On the Efficiency of Differential Transformation Method to the Solutions of Large Amplitude Nonlinear Oscillation Systems

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ABSTRACT

In this work, the efficiency of differential transformation method to the solutions of large amplitude nonlinear oscillatory systems is further established. Two cases of oscillation systems, nonlinear plane pendulum and pendulum in a rotating plane are considered. Without any linearization, discretization or series expansion of the sine and cosine of the angular displacement in the nonlinear models of the systems, the differential transformation method with Padé approximant is used to provide analytical solutions to the nonlinear problems. Also, the increased predictive power and the high level of accuracy of the differential transformation method over the previous methods are presented. The extreme accuracy and validity of the analytical solutions obtained by the differential transformation method are shown through comparison of the results of the solution with the corresponding numerical solutions obtained by fourth-fifth-order Runge-Kutta method. Also, with the aid of the analytical solutions, parametric studies were carried to study the impacts of the model parameters on the dynamic behavior of the large-amplitude nonlinear oscillation system. The method avoids any numerical complexity and it is very simple, suitable and useful as a mathematical tool for dealing the nonlinear problems.

Keywords: Large amplitude, Oscillation system, Strong nonlinearity, Analytical solution, Differential transformation method
1. INTRODUCTION

The wide areas of applications of oscillation systems such as mass on moving belt, simple pendulum, pendulum in a rotating plane, chaotic pendulum, elastic beams supported by two springs, vibration of a milling machine [1–3] have made them to be extensively investigated. Because, many practical physics and engineering components consist of vibrating systems that can be modeled using the oscillator systems [4–6], the importance of the systems has been established in physics and engineering. In order to simply describe the motion of the oscillation systems and provide physical insights into the dynamics, the nonlinear equation governing the pendulum motion has been linearized and solved [7-9].

The applications of such linearized model are found in the oscillatory motion of a simple harmonic motion, where the oscillation amplitude is small and the restoring force is proportional to the angular displacement and the period is constant. However, when the oscillation amplitude is large, the pendulum’s oscillatory motion is nonlinear and its period is not constant, but a function of the maximum angular displacement. Consequently, much attentions have been devoted to develop various analytical and numerical solutions to the nonlinear vibration of large-amplitude oscillation system. In one of the works, Beléndez et al. [10] developed an analytical solution to conservative nonlinear oscillators using cubication method while Gimeno and Beléndez [11] adopted rational-harmonic balancing approach to the nonlinear pendulum-like differential equations. Lima [12] submitted a trigonometric approximation for the tension in the string of a simple pendulum. Amore and Aranda [13] utilized improved Lindstedt–Poincaré method to provide approximate analytical solution of the nonlinear problems.

Momeni et al. [14] applied He’s energy balance method to Duffing harmonic oscillators. With the aid of the He’s energy balance method, Ganji et al. [15] provided a periodic solution for a strongly nonlinear vibration system. The He’s energy balance method as well as He’s Variational Approach was used by Askari et al. [16] for the nonlinear oscillation systems. The numerical method was also adopted by Babazadeh et al. [17] for the analysis of strongly nonlinear oscillation systems. Biazar and Mohammadi [18] explored multistep differential transform method to describe the nonlinear behaviour of oscillatory systems while Fan [19] and Zhang [20] applied He’s frequency–amplitude formulation for the Duffing harmonic oscillator.

The above reviewed works are based on approximate solutions. In order to gain better physical insight into the nonlinear problems, Beléndez et al. [21] presented the exact solution for the nonlinear pendulum using Jacobi elliptic functions. However, an exact analytic solution, as well as an exact expression for the period, both in terms of elliptic integrals and functions, is known only for the oscillatory regime ($0 < \theta_0 < \pi$). In this regime, the system oscillates between symmetric limits. Consequently, the quest for analytical and numerical solutions for the large-amplitude nonlinear oscillation systems continues. Therefore, with the aid of a modified variational iterative method, Herisanu and Marinca [22] described the behavior of the strongly nonlinear oscillators.

Kaya et al. [23] applied He’s variational approach to analyze the dynamic behaviour of a multiple coupled nonlinear oscillators while Khan and Mirzabeigy [24] submitted an improved accuracy of He’s energy balance method for the analysis of conservative nonlinear oscillator. A modified homotopy perturbation method was utilized by Khan et al. [25] to solve the nonlinear oscillators.
In another work, a coupling of homotopy and the variational approach was used by Khan et al. [26] to provide a physical insight into the vibration of a conservative oscillator with strong odd-nonlinearity. Wu et al. [27] presented analytical approximation to a large amplitude oscillation of a nonlinear conservative system while homotopy perturbation method was used by He [28] to provide an approximation analytical solution for nonlinear oscillators with discontinuous.

Belato et al. [29] analyzed a non-ideal system, consisting of a pendulum whose support point is vibrated along a horizontal guide. Few years later, Cai et al. [30] used methods of multiple scales and Krylov-Bogoliubov-Mitropolsky to developed the approximate analytical solutions to a class of nonlinear problems with slowly varying parameters. In some years after, Eissa and Sayed [31] examined the vibration reduction of a three-degree-of-freedom spring-pendulum system, subjected to harmonic excitation.

Meanwhile, Amore and Aranda [32] utilized Linear Delta Expansion to the Linstedt–Poincaré method and developed improved approximate solutions for nonlinear problems. The chaotic behavior and the dynamic control of oscillation systems has been a subject of major concern as presented in the literature. Therefore, with the aid of a shooting method, Idowu et al. [33] studied chaotic solutions of a parametrically undamped pendulum. Also, Amer and Bek [34] applied multiple scales method to study the chaotic response of a harmonically excited spring pendulum moving in a circular path. In the previous year, Anh et al. [35] examined the vibration reduction for a stable inverted pendulum with a passive mass-spring-pendulum-type dynamic vibration absorber. Ovseyevich [36] analyzed the stability of the upper equilibrium position of a pendulum in cases where the suspension point makes rapid random oscillation with small amplitude.

The relatively new approximate analytical method, differential transformation method (DTM) has proven to be more effective than most of the other approximate analytical methods as it does not require many computations as carried out in Adomian decomposition method, homotopy analysis method, regular and singular perturbation methods (method of matched asymptotic, method of multiple scales, etc), homotopy perturbation method, optimal homotopy asymptotic method, optimal homotopy perturbation method, variational iteration method, variation of parameter method, double decomposition method, Daftardar-Gejiji and Jafari method, Temini and Ansari method etc.

Also, the differential transformation method as introduced by Zhou [37] has fast gained ground as it appears in many engineering and scientific research papers because of its comparative advantages over the other approximate analytical methods. It is a method that solves differential equations, difference equation, differential-difference equations, fractional differential equation, pantograph equation and integro-differential equation without linearization, discretization, restrictive assumptions, perturbation and discretization or round-off error. It reduces complexity of expansion of derivatives and the computational difficulties of the other traditional and approximation analytical methods.

Using DTM, a closed form solution or approximate solution can be obtained as it provides excellent approximations to the solution of non-linear equation with a very high accuracy. It is capable of greatly reducing the size of computational work while still accurately providing the series solution with fast convergence rate. It is a more convenient method for engineering calculations than other approximate analytical or numerical methods [38].

It appears more appealing than the numerical solution as it comparatively reduces the computation costs, simulations and task in the analysis of nonlinear problems. Moreover, the
need for small perturbation parameter as required in traditional perturbation methods (PMs), the rigour of the determination of Adomian polynomials as carried out in ADM, the restrictions of HPM to weakly nonlinear problems as established in literatures are overcome in DTM [38].

Also, the lack of rigorous theories or proper guidance for choosing initial approximation, auxiliary linear operators, auxiliary functions, auxiliary parameters, and the requirements of conformity of the solution to the rule of coefficient ergodicity as done in HAM, the search Langrange multiplier as carried in VIM, the challenges associated with proper construction of the approximating functions for arbitrary domains or geometry of interest as in Galerkin weighted residual method (GWRM), least square method (LSM) and collocation method (CM) are some of the difficulties that DTM overcomes [38].

Ghafoori et al. [39] displayed the efficiency of the differential transformation method over homotopy perturbation method (HPM) and variation iteration method (VIM). In their work, it was shown that the results obtained by DTM are much more accurate in comparison with VIM and HPM; where for $t > 0$ the values achieved by HPM and VIM are not reliable. On the other hand, DTM solution presents very accurate results even for times much greater than zero [39]. However, the solution of the DTM in their work is based on series expansions of sine and cosine of the angular displacement in the nonlinear model. The expansions were truncated after the fourth terms. Indisputably, such approximated series expansion limits the predictive power of the DTM.

Therefore, in this work, the efficiency of differential transformation method for the analytical solutions of nonlinear oscillatory system is further established. Without any series expansion of the included sine and cosine of the angular displacement in the nonlinear model of the system, the differential transform method was successfully applied. Using the differential transformation method, the increased predictive power and the high level of accuracy of the differential transformation method with Padé approximant in this present analysis over the one presented by Ghafoori et al. [39] are presented.

Two cases of oscillation systems (nonlinear plane pendulum and pendulum with a rotating plane) were considered. Also, parametric studies were carried to study the impacts of the model parameters on the dynamic behavior of the large-amplitude nonlinear oscillation system.

2. MODEL FORMULATION FOR THE LARGE AMPLITUDE NONLINEAR OSCILLATION SYSTEMS

In this section, the equations of motion for the two cases of oscillation systems (nonlinear plane pendulum and pendulum with a rotating plane, with a rigid frame which is forced to rotate at the fixed rate and then makes the embedded simple pendulum to oscillate) are presented.

2. 1. Plane pendulum

Consider an oscillating system which freely oscillates in a vertical plane about the equilibrium position (P) on a vertical straight line which passes through the pivot point under the action of the gravitational force as shown in Fig. 1.
The system consists of a particle of mass $m$ attached to the end of an inextensible string, with the motion taking place in a vertical plane. Using Newton’s law for the rotational system, the differential equation modelling the free undamped simple pendulum is

$$
\tau = I \alpha \quad \Rightarrow \quad -mgsin\theta L = mL^2 \frac{d^2 \theta}{dt^2}
$$

(1)

and after rearrangement, we arrived at the nonlinear equation of motion of the pendulum as

$$
\frac{d^2 \theta}{dt^2} + \frac{g}{L} sin\theta = 0
$$

(2)

the initial conditions are

$$
\theta(0) = \theta_0, \quad \dot{\theta}(0) = 0
$$

(3)

$\theta_0$ is the initial angular displacement or the amplitude of the oscillations.
If the amplitude of the angular displacement is small, the equation of motion reduces to the equation of simple harmonic motion, which is given as

$$\frac{d^2 \theta}{dt^2} + \frac{g}{L} \theta = 0$$

(4)

The exact analytical solution is given as

$$\theta = \theta_0 \cos \left( \sqrt{\frac{g}{L}} t \right) = \theta_0 \cos \omega t$$

(5)

where $\theta$ is the angular displacement, $t$ is the time, $L$ is the length inextensible string (the pendulum), $g$ is the acceleration due to gravity and $\omega = \sqrt{\frac{g}{L}}$ is the natural frequency of the motion.

Our interest in the present study is the analysis of the nonlinear oscillations of the pendulum. It should be noted that the presence of the trigonometry function $\sin \theta$ in Eq. (2) introduces a high nonlinearity in the equation. However, Beléndez et al. [21] presented the exact solution for the large-amplitude nonlinear equation using the complete and incomplete elliptic integrals of the first kind as

$$\theta(t) = 2 \arcsin \left( \frac{\sin \frac{\theta_0}{2}}{2} \sin \left[ K \left( \sin^2 \frac{\theta_0}{2} - \omega t; \sin^2 \frac{\theta_0}{2} \right) \right] \right)$$

(6)

where

$$\omega_0 = \sqrt{\frac{g}{L}}$$

is the frequency for the small-angle regime.

and

$K(m)$ is the complete elliptic integral of the first kind which is given as

$$K(m) = \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-m^2z^2)}}$$

(7)

where

$$z = \sin \phi$$

It should be pointed out that the exact analytic solution by Beléndez et al. [21], as well as the exact expression for the period, both in terms of elliptic integrals and functions, is only known for the oscillatory regime. In this regime ($0 < \theta_0 < \pi$), the system oscillates between
symmetric limits. In fact, the developed exact solution provides a high accurate solution to the nonlinear oscillation system with initial amplitudes as high as 135°. Therefore, in another work, Beléndez et al. [40] applied a rational harmonic representation to develop an approximation solution for the pendulum as

\[ \theta(t) = \frac{(960 - 49\theta_0^2)\theta_0 \cos \omega t}{960 - 69\theta_0^2 + 20\theta_0^2 \cos^2 \omega t} \]  

(8)

where

\[ \omega = \frac{1}{4} \sqrt{\frac{g}{l}} \left( 1 + \sqrt{\frac{\cos \theta_0}{2}} \right)^2 \]  

(9)

After substitution of Eq. (9) into Eq. (8), we have

\[ \theta(t) = \frac{(960 - 49\theta_0^2)\theta_0 \cos \left[ \frac{1}{4} \sqrt{\frac{g}{l}} \left( 1 + \sqrt{\frac{\cos \theta_0}{2}} \right)^2 \right] t}{960 - 69\theta_0^2 + 20\theta_0^2 \cos^2 \left[ \frac{1}{4} \sqrt{\frac{g}{l}} \left( 1 + \sqrt{\frac{\cos \theta_0}{2}} \right)^2 \right] t} \]  

(10)

From Eq. (9), it is also established that for a nonlinear oscillation system, the angular frequency of the oscillation depends on the amplitude of the motion. However, the developed approximate analytical expression in Eq. (10) provides a high accurate solution to the nonlinear oscillation system with initial amplitudes up to 170°. Therefore, the quest for analytical and numerical solutions for large-amplitude nonlinear oscillation systems over a wider range continues.

2.2. Pendulum with a rotating plane

Consider a pendulum with a rotating plane, which has a rigid frame as shown in Fig. 2, is forced to rotate at the fixed rate \( U \). While the frame rotates, the simple pendulum oscillates [39, 41-44]. The system consists of a particle of mass \( m \) attached to the end of an unstretchable rod. Using the Newton’s law for the rotational system, the nonlinear differential equation for the free undamped pendulum is given as [31]

\[ \frac{d^2 \theta}{dt^2} + (1 - \beta \cos \theta) \sin \theta = 0 \]  

(11)

the initial conditions are

\[ \theta(0) = \theta_0, \quad \dot{\theta}(0) = 0 \]  

(12)
where

\[ \theta_0 \] is the initial angular displacement or the amplitude of the oscillations and \( \beta = \frac{\Omega^2 r}{g} \).

**Fig. 2. Pendulum with a rotating plane**

The exact analytic solution for this second case is very difficult to be developed. Therefore, Ghafoori et al. [39] showed the efficiency of the differential transform method in providing approximate analytical solution to the pendulum. In their work, it was shown that the results obtained by DTM are much more accurate in comparison with VIM and HPM; where for \( t > 0 \), the values achieved by HPM and VIM are not reliable. On the other hand, DTM solution presents very accurate results even for times much greater than zero [39].

However, the solution of the DTM in their work is based on series expansions of sine and cosine of the angular displacement in the nonlinear model of the system. Also, the expansions were truncated after the fourth terms. Indisputably, such approximated series expansion limits the predictive power of the DTM. Therefore, a higher level of accuracy and validity of the proposed method is required.

### 3. METHOD OF SOLUTION: DIFFERENTIAL TRANSFORM METHOD

The nonlinear equations (2) and (4) are solved using differential transformation method as introduced by Zhou [37]. The basic definitions and the operational properties of the method are as follows:

If \( u(t) \) is analytic in the domain \( T \), then the function \( u(t) \) will be differentiated continuously with respect to time \( t \).
\[
\frac{d^p u(t)}{dt^p} = \varphi(t, p) \quad \text{for all} \quad t \in T
\]  

(13)

for \( t = t_i \), then \( \varphi(t, p) = \varphi(t_i, p) \), where \( p \) belongs to the set of non-negative integers, denoted as the \( p \)-domain. We can therefore write Eq. (13) as

\[
U(p) = \varphi(t_i, p) = \left[ \frac{d^p u(t)}{dt^p} \right]_{t=t_i}
\]  

(14)

where \( U_p \) is called the spectrum of \( u(t) \) at \( t = t_i \).

Expressing \( u(t) \) in Taylor’s series as

\[
u(t) = \sum_{p=0}^{\infty} \left[ \frac{(t-t_i)^p}{p!} \right] U(p)
\]  

(15)

where Equ. (13) is the inverse of \( U(k) \) us symbol ‘D’ denoting the differential transformation process and combining (14) and (15), we have

\[
u(t) = \sum_{p=0}^{\infty} \left[ \frac{(t-t_i)^p}{p!} \right] U(p) = D^{-1} U(p)
\]  

(16)

**Table 1.** Operational properties of differential transformation method.

<table>
<thead>
<tr>
<th>S/N</th>
<th>Function</th>
<th>Differential transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( u(t) \pm v(t) )</td>
<td>( U(p) \pm V(p) )</td>
</tr>
<tr>
<td>2</td>
<td>( \alpha u(t) )</td>
<td>( \alpha U(p) )</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{du(t)}{dt} )</td>
<td>((p+1) U(p+1))</td>
</tr>
<tr>
<td>4</td>
<td>( u(t)v(t) )</td>
<td>( \sum_{r=0}^{p} V(r)U(p-r) )</td>
</tr>
<tr>
<td>5</td>
<td>( u^{(m)}(t) )</td>
<td>( \sum_{r=0}^{p} U^{(m-1)}(r)U(p-r) )</td>
</tr>
</tbody>
</table>
3. 1. Differential transformation method to the nonlinear simple pendulum

The application of differential transform method to the nonlinear simple pendulum is demonstrated in this section.

Eq. (2) can be written as

$$\dot{\theta} + \frac{g}{L} \sin \theta = 0$$  \hspace{1cm} (17)

In order to apply differential transformation method without series expansion of the included sine and cosine of the angular displacement in the nonlinear simple pendulum,

let  \( f = \sin \theta \)  \hspace{1cm} (18)

then

$$\frac{df}{d\theta} = \cos \theta, \quad \text{i.e.} \quad \frac{df}{d\theta} = \frac{df}{dt} \frac{dt}{d\theta} = \cos \theta$$  \hspace{1cm} (19)

which gives

$$\frac{df}{dt} = \cos \theta \frac{d\theta}{dt}$$  \hspace{1cm} (20)

Eq. (20) can be written as

$$\dot{f} = \dot{\theta} \cos \theta$$  \hspace{1cm} (21)

From Eq. (21), we have

$$\cos \theta = \frac{\dot{f}}{\dot{\theta}}$$  \hspace{1cm} (22)
The application of chain rule to Eq. (20) gives

\[
\frac{d^2 f}{dt^2} = \cos \theta \frac{d^2 \theta}{dt^2} - \sin \theta \left( \frac{d \theta}{dt} \right)^2
\]  

(23)

which can be written as

\[
\ddot{f} = \dot{\theta} \cos \theta - f \left( \dot{\theta} \right)^2
\]  

(24)

Substitution of Eq. (22) into Eq. (17) yields

\[
\dot{\theta} + \frac{g}{L} f = 0
\]  

(25)

and when Eq. (22) is substituted into Eq. (24), we have

\[
\dot{\theta} \ddot{f} - \dot{f} \dot{\theta} + \left( \dot{\theta} \right)^3 f = 0
\]  

(26)

The obtained Eqs. (25) and (26) are the desired conserved forms of Eq. (17) that are to be solved analytically using differential transform method (DTM). The DTM recursive relations for the fully coupled Eqs, (25) and (25) are

\[
(k + 1)(k + 2) \theta_{k+1} + \omega^2 F_k = 0
\]  

(27)

\[
\sum_{l=0}^{k} (l + 1)(l + 2) F_{l+2} (k - l + 1) \theta_{k-l+1} - \sum_{l=0}^{k} (l + 1)(l + 2) \theta_{l+2} (k - l + 1) F_{k-l+1} \\
+ \sum_{n=0}^{k} \left( \sum_{m=0}^{n} \sum_{l=0}^{m} (l + 1) \theta_{l+1} (m - l + 1) \theta_{m-l+1} (n - m + 1) \theta_{n-m+1} F_{k-n} \right) = 0
\]  

(28)

subject to

\[
\theta[0] = a, \quad \theta[1] = c, \quad F[0] = b, \quad F[1] = d
\]  

(29)

The evaluations of Eq. (27), (28) and (29) gives the term by term solutions as:

\[
\theta_0 = a,
\]

\[
\theta_1 = c,
\]

\[
\theta_2 = -\frac{\omega^2 b}{2}
\]
\[
\theta_3 = -\frac{d\omega^2}{12}
\]
\[
\theta_4 = \frac{\omega^2 b (c^3 + d\omega^2)}{72c}
\]
\[
\theta_5 = -\frac{\omega^2 \left(6b^2c^2\omega^2 - 2c^3d - d^2\omega^2\right)}{960c}
\]
\[
\theta_6 = \frac{\omega^2 \left(9b^2c^2\omega^4 - 3c^6 - 25c^3d\omega^2 - d^2\omega^4\right)}{10800c^2}
\]
\[
\theta_7 = \frac{\omega^2 \left(280b^2c^5\omega^2 + 538b^2c^2d\omega^4 - 24c^6d - 126c^3d^2\omega^2 - 3d^3\omega^4\right)}{725760c^2}
\]
\[
\theta_8 = -\frac{\omega^2 b \left(3270b^2c^5\omega^4 + 1236b^2c^2d\omega^6 - 60c^9 - 1902c^6d\omega^2 - 2395c^3d^2\omega^4 - 4d^3\omega^6\right)}{16934400c^3}
\]

Since \( \theta \) is our interest, we only present the term by term solutions of \( \theta \).

From the definition of DTM \( \left( \theta(t) = \sum_{\xi=0}^{N} \theta_{\xi} t^{\xi} \right) \), the solution of the nonlinear simple pendulum model in Eq. (2) is given as:

\[
\theta(t) = a + ct - \left(1/2\omega^2\right)t^2 - \left(1/12d\omega^2\right)t^3 + \frac{\omega^2 b \left( c^3 + d\omega^2 \right)}{72c} t^4 - \frac{\omega^2 \left( 6b^2c^2\omega^2 - 2c^3d - d^2\omega^2 \right)}{960c} t^5
\]
\[
+ \frac{\omega^2 b \left( 9b^2c^2\omega^4 - 3c^6 - 25c^3d\omega^2 - d^2\omega^4 \right)}{10800c^2} t^6 + \frac{\omega^2 \left( 280b^2c^5\omega^2 + 538b^2c^2d\omega^4 - 24c^6d \right)}{725760c^2} t^7
\]
\[
- \frac{\omega^2 b \left( 3270b^2c^5\omega^4 + 1236b^2c^2d\omega^6 - 60c^9 - 1902c^6d\omega^2 - 2395c^3d^2\omega^4 - 4d^3\omega^6 \right)}{16934400c^3} t^8 + ...
\]

3.2. Differential transformation method to the pendulum with a rotating plane

For the second case under consideration, which is pendulum with a rotating plane, we can write Eq. (10) as

\[
\dot{\theta} + (1 - \beta \cos \theta) \sin \theta = 0
\]
Also, the application of differential transformation method without series expansion of the included sine and cosine of the angular displacement in the nonlinear model of the pendulum is demonstrated as follows,

let \( f = \sin \theta \) \hspace{1cm} (33)

then

\[
\frac{df}{d\theta} = \cos \theta, \quad \text{i.e.} \quad \frac{df}{d\theta} = \frac{df}{dt} = \cos \theta
\] \hspace{1cm} (34)

which gives

\[
\frac{df}{dt} = \cos \theta \frac{d\theta}{dt}
\] \hspace{1cm} (35)

Eq. (35) can be written as

\[ \dot{f} = \dot{\theta} \cos \theta \] \hspace{1cm} (36)

From Eq. (36), we have

\[
\cos \theta = \frac{\dot{f}}{\dot{\theta}}
\] \hspace{1cm} (37)

The application of chain rule on Eq. (35) gives

\[
\frac{d^2f}{dt^2} = \cos \theta \frac{d^2\theta}{dt^2} - \sin \theta \left( \frac{d\theta}{dt} \right)^2
\] \hspace{1cm} (38)

Which can be written as

\[ \ddot{f} = \dot{\theta} \cos \theta - f \left( \dot{\theta} \right)^2 \] \hspace{1cm} (39)

Substitution of Eqs. (33) and (37) into Eq. (32) yields

\[ \dot{\theta} \ddot{\theta} + \dot{\theta} f - \beta \ddot{f} = 0 \] \hspace{1cm} (40)

and when Eq. (37) is substituted into Eq. (39), we have

\[ \dot{\theta} \dddot{f} - \ddot{f} \dot{\theta} + \left( \dot{\theta} \right)^3 f = 0 \] \hspace{1cm} (41)
Also, the obtained Eqs. (40) and (41) are the desired conserved forms of Eq. (32) which are to be solved analytically using differential transform method (DTM).

The DTM recursive relation for the fully coupled Eqs. (40) and (41) are

\[
\sum_{l=0}^{k}(l+1)(l+2)\theta_{l+2}(k-l+1)\theta_{k-l+1} + \sum_{l=0}^{k}(l+1)\theta_{l+1}F_{k-l} - \beta \sum_{l=0}^{k}(l+1)F_{l+1}F_{k-l} = 0 \tag{42}
\]

\[
\left\{ \sum_{l=0}^{k}(l+1)(l+2)F_{l+2}(k-l+1)\theta_{k-l+1} - \sum_{l=0}^{k}(l+1)(l+2)\theta_{l+2}(k-l+1)F_{k-l+1} + \sum_{n=0}^{k} \left( \sum_{m=0}^{n} (l+1)\theta_{l+1}(m-l+1)\theta_{m-l+1}(n-m+1)\theta_{n-m+1}F_{k-n} \right) \right\} = 0 \tag{43}
\]

Using the initial conditions in DTM domain in Eq. (6) in Eqs. (42) and (43), the term by term solution becomes;

\[
\theta_0 = a,
\]

\[
\theta_1 = c,
\]

\[
\theta_2 = \frac{b(\beta d - c)}{2c}
\]

\[
\theta_3 = -\frac{b^2 \beta c^2 - \beta d^2 + cd}{6c}
\]

\[
\theta_4 = -\frac{b(b^2 \beta c^2 d - b^2 \beta c^3 + 4\beta c^4 d - \beta^2 d^3 - c^5 + 2\beta cd^2 - c^2 d)}{24c^3}
\]

\[
\theta_5 = \frac{\left( b^4 \beta^2 c^4 + 4b^2 \beta c^6 - 14b^2 \beta^2 c^2 d^2 + 17b^2 \beta c^3 d - 4\beta c^4 d^2 - 3b^2 c^4 + \beta^2 d^4 + c^5 d - 2\beta cd^3 + c^2 d^2 \right)}{120c^3}
\]

\[
\theta_6 = \left[ b \left( b^4 \beta^3 c^4 d - b^4 \beta^2 c^5 + 4b^2 \beta^2 c^6 d - 14b^2 \beta^3 c^2 d^3 - 29b^2 \beta c^7 + 16\beta c^8 d + 31b^2 \beta^2 c^3 d^2 - 44 \beta^2 c^4 d^3 \right) \right] = \frac{-c^9 - 20b^2 \beta c^4 d + \beta^3 d^5 + 55 \beta c^5 d^2 + 3b^2 c^5 - 3\beta c^6 d^3 - 11c^6 d + 3\beta c^2 d^5 - c^3 d^2}{720c^5}
\]
\[ \theta_7 = -\frac{1}{5040c^5} \left( b^6 \beta^3 c^6 + 44b^4 \beta^2 c^8 - 135b^4 \beta^4 c^4 d^2 + 16b^2 \beta c^{10} + 213b^4 \beta^2 c^6 d^2 - 328b^2 \beta^2 c^6 d^2 - 78b^4 \beta c^6 \right) \\
+135b^2 \beta^3 c^2 d^4 + 265b^2 \beta c^6 d^2 - 16b \beta c^8 d^2 - 303b^2 \beta^2 c^3 d^3 - 15b^2 c^6 + 44 \beta^2 c^4 d^4 + c^9 d + 201b^2 \beta c^4 d^2 - \beta^3 d^6 - 55 \beta^2 c^5 d^3 - 33b^2 c^6 d^3 + 3 \beta^2 c^2 d^3 + 11c^6 d^2 - 3 \beta c^2 d^4 + c^3 d^3 \right) \\
\]

\[ \theta_8 = -\frac{b}{40320c^7} \left( b^6 \beta^4 c^6 d - b^6 \beta^3 c^7 + 408b^4 \beta^3 c^8 d - 135b^4 \beta^4 c^4 d^3 - 345b^4 \beta^2 c^9 + 912b^2 \beta c^{10} d + 348b^4 \beta^3 c^5 d^2 \\
-2064b^2 \beta^3 c^6 d^3 - 345b^2 \beta c^{11} + 64 \beta c^{12} d - 291b^4 \beta^2 c^6 d + 135b^2 \beta^4 c^3 d^5 + 3540b^2 \beta^2 c^7 d^2 \\
-912 \beta^2 c^8 d^3 - c^{13} + 78b^4 \beta c^7 - 438b^2 \beta^2 c^3 d^4 - 1554b^2 \beta c^8 d + 408 \beta^3 c^4 d^2 + 780 \beta c^9 d^2 \\
+504b^2 \beta^2 c^4 d^3 + 78b^2 c^9 - \beta^4 d^7 - 918 \beta^2 c^5 d^4 - 57c^{10} d - 234b^2 \beta c^5 d^2 \\
+4 \beta^3 c^6 d^6 + 612 \beta c^6 d^3 + 33b^2 c^6 d - 6 \beta^2 c^2 d^5 - 102c^7 d^2 + 4 \beta c^3 d^4 - c^4 d^3 \right) \\
\]

Applying the principle of DTM inversion, the desired solution becomes;

\[ \theta(t) = \sum_{i=0}^{N} \theta_i t^i \], which in expanded form becomes;

\[ \theta(t) = \left( a + ct + \left( \frac{b(\beta d - c)}{2c} \right) t^2 - \left( \frac{b^2 \beta c^2 - \beta d^2 + cd}{6c} \right) t^3 - \left( \frac{b^2 \beta c^3 d - b^2 \beta c^3 + 4 \beta c^4 d}{24c^3} \right) t^4 \right) + \left( \frac{b^2 \beta^2 c^4 d - b^2 \beta^2 c^4 + 4 \beta^2 c^5 d - 2b \beta c^6 d^2 - 3b^2 c^6 + \beta^2 c^6 d^2 + c^9 d - 2 \beta cd^3 + c^3 d^2}{120c^5} \right) t^5 \\
+ \left( \frac{b^2 \beta \beta c^5 d - b^2 \beta \beta c^5 + 44b^2 \beta^2 c^6 d - 14b^2 \beta^2 c^6 d - 29b^2 \beta c^7 + 16 \beta c^8 d + 31 \beta^3 \beta^2 c^8 d^2 - 44 \beta^3 \beta^2 c^8 d^2}{720c^8} \right) t^6 \\
+ \left( \frac{b^6 \beta^4 c^6 d - b^6 \beta^4 c^6 + 44b^4 \beta^3 c^8 d - 135b^4 \beta^4 c^4 d^3 + 16 \beta^2 c^{10} + 213b^4 \beta^2 c^6 d - 328b^2 \beta^2 c^6 d^2 - 78b^4 \beta c^6 + 135 \beta^2 \beta^2 c^6 d^2 \right) \\
+265b^2 \beta^2 c^6 d - 16 \beta^2 c^6 d - 303b^2 \beta^2 c^3 d^4 - 15b^2 c^6 + 44 \beta^2 c^4 d^4 + c^9 d + 201b^2 \beta^2 c^2 d^2 - \beta^4 d^7 - 55 \beta^2 c^3^3 \\
-33b^2 c^6 d + 3 \beta^2 c^2 d^3 + 11c^6 d^2 - 3 \beta c^2 d^4 + c^3 d^5 \right) t^7 \\
+ \left( \frac{b^6 \beta^4 c^6 d - b^6 \beta^4 c^6 + 44b^4 \beta^3 c^8 d - 135b^4 \beta^4 c^4 d^3 - 135b^4 \beta^4 c^4 d^3 - 345b^4 \beta^2 c^9 + 912b^2 \beta c^{10} d + 348b^4 \beta^3 c^5 d^2 \\
-2064b^2 \beta^3 c^6 d^3 - 345b^2 \beta c^{11} + 64 \beta c^{12} d - 291b^4 \beta^2 c^6 d + 135b^2 \beta^4 c^3 d^5 + 3540b^2 \beta^2 c^7 d^2 \\
-912 \beta^2 c^8 d^3 - c^{13} + 78b^4 \beta c^7 - 438b^2 \beta^2 c^3 d^4 - 1554b^2 \beta c^8 d + 408 \beta^3 c^4 d^2 + 780 \beta c^9 d^2 \\
+504b^2 \beta^2 c^4 d^3 + 78b^2 c^9 - \beta^4 d^7 - 918 \beta^2 c^5 d^4 - 57c^{10} d - 234b^2 \beta c^5 d^2 \\
+4 \beta^3 c^6 d^6 + 612 \beta c^6 d^3 + 33b^2 c^6 d - 6 \beta^2 c^2 d^5 - 102c^7 d^2 + 4 \beta c^3 d^4 - c^4 d^3 \right) t^8 + ... \\
\]

(44)
4. THE BASIC CONCEPT AND THE PROCEDURE OF PADÉ APPROXIMANT

The limitation of power series methods to a small domain have been overcome by after-treatment techniques. These techniques increase the radius of convergence and also, accelerate the rate of convergence of a given series. Among the so-called after treatment techniques, Padé approximant technique has been widely applied in developing accurate analytical solutions to nonlinear problems of large or unbounded domain problems of infinite boundary conditions [45]. The Padé-approximant technique manipulates a polynomial approximation into a rational function of polynomials. Such a manipulation gives more information about the mathematical behaviour of the solution. The basic procedures are as follows.

Suppose that a function \( f(\eta) \) is represented by a power series.

\[
 f(\eta) = \sum_{i=0}^{\infty} c_i \eta^i
\]  

This expression is the fundamental point of any analysis using Padé approximant. The notation \( c_i, i = 0, 1, 2, \ldots \) is reserved for the given set of coefficient and \( f(\eta) \) is the associated function. \([L/M]\) Padé approximant is a rational function defined as

\[
 f(\eta) = \frac{\sum_{i=0}^{L} a_i \eta^i}{\sum_{i=0}^{M} b_i \eta^i} = \frac{a_0 + a_1 \eta + a_2 \eta^2 + \ldots + a_L \eta^L}{b_0 + b_1 \eta + b_2 \eta^2 + \ldots + b_M \eta^M}
\]  

which has a Maclaurin expansion, agrees with equation (1) as far as possible. It is noticed that in Eq. (47), there are \( L+1 \) numerator and \( M+1 \) denominator coefficients. So there are \( L+1 \) independent number of numerator coefficients, making \( L+M+1 \) unknown coefficients in all. This number suggest that normally \([L/M]\) out of fit the power series Eq. (46) through the orders \( 1, \eta, \eta^2, \ldots, \eta^{L+M} \).

In the notation of formal power series

\[
 \sum_{i=0}^{\infty} c_i \eta^i = \frac{a_0 + a_1 \eta + a_2 \eta^2 + \ldots + a_L \eta^L}{b_0 + b_1 \eta + b_2 \eta^2 + \ldots + b_M \eta^M} + O(\eta^{L+M+1})
\]  

which gives

\[
 (b_0 + b_1 \eta + b_2 \eta^2 + \ldots + b_M \eta^M)(c_0 + c_1 \eta + c_2 \eta^2 + \ldots) = a_0 + a_1 \eta + a_2 \eta^2 + \ldots + a_L \eta^L + O(\eta^{L+M+1})
\]  

Expanding the LHS and equating the coefficients of \( \eta^{L+1}, \eta^{L+2}, \ldots, \eta^{L+M} \) we get

\[
 b_M c_{L-M+1} + b_{M-1} c_{L-M+2} + b_{M-2} c_{L-M+3} + \ldots + b_2 c_{L-1} + b_1 c_L + b_0 c_{L+1} = 0
\]

\[
 b_M c_{L-M+2} + b_{M-1} c_{L-M+3} + b_{M-2} c_{L-M+4} + \ldots + b_2 c_L + b_1 c_{L+1} + b_0 c_{L+2} = 0
\]
If \( i < 0, c_i = 0 \) for consistency. Since \( b_0 = 1 \), Eqn. (50) becomes a set of \( M \) linear equations for \( M \) unknown denominator coefficients

\[
\begin{pmatrix}
  c_{L-M+1} & c_{L-M+2} & c_{L-M+3} & \cdots & c_L \\
  c_{L-M+2} & c_{L-M+3} & c_{L-M+4} & \cdots & c_{L+1} \\
  c_{L-M+3} & c_{L-M+4} & c_{L-M+4} & \cdots & c_{L+2} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  c_L & c_{L+1} & c_{L+2} & \cdots & c_{L+M-1}
\end{pmatrix}
\begin{pmatrix}
  b_{M} \\
  b_{M-1} \\
  b_{M-2} \\
  \vdots \\
  b_1
\end{pmatrix}
= \begin{pmatrix}
  c_{L+1} \\
  c_{L+2} \\
  c_{L+3} \\
  \vdots \\
  c_{L+M}
\end{pmatrix}
\tag{51}
\]

From the above Eq. (51), \( b_i \) may be found. The numerator coefficient \( a_0, a_1, a_2, \ldots, a_L \) follow immediately from Eq. (49) by equating the coefficient of \( 1, \eta, \eta^2, \ldots, \eta^{L+M} \) such that

\[
a_0 = c_0
\]
\[
a_1 = c_1 + b_1 c_0
\]
\[
a_2 = c_2 + b_1 c_1 + b_2 c_0
\]
\[
a_3 = c_3 + b_1 c_2 + b_2 c_1 + b_3 c_0
\]
\[
a_4 = c_4 + b_1 c_3 + b_2 c_2 + b_3 c_1 + b_4 c_0
\]
\[
a_5 = c_5 + b_1 c_4 + b_2 c_3 + b_3 c_2 + b_4 c_1 + b_5 c_0
\]
\[
a_6 = c_6 + b_1 c_5 + b_2 c_4 + b_3 c_3 + b_4 c_2 + b_5 c_1 + b_6 c_0
\]
\[
\vdots
\]
\[
a_L = c_L + \sum_{i=1}^{\min\{L/M\}} b_i c_{L-i}
\]

The Eq. (51) and Eq. (52) normally determine the Padé numerator and denominator and are called Padé equations. The \([L/M]\) Padé approximant is constructed which agrees with the equation in the power series through the order \( \eta^{L+M} \). To obtain a diagonal Padé approximant of order \([L/M]\), the symbolic calculus software Maple is used.
It should be noted as mentioned previously that $\Delta$ and $\Omega$ in the solutions are unknown constants. In order to compute their values for extended large domains solutions, the power series as presented in Eqs. (31) and (45) are converted to rational functions using [18/19] Pade approximation through the software Maple and then, the large domains are applied. The resulting simultaneous equations are solved to obtain the values of $\Delta$ and $\Omega$ for the respective values of the model parameters under considerations.

5. RESULTS AND DISCUSSION

The increased predictive power and extreme accuracy of the differential transformation method with Padé approximant is displayed in Tables 1-5. The Tables depicted the high level of accuracy and agreements of the symbolic solutions of the modified DTM when compared to the numerical solutions of the fourth-fifth-order Runge-Kutta method. From the results in the Tables, it could be stated that the differential transformation method gives highly accurate results and avoids any numerical complexity. The higher accuracy of the differential transformation method over homotopy perturbation method (HPM) and variation iteration method (VIM) has been pointed out by Ghafoori et al. [39]. Also, in their work, it was stated that the values achieved by HPM and VIM are not reliable. The HPM and VIM fail to handle strongly nonlinear differential systems. This limitation was overcome by the DTM. However, the solution of DTM obtained by Ghafoori et al. [39] is based on series expansion of sine and cosine of the angular displacement in the nonlinear model.

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Table 3. Comparison of results when $\beta = 0.25$

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Table 4. Comparison of results when $\beta = 0.50$

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Table 5. Comparison of results when $\beta = 0.75$

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Table 6. Comparison of results when $\beta = 1.00$

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The expansions were truncated after the fourth terms. Indisputably, such approximated series expansion limits the predictive power of the DTM. Without any series expansion of the included sine and cosine of the angular displacement in the nonlinear model of the system, an improved solution with increased predictive power and high level of accuracy is presented in this work.

5. Results of the nonlinear simple pendulum

In the section, the effects of length of pendulum and the initial angular displacement or the amplitude of the oscillations on the nonlinear vibration of the simple pendulum are presented in Figs. 3-8 and Figs. 10-34 while Figs. 9a-d presents the verifications of the differential transformation method.

5.1. Effects of the length of pendulum on the angular displacement of the simple pendulum (\( \theta_0 = 1.0 \))

Figs 3a-3d shows the effect of length of pendulum on the angular displacement of the nonlinear vibration of the simple pendulum when the initial angular displacement or amplitude of the system is 1.0. As expected, the results show that by increasing the length of the pendulum, the system frequency decreases as shown in Fig. 3. For high amplitudes, the periodic motion exhibited by a simple pendulum is practically harmonic but its oscillations are not isochronous i.e. the period is a function of the amplitude of oscillations.

![Fig. 3a. Time response of the system when L = 5 m](image_url)
Fig. 3b. Time response of the system when $L = 10$ m

Fig. 3c. Time response of the system when $L = 15$
5. 1. 2. Effects of the length of pendulum on the angular frequency of the simple pendulum 
($\theta_0 = 2.0$)

Fig. 3d. Time response of the system when $L = 20$ m

Fig. 4a. Time response of the system when $L = 5$
**Fig. 4b.** Time response of the system when $L = 10$ m

**Fig. 4c.** Time response of the system when $L = 15$
Fig. 4d. Time response of the system when $L = 20$ m

Fig. 5. Time response of the simple pendulum when $\theta_0 = 1.0$
The effect of the pendulum length on the angular displacement of the nonlinear vibration of the simple pendulum when the initial angular displacement of the system is 2.0 is shown in Figs. 4a-d. The same effect as in Figs. 3 is noted. However, when Figs. 3 and 4 are compared, it is shown that as the amplitude of the system increases, the system frequency decreases while the angular displacement increases. The results show that the effect of large initial amplitude on the system frequency is different as compared to small initial amplitude. The results show that the nonlinear frequency increases by increasing the amplitude of pendulum for smaller values of the initial amplitude. However, the nonlinear frequency of the pendulum decreases by increasing the amplitude for the higher values of the amplitude.

From Figs. 5 and 6, the effects of pendulum length and amplitude on the angular displacement of the plane pendulum are presented; it can be seen that increasing the length of the pendulum makes it oscillate with the lower frequency. It could also be noted as the amplitude of the system increases, the nonlinear frequency of the pendulum decreases.

5. 2. Effects of the length of pendulum and amplitude of oscillation on the phase plots of the system

The impacts of the pendulum length and the amplitude of the oscillations on the phase plots of the nonlinear simple pendulum are displayed in Figs. 7a-d and 8a-d.

5. 2. 1. Effects of the length of pendulum on the phase plot of the nonlinear simple pendulum \((\theta_0 = 1.0)\)

Figs. 7a-d and 9 show the effects of the pendulum length on the phase plot when amplitude of the system is 1.0 while Figs. 8a-d and 10 depict the effects of the pendulum length.
on the phase plot when amplitude of the system is 2.0. The nearly circular curves around (0,0) in figures show that the pendulum goes into a stable limit cycle.

**Fig. 7a.** Phase plot in the \((\theta(t), \frac{d\theta}{dt})\) plane when \(L = 5\) m

**Fig. 7b.** Phase plot in the \((\theta(t), \frac{d\theta}{dt})\) plane when \(L = 10\) m
Fig. 7c. Phase plot in the ($\theta(t), \frac{d\theta}{dt}$) plane when $L = 15$ m

Fig. 7d. Phase plot in the ($\theta(t), \frac{d\theta}{dt}$) plane when $L = 20$ m
5. 2. 2. Effects of the length of pendulum on the phase plots of the nonlinear simple pendulum ($\theta_0 = 2.0$)

\[ \theta(t), \frac{d\theta}{dt} \]

**Fig. 8a.** Phase plot in the $(\theta(t), \frac{d\theta}{dt})$ plane when $L = 5$ m

**Fig. 8b.** Phase plot in the $(\theta(t), \frac{d\theta}{dt})$ plane when $L = 10$ m
Fig. 8c. Phase plot in the \((\theta(t), d\theta/dt)\) plane when \(L = 15\) m

Fig. 8d. Phase plot in the \((\theta(t), d\theta/dt)\) plane when \(L = 20\) m
Fig. 9. Phase plot in the \((\theta(t), d\theta/dt)\) plane when \(\theta_0 = 1.0\).

Fig. 10. Phase plot in the \((\theta(t), d\theta/dt)\) plane when \(\theta_0 = 2.0\).
5. 3. Effects of the length of pendulum and amplitude of oscillation on the angular velocity

The effects of pendulum length on the angular velocity is displayed in this section.

5. 3. 1. Effects of the length of pendulum on the angular velocity \((\theta_i = 1.0)\)

![Fig. 11a. Velocity variation with time when L = 5 m](image1)

![Fig. 11b. Velocity variation with time when L = 10 m](image2)
Fig. 11c. Velocity variation with time when $L = 15$ m

Fig. 11d. Velocity variation with time when $L = 20$ m
5. 3. 2. Effects of the length of pendulum on the angular velocity $\theta_0 = 2.0$

Fig. 12a. Velocity variation with time when L = 5 m

Fig. 12b. Velocity variation with time when L = 10 m
Fig. 12c. Velocity variation with time when $L = 15$ m

Fig. 12d. Velocity variation with time when $L = 20$ m
Fig. 13. Velocity variation with time when $\theta = 1.0$

Fig. 14. Velocity variation with time when $\theta = 2.0$
The impacts of the length of pendulum and the amplitude of the oscillations on the angular velocity of the nonlinear simple pendulum are displayed in Figs. 11a-d and 13 when amplitude of the system is 1.0 while Figs. 12a-d and 14 show the effects of the length of pendulum and the amplitude of the oscillations on the angular velocity of the nonlinear simple pendulum when amplitude of the system is 1.0. The results depict that the maximum velocity of the system decreases as the length of the pendulum increases.

5.4. Verification of results

In this section, the comparison of the results of the differential transformation and numerical methods using fourth-fifth-order Runge–Kutta methods are presented. The extreme accuracy and validity of the solutions obtained by the differential transformation method are established as shown in the Figs. 15-20.

![Graph showing time response of the simple pendulum](image)

**Fig. 15.** Time response of the simple pendulum when L=5m and \( \theta_0 = 1.0 \)
Fig. 16. Time response of the simple pendulum when $L=5\text{m}$ and $\theta_0 = 2.0$

Fig. 17. Phase plot in the $(\theta(t), \frac{d\theta}{dt})$ plane when $L=5\text{m}$ and $\theta_0 = 1.0$
Fig. 18. Phase plot in the $(\theta(t), \frac{d\theta}{dt})$ plane when $L=5m$ and $\theta_0 = 2.0$

Fig. 19. Velocity variation with time when $L=5m$ and $\theta_0 = 1.0$
5.5. Results of the pendulum with a rotating plane

5.5.1. Effects of the length of pendulum and amplitude of oscillation on the angular displacement

This section presents the results of the effects of length of pendulum and the initial angular displacement or the amplitude of the oscillations on the angular displacement of the pendulum with a rotating plane.

5.5.2. Effects of the length of pendulum on the angular displacement of the pendulum
\( (\theta_0 = 1.0) \)

As the system is nonlinear, effects of nonlinearity on the frequency of vibration are analyzed. The source of nonlinearity which is the large amplitude of motion of the pendulum support. In the Figs. 21a-e, 22 and 23a-e, the dependency of system frequency on initial amplitude is shown. It can be easily established from the results that by increasing the initial amplitude of the nonlinear vibration, the system frequency is decreased.

Fig. 20. Velocity variation with time when L=5m and \( \theta_0 = 2.0 \)
Fig. 21a. Time response of the system when $\beta = 0.00$

Fig. 21b. Time response of the system when $\beta = 0.25$
Fig. 21c. Time response of the system when $\beta = 0.50$

Fig. 21d. Time response of the system when $\beta = 0.75$
Fig. 21e. Time response of the system when $\beta = 1.00$

![Graph showing time response of the system with different $\beta$ values](image)

Fig. 22. Effects of $\beta$ on the time response of the system
The effect of nonlinearity shows that as the resonance causes the amplitude of the motion to increase, the relation between period and amplitude (which is a characteristic effect of nonlinearity) causes the resonance to detune, decreasing its tendency to produce large motions. It is evident that when $\beta$ approaches zero, the nonlinear frequency approaches the linear frequency of the simple harmonic motion displayed in Eq. (3).

5. 5. 3. Effects of the length of pendulum on the angular frequency of the pendulum

$(\theta_a = 2.0)$

For a relatively large initial displacement value, it can be seen that the time-displacement graphs have a consistent harmonic pattern. From the resulting phase diagrams, it can be seen that the system behaviour is stable, in the mean that there is no system solution that leads to infinity or towards zero. This indicates that the method used can provide a good solution to nonlinear vibration of the harmonic oscillation equation. This is a pure harmonic oscillation and the equilibrium point is referred to as the center since the it is an undamped harmonic oscillator. With the aid of Lyapunov stability criteria (an equilibrium point in a nonlinear system is asymptotically Lyapunov stable if all eigenvalues of the linear variational equations have negative real parts and unstable if there exists at least one eigenvalue of the linear variational equations which has a positive real part).

![Graph](image_url)

**Fig. 23a.** Time response of the system when $\beta = 0.00$
Fig. 23b. Time response of the system when $\beta = 0.25$

Fig. 23c. Time response of the system when $\beta = 0.50$
Fig. 23d. Time response of the system when $\beta = 0.75$

Fig. 23e. Time response of the system when $\beta = 1.00$
Fig. 24. Effects of $\beta$ on the time response of the system

5. 5. 4. Effects of the length of pendulum and amplitude of oscillation on the phase plots ($\theta_0 = 1.0$)

Fig. 25a. Phase plot in the $(\theta(t), d\theta/dt)$ plane when $\beta = 0.00$
Fig. 25b. Phase plot in the $(\theta(t), d\theta/dt)$ plane when $\beta = 0.25$

Fig. 25c. Phase plot in the $(\theta(t), d\theta/dt)$ plane when $\beta = 0.50$
Fig. 25d. Phase plot in the $(\theta(t), d\theta/dt)$ plane when $\beta = 0.75$

Fig. 25e. Phase plot in the $(\theta(t), d\theta/dt)$ plane when $\beta = 1.00$
Fig. 26. Effects of $\beta$ on the Phase plot in the $(\theta(t), d\theta/dt)$ plane

5.5.5. Effects of the length of pendulum and amplitude of oscillation on the phase plots ($\theta_0 = 2.0$)

Fig. 27a. Phase plot in the $(\theta(t), d\theta/dt)$ plane when $\beta = 0.00$
Fig. 27b. Phase plot in the $(\theta(t), \frac{d\theta}{dt})$ plane when $\beta = 0.25$

Fig. 27c. Phase plot in the $(\theta(t), \frac{d\theta}{dt})$ plane when $\beta = 0.50$
Fig. 27d. Phase plot in the \( (\theta(t), \dot{\theta}(t)) \) plane when \( \beta = 0.75 \)

Fig. 27e. Phase plot in the \( (\theta(t), \dot{\theta}(t)) \) plane when \( \beta = 1.00 \)
Fig. 28. Effects of $\beta$ on the Phase plot in the $(\theta(t), \frac{d\theta}{dt})$ plane

Fig. 25-28 explain that starting points near the origin, and small velocities, the pendulum goes into a stable limit cycle as in.

5. 6. Phase plane diagrams for different trajectories of the pendulum with a rotating plane

Fig. 29. Phase plot for different trajectories in the $(\theta(t), \frac{d\theta}{dt})$ plane for the simple pendulum

-53-
**Fig. 30.** Phase plot for different trajectories in the \((\theta(t), \frac{d\theta}{dt})\) plane for the pendulum with a rotating plane when \(\beta = 1.0\)

**Fig. 31.** Phase plot for different trajectories in the \((\theta(t), \frac{d\theta}{dt})\) plane for the pendulum with a rotating plane when \(\beta = 2.0\)
Fig. 32. Phase plot for different trajectories in the $(\theta(t), d\theta/dt)$ plane for the pendulum with a rotating plane when $\beta = 3.0$.

Fig. 33. Phase plot for different trajectories in the $(\theta(t), d\theta/dt)$ plane for the pendulum with a rotating plane when $\beta = 4.0$.
Fig. 34. Phase plot for different trajectories in the $(\theta(t), \frac{d\theta}{dt})$ plane for the pendulum with a rotating plane when $\beta = 10.0$

The phase plane diagrams for different trajectories of the pendulum with a rotating plane are shown in Fig. 29-34. The arrows in the figures show the direction of motion of the two variables as the pendulum swings. The curves in the plots show particular ways that the pendulum swing.

The circular curves around the points $(\pi,0)$ and $(-\pi,0)$ really represent the same motion. It is shown that when a pendulum starts off with high enough velocity at $\theta = 0$, it goes all the way around. It is very obvious that its velocity will slow down on the way up but then it will speed up on the way down again. In such motion, the trajectory appears to fly off into $\theta$ space. $\theta$ is an angle that has values from $-\pi$ to $+\pi$. The $\theta$ data in this case is not wrapped around to be in this range. In the absence of damping or friction, it just keeps spinning around indefinitely.

The counterclockwise motions of the pendulum of this kind are shown in the graph by the wavy lines at the top that keep going from left to right indefinitely, while the curves on the bottom which go from right to left represent clockwise rotations. However, if damping or friction, the pendulum will gradually slow down, taking smaller and smaller swings, instead of swinging with fixed amplitude, back and forth. As presented in the phase-plots, the motion comes out as a spiral. In each swing, the pendulum angle $\theta$ goes to a maximum, then the pendulum stops momentarily, then swings back gaining speed. But the speed when it comes back to the middle is slightly less.

The result is that on the phase plot, it follows a spiral, getting closer and closer to stopping at $(0,0)$. This will be presented in our subsequent works.
6. CONCLUSIONS

In this work, the efficiency of differential transformation method is further displayed for the nonlinear oscillatory system has been demonstrated. The method was successfully implemented to illustrate its effectiveness and convenience to the nonlinear problems. Two cases of oscillation systems (nonlinear plane pendulum and pendulum with a rotating plane) were analyzed. Without any linearization, discretization or series expansion of the included sine and cosine of the angular displacement in the nonlinear models of the systems, the differential transformation method with Padé approximant was used to provide analytical solutions to the nonlinear problems.

Also, the increased predictive power and extreme accuracy of the modified differential transformation method in this present analysis over the previous work were further established. Additionally, the comparison of the results of the differential transformation method-Padé approximant with the corresponding numerical solutions obtained by the fourth-fifth-order Runge-Kutta method shows the extreme accuracy of the method. It was noted that the differential transformation method gives highly accurate results and avoids any numerical complexity. From the analysis, it could be stated that the method is more computationally efficient than the traditional perturbation method (regular and singular perturbation methods), He’s homotpy perturbation and variational iteration methods. It could be stated that the differential transformation method is more intuitive than the other methods as it gives more information about behaviour of the nonlinear systems even when chaos occurs. Therefore, it could be concluded that the differential transform method is very effective for solving the nonlinear equations for the behaviours of the physical problems and it can serve as a very useful mathematical tool for dealing with nonlinear equations.

References


