}^{n+1}\left[A_1^{n+1-i}A_2A_3^{i-1}\right]$$

$$= \sum_{i=0}^{l_2-1}\left[A_1^{n-i}A_2A_3^i\right] + \sum_{i=l_2}^{n}\left[A_1^{n-i}A_2A_3^i\right]$$

$$= \sum_{i=0}^{n}\left[A_1^{n-i}A_2A_3^i\right] = S(n+1).$$

That is

$$A_1^{n+2}Y + S(n+2)A_3^D = S(n+1),$$

hence

$$\begin{bmatrix} A_1 & A_2 \\ O & A_3 \end{bmatrix}^{n+2} \begin{bmatrix} A_1^D & Y \\ O & A_3^D \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ O & A_3 \end{bmatrix}^{n+1}.$$

So the third condition is satisfied.

If we take the transposes we get the following corollary.

Corollary 1. Let

$$G=\begin{bmatrix} A_3 & O \\ A_2 & A_1 \end{bmatrix}\in\mathbb{C}^{n\times n},$$

where A_1, A_3 are singular square with $rank(A_1)=rank(A_3)=1$, $Tr(A_1)\neq 0$ and $Tr(A_3)\neq 0$, $Ind(A_1)=l_1$ and $Ind(A_3)=l_2$, then

$$G^D=\begin{bmatrix} A_3^D & O \\ Y & A_1^D \end{bmatrix},$$

where A_1^D, A_3^D, and Y are the matrices given in (1).

There are many cases under the same conditions by Theorem 3.1.

Corollary 2. Let

$$H_1=\begin{bmatrix} A_1 & A_2 \\ O & O \end{bmatrix},\quad H_2=\begin{bmatrix} A_1 & O \\ A_2 & O \end{bmatrix},\ H_3=\begin{bmatrix} O & O \\ A_2 & A_1 \end{bmatrix},\quad H_4=\begin{bmatrix} O & A_2 \\ O & A_1 \end{bmatrix}$$

where $H_i\in\mathbb{C}^{n\times n}$ and A_1 is singular square. Then

$$H_1^D=\begin{bmatrix} A_1^D & Y_1 \\ O & O \end{bmatrix},\quad H_2^D=\begin{bmatrix} A_1^D & O \\ Y_2 & O \end{bmatrix},\ H_3^D=\begin{bmatrix} O & O \\ Y_3 & A_1^D \end{bmatrix},\quad H_4^D=\begin{bmatrix} O & Y_4 \\ O & A_1^D \end{bmatrix}$$

where

$$A_1^D=\frac{1}{[Tr(A_1)]^2}A_1,\quad Y_1=\left(\frac{1}{[Tr(A_1)]^4}A_1^2\right)A_2,\quad Y_2=A_2\left(\frac{1}{[Tr(A_1)]^4}A_1^2\right),$$

$$Y_3=\left(\frac{1}{[Tr(A_1)]^4}A_1^2\right)A_2,\ Y_4=A_2\left(\frac{1}{[Tr(A_1)]^4}A_1^2\right).$$

Each case of these cases follows directly from Theorem 3.1 and corollary 1.

Theorem 3.2. Let

$$H=\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\in\mathbb{C}^{n\times n},$$

be an EP matrix where A_1, A_2, A_3 and A_4 are matrices such that A_1 is square and non-singular and $rank(A_1)=rank(H)$. If P and Q are any matrices such that.

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} A_1 [I \quad Q],$$

then

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}^D = \begin{bmatrix} I \\ Q^* \end{bmatrix} ([I - P^*P]A_1[I + QQ^*])^{-1} [I \quad P^*],$$

where P and Q are given by Lemma 2.1.

Proof. Let $A_1 \in \mathbb{C}^{n\times n}$,

$$B = \begin{bmatrix} I_n \\ P \end{bmatrix} A_1, \;\; G = [I_n \quad Q].$$

Note that $rank(B) = rank(G) = rank(A_1) = rank(H) = n$ = number of columns of B = number of rows of G.

Thus we can apply Theorem 2.1 to get

$$H^D = (BG)^D = G^*(GG^*)^{-1}(B^*B)^{-1}B^*.$$

Since

$$(B^*B)^{-1}B^* = [A_1^*(I + P^*P)A_1]^{-1} A_1^* \, [I \quad P^*]$$

$$= A_1^{-1}(I + P^*P)^{-1} \, [I \quad P^*],$$

and

$$G^*(GG^*)^{-1} = \begin{bmatrix} I \\ Q^* \end{bmatrix} (I + QQ^*)^{-1},$$

then H^D is obtained.

Theorem 3.3. Let

$$H = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \in \mathbb{C}^{n\times n},$$

where A_1 is square and singular with $rank(A_1) = rank(H) = 1, Tr(H) \neq 0$, then

$$H^D = \frac{1}{[Tr(A_1)+Tr(A_4)]^2} H.$$

The proof of this theorem is similar to proof of Theorem 3.3 in [2].

Corollary 3. Let

$$H = \begin{bmatrix} A_1 & A_2 \\ A_3 & O \end{bmatrix} \in \mathbb{C}^{n\times n}$$

where A_1 is square and singular with $rank(A_1) = rank(H) = 1$, $Tr(A_1) \neq 0$, then

$$H^D = \frac{1}{[Tr(A_1)]^2} H.$$

The next theorem gives representation of a partitioned matrix which has full rank factorization.

4. NUMERICAL EXAMPLES

In this section, we give some numerical examples to illustrate our results.

Example 4.1. Let

$$H = \begin{bmatrix} 1 & -1 & 0 & 1 \\ -1 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ O & A_3 \end{bmatrix} \in \mathbb{C}^{4\times 4}$$

where

$$A_1 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix},\ A_3 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix},\ O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Note that A_1, A_3 are square singular with $rank(A_1) = rank(A_3) = 1$, and $Tr(A_1) = 1 + 1 = 2 \neq 0$, $Tr(A_3) = 1 \neq 0$. Note also, $Ind(A_1) = 1$, and $Ind(A_3) = 1$.

So we can apply Theorem 3.1.

$$A_1^D = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, A_3^D = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, K_1 = \begin{bmatrix} -1/2 & -1/2 \\ -1/2 & -1/2 \end{bmatrix}, K_2 = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix},$$

$$Y = (A_1^D)^2[A_2 + A_1^D A_2 A_3](-K_2) + (-K_1)[A_2 + A_1 A_2 A_3^D](A_3^D)^2 - A_1^D A_2 A_3^D$$

$$= \begin{bmatrix} -10/32 & 22/32 \\ -22/32 & 10/32 \end{bmatrix}$$

hence

$$H^D = \begin{bmatrix} 1/4 & -1/4 & -10/32 & 22/32 \\ -1/4 & 1/4 & -22/32 & 10/32 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Example 4.2. Let

$$H = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \in \mathbb{C}^{4\times 4}$$

be an EP matrix, where

$$A_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \qquad A_4 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Note that A_1 is square and non-singular, and $rank(A_1) = rank(H) = 2$. Using Lemma 2.1 we see that $A_4 = A_3 A_1^{-1} A_2$, where $A_1^{-1} = A_1$,

$$P = A_3 A_1^{-1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, Q = A_1^{-1} A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

Now, we can apply Theorem 3.2, to obtain the representation of H^D as following

$$H^D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

You can satisfy that H^D we get satisfies Definitions of the Drazin inverse easily.

Example 4.3. Let

$$H = \begin{bmatrix} 1 & -1 & 0 & -1 \\ -1 & 1 & 0 & 1 \\ 2 & -2 & 0 & -2 \\ -1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \in \mathbb{C}^{4\times 4}$$

where

$$A_1 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix},\ A_3 = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix},\ A_4 = \begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix}.$$

Note that A_1 is square and singular, and $rank(A_1) = rank(H) = 1, Tr(H) = 1 + 1 + 1 = 3 \neq 0$, so we can apply Theorem 3.3

$$H^D = \frac{1}{9}\begin{bmatrix} 1 & -1 & 0 & -1 \\ -1 & 1 & 0 & 1 \\ 2 & -2 & 0 & -2 \\ -1 & 1 & 0 & 1 \end{bmatrix}.$$

5. CONCLUDING REMARKS

In this paper we introduced the representations of the Drazin inverse for the partitioned matrices $\begin{bmatrix} A_1 & A_2 \\ O & A_3 \end{bmatrix}$ and $\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$ under some conditions. We hope that this paper will be useful for further study of solving a class of second-order singular differential equations [7].

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