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A comparative study of differential transformation and homotopy perturbation methods for transient combustion analysis for iron micro-particles in a gaseous oxidizing medium

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ABSTRACT

In this paper, a comparative study of differential transformation and homotopy perturbation methods for transient combustion analysis of iron micro-particles in a gaseous oxidizing medium is presented. Also, parametric studies are carried out to properly understand the reaction of the process and the associated burning time. Thermal radiation effect from the external surface of burning particle and variation of iron particle density with temperature are considered. The solutions obtained by DTM and HPM are compared with those of the fourth order Runge-Kutta numerical method. Results show that DTM has more accurate results between the two approximate analytical methods considered. Also, results show that by increasing the heat realized parameter, combustion temperature increased and it faster reaches to its constant value. It is envisaged that the present study will create tremendous insight into means of properly managing combustible micro particles exiting factories and production process organization.

***Keywords*:** Iron particle combustion, Thermal radiation, Temperature distribution, Differential transformation method, Homotopy perturbation method

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1. INTRODUCTION

One of the most challenging in industries is combustion of metallic particles such as iron particles. In many of industrial applications, manufacture, process, generate, or use combustible dusts, an accurate knowledge of their explosion hazards is essential. Many researchers have worked on estimating and modeling the particle and dust combustion such as Haghiri and Bidbadi [1] which investigated the dynamic behavior of particles across flame propagation through a two-phase mixture consisting of micro-iron particles and air. They considered three zones for flame structure namely preheat, reaction, and post flame (burned) regions. Liu et al. [2] analyzed the flame propagation through hybrid mixture of coal dust and methane in a combustion chamber. A one-dimensional, steady-state theoretical analysis of flame propagation mechanism through micro-iron dust particles based on dust particles' behavior with special remark on the thermophoretic force in small Knudsen numbers is presented by Bidabadi et al. [3]. A mathematical model for analyzing the structure of flame propagating through a two-phase mixture consisting of organic fuel particles and air is performed by Haghiri and Bidabadi [4]. In contrast to previous analytical studies, they take thermal radiation effect in to consideration, which has not been attempted before. Recently, Hatami et al. [5] solved the nonlinear energy equation resulted from particle combustion modeling based on Bidabadi and Mafi'swork [6] by using differential transformation method (DTM) and BPES and they presented equations for calculating the convective heat transfer coefficient and burning time for iron particles. Polynomial expansion methods are extensively used in many mathematical and engineering fields to yield meaningful results for both numerical and analytical analysis. Among the most frequently used polynomials, weighted residual methods (WRMs) are one of the interesting tools due to their simplicity and high accuracy. Collocation, Galerkin and least square are examples of the WRMs. Stern and Rasmussen [7] used collocation method for solving a third order linear differential equation. Vaferi et al. [8] have studied the feasibility of applying of orthogonal collocation method to solve diffusivity equation in the radial transient flow system. Recently Hatami et al. [9] used collocation and Galerkin methods for heat transfer study through porous fins. Also least square method is introduced by Aziz and Bouaziz [10, 11] for predicting the performance of longitudinal fins. They found that least squares method is simple compared with other analytical methods. Shaoqin and Huoyuan[12] developed and analyzed least-squares approximations for the incompressible magneto-hydrodynamic equations also Hatami et al. [13-15], Hatami and Ganji [16-18], Hatami and Domairry [19,20] and Ahmadi et al. [21] applied these analytical methods in different engineering problems.

Motivated by above mentioned works, this paper aims to introduce two analytical methods for obtaining the temperature of iron particle during combustion, so DTM and HPM are applied. These methods have an excellent agreement with numerical Runge-Kutta method; also they have very low errors without any needing to perturbation or discretization compared to previous analytical methods in the literature

2. PROBLEM DESCRIPTION AND GOVERNING EQUATION

As seen in Fig. 1, consider an iron spherical particle which is combusted in the gaseous oxidizing medium due to high reaction with oxygen. The particle is considered to be isothermal and the Biot number is small (Bi_H << 0.1) due to high value of the thermal diffusivity of substance.

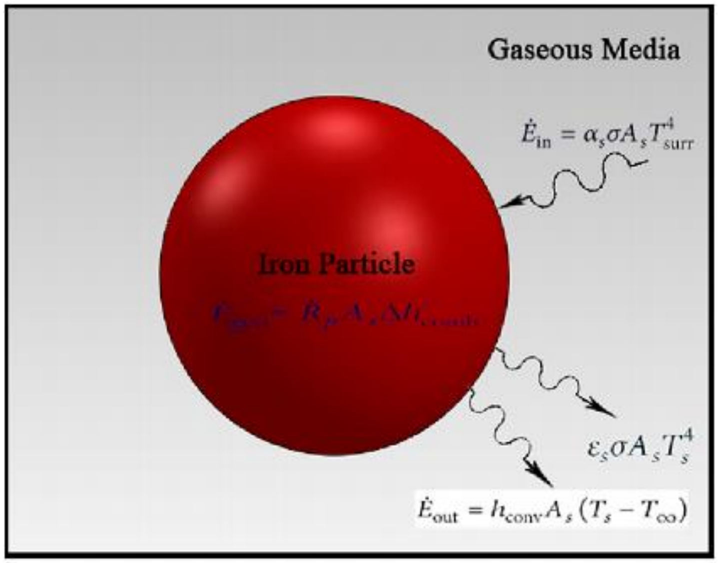

Figure 1. Schematic of combusted iron particle in gaseous media (Hatami et al 2014)

In this study, a lumped system analysis is applied. When this condition is satisfied, the variation of temperature with particle's radius will be not sensible and can be approximated as a constant value, so the particle's temperature is just a function of time, $T = T(t)$, and it is not a function of radial coordinate, $T \neq T(r)$. The assumptions used in this modeling are [5, 6]:

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_{gen} = \left(\frac{dE}{dt}\right)_p \tag{1}$$

where E_{in} is the rate of the energy entering the system which is due to absorption of total radiation occurred on the particle's surface from the surrounding, E_{out} is the rate of energy leaving the system by convection mechanisms on the particle's surface and thermal radiation which emits from the outer surface of particle, E_{gen} is the rate of energy generation inside the particle due to the combustion process and equals to the released heat from the chemical reaction, and (dE/dt) shows the rate of total energy changes in the particle. These energy terms can be calculated by following equations [3, 5]:

$$\dot{E}_{in} = \alpha_s \sigma A_s T^4_{surr} \tag{2}$$

$$\dot{E}_{out} = h_{conv} A_s \left(T_s - T_{\infty}\right) + \varepsilon_s \sigma A_s T_s^4 \tag{3}$$

$$\dot{E}_{gen} = \dot{Q}_{comb} = \dot{R}_p \, A_s \Delta h^o_{comb} \tag{4}$$

$$\left(\frac{dE}{dt}\right)_p = \rho_p V_p c_p \frac{dT_s}{dt} \tag{5}$$

by substituting the Eqs. (2) - (5) into Eq. (1),

$$\alpha_s \sigma A_s T_{surr}^4 - \left(h_{conv} A_s \left(T_s - T_\infty\right) + \varepsilon_s \sigma A_s T_s^4\right) + \dot{R}_p A_s \Delta h_{comb}^o = \rho_p V_p c_p \frac{dT_s}{dt} \tag{6}$$

Three reasonable assumptions are used to improve Eq. (6):

(I) Absorptivity and emissivity of the surface are temperature and radiation wavelength-dependent. Kirchhoff's law of radiation reveals that the absorptivity and the emissivity of a surface at a given temperature and wavelength are equal. ($\varepsilon_s = \alpha_s$).

(II) The initial temperature of the particle at the beginning of combustion can be regarded as the initial condition. This temperature is known as ignition temperature. ($T(0) = T_{ig}$)

(III) The density of particle is a function of particle's temperature, so it is considered as a linear function

$$\rho_p = \rho_p\left(T\right) = \rho_{p,\infty}\left[1 + \beta\left(T - T_\infty\right)\right]$$

by applying the above assumptions, Eq. (6) will be changed to the following,

$$\rho_{p,\infty}\left[1 + \beta\left(T - T_\infty\right)\right]\rho_p V_p c_p \frac{dT_s}{dt} + h_{conv} A_s \left(T_s - T_\infty\right) + \varepsilon_s \sigma A_s \left(T_s^4 - T_{surr}^4\right) - \dot{R}_p A_s \Delta h_{comb}^o = 0 \tag{7}$$

For solving this nonlinear differential equation, it is better to change it to a dimensionless form. So the following set of dimensionless variables are defined,

$$\begin{cases} \theta = \dfrac{T}{T_{ig}}, \quad \theta_\infty = \dfrac{T_\infty}{T_{ig}}, \quad \theta_{surr} = \dfrac{T_{surr}}{T_{ig}}, \quad \varepsilon_1 = \beta T_{ig} \\ \tau = \dfrac{t}{\left(\rho_{p,\infty} V_p c_p \big/ h_{onv} A_s\right)}, \quad \psi = \dfrac{\dot{Q}_{comb}}{h_{conv} A_s T_{ig}}, \quad \varepsilon_2 = \dfrac{\varepsilon_s \sigma T_{ig}^3}{h_{conv}} \end{cases} \tag{8}$$

Consequently, the nonlinear differential equation and its initial condition can be expressed in the dimensionless form as,

$$\varepsilon_1 \theta \frac{d\theta}{d\tau} + \left(1 - \varepsilon_1 \theta_\infty\right)\frac{d\theta}{d\tau} + \varepsilon_1\left(\theta^4 - \theta_{surr}^4\right) + \theta - \psi - \theta_\infty = 0, \tag{9}$$

and the initial condition is

$$\theta\left(0\right) = 1. \tag{10}$$

3. METHODS OF SOLUTION

Due to the nonlinear terms in Eq. (9), it is very difficult to develop a closed form or an exact analytical solution to the nonlinear equation. Therefore, the common practice is to make recourse to numerical method. However, in recent time, several semi- or approximate analytical methods have been developed to solve nonlinear equations. In this present study, the nonlinear equation in Eq. (9) is be solved analytically using differential transformation and homotopy perturbation methods

3. 1. Method of solution: Differential transformation method

As pointed previously, the differential transformation method is an approximate analytical method for solving differential equations. However, a closed form series solution or approximate solution can be obtained for non-linear differential equations with the use of DTM.

The basic definitions of the method is as follows**.**

If $u(t)$ is analytic in the domain T, then it will be differentiated continuously with respect to time *t.*

$$\frac{d^K u(t)}{dt^K} = \phi(t,k) \qquad \text{for} \qquad \text{all } t \in T \tag{11}$$

for $t = t_i$, then $\phi(t,k) = \phi(t_i, K)$, where K belongs to the set of non-negative integers, denoted as the K-domain. Therefore Eq. (11) can be rewritten as

$$U(k) = \phi(t_i, k) = \left[\frac{d^k u(t)}{dt^K}\right]_{t=t_i} \tag{12}$$

where U_k is called the spectrum of $u(t)$ at $t = t_i$

If $u(t)$ can be expressed by Taylor's series, the $u(t)$ can be represented as

$$u(t) = \sum_{k}^{\infty}\left[\frac{(t-t_i)^K}{k!}\right] U(k) \tag{13}$$

where Equ. (12) is called the inverse of $U(k)$ using the symbol 'D' denoting the differential transformation process and combining (12) and (13), it is obtained that

$$u(t) = \sum_{K=0}^{\infty}\left[\frac{(t-t_i)^K}{K!}\right] U(k) = D^{-1}U(k) \tag{14}$$

3. 1. 1. Operational properties of differential transformation method

If $u(t)$ *and* $v(t)$ are two independent functions with time (t) where $U(k)$ and $V(k)$ are the transformed function corresponding to $u(t)$ and $v(t)$, then it can be proved from the fundamental mathematics operations performed by differential transformation that.

i. If $z(t) = u(t) \pm v(t)$, then $Z(k) = U(k) \pm V(k)$

ii. If $z(t) = \alpha u(t)$, then $Z(k) = \alpha U(k)$

iii. If $z(t) = \dfrac{du(t)}{dt}$, then $Z(k) = (k-1)\,U(k+1)$

iv. If $z(t) = u(t)v(t)$, then $Z(t) = \sum_{i=0}^{K} V(l)U(k-l)$

v. If $z(t) = u^{m}(t)$, then $Z(t) = \sum_{I=0}^{K} U^{m-1}(l)U(k-l)$

vi. If $z(t) = u^{n}(t)v^{n}(t)$, then $Z(t) = \sum_{l-0}^{k} \left[\sum_{j=0}^{l} \left[V(j)U(l-j)\right] \;\; \sum_{j=0}^{k-l} \left[V(j)U(k-l-j)\right] \right]$

vii. If $z(t) = u(t)v(t)$, then $Z(k) = \sum_{l=0}^{k} (l+1)V(l+1)U(k-l)$

Using the operational properties above, the differential transformed DTM of Eq. (9) is

$$\check{n}_1 \sum_{l=0}^{k} (l+1)\theta[l+1]\theta[k-l] + (1-\check{n}_1\theta_\infty)(k+1)\theta[k+1] + \check{n}_2 \left(\sum_{p}^{k} \sum_{m}^{p} \sum_{l=0}^{m} \theta[l]\theta[m-l]\theta[p-m]\theta[k-p] - \theta_{surr}^{4}\delta[k] \right) + \theta[k] - \psi\delta[k] - \theta_\infty\delta[k] = 0 \tag{15}$$

with,

$$\theta[0] = 1$$

using the initial condition given, the leading term is obtain as

$$\theta_0 = 1$$

by varying the counter in the Eq, (9), the other terms may be obtained as:

$$\theta_1 = -\left(\theta_{surr}{}^{4}\check{n}_2 + \psi - \check{n}_2 + \theta_\infty - 1\right) / \left(\check{n}_1\theta_\infty - \check{n}_1 - 1\right)$$

$$\theta_2 = (1/2)\left(\theta_{surr}{}^{4}\check{n}_2 + \psi - \check{n}_2 + \theta_\infty - 1\right)(\theta_{surr}{}^{4}\check{n}_1\check{n}_2 - 4\check{n}_1\check{n}_2\theta_\infty + \psi\check{n}_1 + 3\check{n}_1\check{n}_2 + 4\check{n}_2 + 1) / \left(\check{n}_1\theta_\infty - \check{n}_1 - 1\right)^3$$

$$
\begin{aligned}
\theta_3 = & -(1/6)\left(\theta_{surr}^{4}\check{n}_2+\psi-\check{n}_2+\theta_\infty-1\right)(3\theta_{surr}^{8}\check{n}_1^2\check{n}_2^2-12\theta_{surr}^{4}\check{n}_1^2\check{n}_2^2\theta_\infty^2+8\theta_{surr}^{4}\check{n}_1^2\check{n}_2^2\theta_\infty+6\psi\theta_{surr}^{4}\check{n}_1^2\check{n}_2-2\theta_{surr}^{4}\check{n}_1^2\check{n}_2^2+\\
& 2\theta_{surr}^{4}\check{n}_1^2\check{n}_2\theta_\infty+24\theta_{surr}^{4}\check{n}_1\check{n}_2^2\theta_\infty-2\theta_{surr}^{4}\check{n}_1^2\check{n}_2-8\theta_{surr}^{4}\check{n}_1\check{n}_2^2-12\psi\check{n}_1^2\check{n}_2\theta_\infty^2+4\theta_{surr}^{4}\check{n}_1\check{n}_2-12\theta_{surr}^{4}\check{n}_2^2+28\check{n}_1^2\check{n}_2^2\theta_\infty^2-\\
& 12\check{n}_1^2\check{n}_2\theta_\infty^3+8\psi\check{n}_1^2\check{n}_2\theta_\infty-40\check{n}_1^2\check{n}_2^2\theta_\infty+28\check{n}_1^2\check{n}_2\theta_\infty^2+3\psi^2\check{n}_1^2-2\psi\check{n}_1^2\check{n}_2+2\psi\check{n}_1^2\theta_\infty+24\psi\check{n}_1\check{n}_2\theta_\infty+15\check{n}_1^2\check{n}_2^2-\\
& 22\check{n}_1^2\check{n}_2\theta_\infty-56\check{n}_1\check{n}_2^2\theta_\infty+24\check{n}_1\check{n}_2\theta_\infty^2-2\psi\check{n}_1^2-8\psi\check{n}_1\check{n}_2+6\check{n}_1^2\check{n}_2+40\check{n}_1\check{n}_2^2-48\check{n}_1\check{n}_2\theta_\infty+4\psi\check{n}_1-12\psi\check{n}_2+20\check{n}_1\check{n}_2+\\
& 2\check{n}_1\theta_\infty+28\check{n}_2^2-12\check{n}_2\theta_\infty-2\check{n}_1+20\check{n}_2+1)/\left(\check{n}_1\theta_\infty-\check{n}_1-1\right)^5
\end{aligned}
$$

$$
\begin{aligned}
\theta_4 = & (1/24)\left(\theta_{surr}^{4}\check{n}_2+\psi-\check{n}_2+\theta_\infty-1\right)(15\theta_{surr}^{12}\check{n}_1^3\check{n}_2^3-24\theta_{surr}^{8}\check{n}_1^3\check{n}_2^3\theta_\infty^3-12\theta_{surr}^{8}\check{n}_1^3\check{n}_2^3\theta_\infty^2-4\theta_{surr}^{8}\check{n}_1^3\check{n}_2^3\theta_\infty+72\theta_{surr}^{8}\check{n}_1^2\check{n}_2^3\theta_\infty^2+\\
& 45\psi\theta_{surr}^{8}\check{n}_1^3\check{n}_2^2-5\theta_{surr}^{8}\check{n}_1^3\check{n}_2^3+20\theta_{surr}^{8}\check{n}_1^3\check{n}_2^2\theta_\infty+24\theta_{surr}^{8}\check{n}_1^2\check{n}_2^3\theta_\infty-48\psi\theta_{surr}^{4}\check{n}_1^3\check{n}_2^2\theta_\infty^3-20\theta_{surr}^{8}\check{n}_1^3\check{n}_2^2+4\theta_{surr}^{8}\check{n}_1^2\check{n}_2^3-\\
& 72\theta_{surr}^{8}\check{n}_1\check{n}_2^3\theta_\infty+240\theta_{surr}^{4}\check{n}_1^3\check{n}_2^3\theta_\infty^3-48\theta_{surr}^{4}\check{n}_1^3\check{n}_2^2\theta_\infty^4-24\psi\theta_{surr}^{4}\check{n}_1^3\check{n}_2^2\theta_\infty^2+25\theta_{surr}^{8}\check{n}_1^2\check{n}_2^2-12\theta_{surr}^{8}\check{n}_1\check{n}_2^3-376\theta_{surr}^{4}\check{n}_1^3\check{n}_2^3\theta_\infty^2+\\
& 72\theta_{surr}^{4}\check{n}_1^3\check{n}_2^2\theta_\infty^3-8\psi\theta_{surr}^{4}\check{n}_1^3\check{n}_2^2\theta_\infty+144\psi\theta_{surr}^{4}\check{n}_1^2\check{n}_2^2\theta_\infty^2+24\theta_{surr}^{8}\check{n}_2^3+232\theta_{surr}^{4}\check{n}_1^3\check{n}_2^3\theta_\infty-40\theta_{surr}^{4}\check{n}_1^3\check{n}_2^2\theta_\infty^2-720\theta_{surr}^{4}\check{n}_1^2\check{n}_2^3\theta_\infty^2+\\
& 144\theta_{surr}^{4}\check{n}_1^2\check{n}_2^2\theta_\infty^3+45\psi^2\theta_{surr}^{4}\check{n}_1^3\check{n}_2-10\psi\theta_{surr}^{4}\check{n}_1^3\check{n}_2^2+40\psi\theta_{surr}^{4}\check{n}_1^3\check{n}_2\theta_\infty+48\psi\theta_{surr}^{4}\check{n}_1^2\check{n}_2^2\theta_\infty-51\theta_{surr}^{4}\check{n}_1^3\check{n}_2^3+16\theta_{surr}^{4}\check{n}_1^3\check{n}_2^2\theta_\infty+\\
& 6\theta_{surr}^{4}\check{n}_1^3\check{n}_2\theta_\infty^2+752\theta_{surr}^{4}\check{n}_1^2\check{n}_2^3\theta_\infty-240\theta_{surr}^{4}\check{n}_1^2\check{n}_2^2\theta_\infty^2-24\psi^2\check{n}_1^3\check{n}_2\theta_\infty^3-40\psi\theta_{surr}^{4}\check{n}_1^3\check{n}_2+8\psi\theta_{surr}^{4}\check{n}_1^2\check{n}_2^2-144\psi\theta_{surr}^{4}\check{n}_1\check{n}_2^2\theta_\infty+\\
& 240\psi\check{n}_1^3\check{n}_2^2\theta_\infty^3-48\psi\check{n}_1^3\check{n}_2\theta_\infty^4-12\theta_{surr}^{4}\check{n}_1^3\check{n}_2\theta_\infty-232\theta_{surr}^{4}\check{n}_1^2\check{n}_2^3+72\theta_{surr}^{4}\check{n}_1^2\check{n}_2^2\theta_\infty+720\theta_{surr}^{4}\check{n}_1\check{n}_2^3\theta_\infty-144\theta_{surr}^{4}\check{n}_1\check{n}_2^2\theta_\infty^2-\\
& 280\check{n}_1^3\check{n}_2^3\theta_\infty^3+240\check{n}_1^3\check{n}_2^2\theta_\infty^4-24\check{n}_1^3\check{n}_2\theta_\infty^5-12\psi^2\check{n}_1^3\check{n}_2\theta_\infty^2+50\psi\theta_{surr}^{4}\check{n}_1^2\check{n}_2-24\psi\theta_{surr}^{4}\check{n}_1\check{n}_2^2-376\psi\check{n}_1^3\check{n}_2^2\theta_\infty^2+72\psi\check{n}_1^3\check{n}_2\theta_\infty^3+\\
& 6\theta_{surr}^{4}\check{n}_1^3\check{n}_2-26\theta_{surr}^{4}\check{n}_1^2\check{n}_2^2+28\theta_{surr}^{4}\check{n}_1^2\check{n}_2\theta_\infty-376\theta_{surr}^{4}\check{n}_1\check{n}_2^3+264\theta_{surr}^{4}\check{n}_1\check{n}_2^2\theta_\infty+580\check{n}_1^3\check{n}_2^3\theta_\infty^2-712\check{n}_1^3\check{n}_2^2\theta_\infty^3+84\check{n}_1^3\check{n}_2\theta_\infty^4-\\
& 4\psi^2\check{n}_1^3\check{n}_2\theta_\infty+72\psi^2\check{n}_1^2\check{n}_2\theta_\infty^2+48\psi\theta_{surr}^{4}\check{n}_2^2+232\psi\check{n}_1^3\check{n}_2^2\theta_\infty-40\psi\check{n}_1^3\check{n}_2\theta_\infty^2-720\psi\check{n}_1^2\check{n}_2^2\theta_\infty^2+144\psi\check{n}_1^2\check{n}_2\theta_\infty^3-28\theta_{surr}^{4}\check{n}_1^2\check{n}_2-\\
& 32\theta_{surr}^{4}\check{n}_1\check{n}_2^2-240\theta_{surr}^{4}\check{n}_2^3+48\theta_{surr}^{4}\check{n}_2^2\theta_\infty-48\psi\theta_{surr}^{4}\check{n}_2^2+232\psi\check{n}_1^3\check{n}_2^2\theta_\infty-40\psi\check{n}_1^3\check{n}_2\theta_\infty^2-720\psi\check{n}_1^2\check{n}_2^2\theta_\infty^2+144\psi\check{n}_1^2\check{n}_2\theta_\infty^3-\\
& 28\theta_{surr}^{4}\check{n}_1^2\check{n}_2-32\theta_{surr}^{4}\check{n}_1\check{n}_2^2-240\theta_{surr}^{4}\check{n}_2^3+48\theta_{surr}^{4}\check{n}_2^2\theta_\infty-420\check{n}_1^3\check{n}_2^3\theta_\infty+808\check{n}_1^3\check{n}_2^2\theta_\infty^2-120\check{n}_1^3\check{n}_2\theta_\infty^3+840\check{n}_1^2\check{n}_2^3\theta_\infty^2-\\
& 720\check{n}_1^2\check{n}_2^2\theta_\infty^3+72\check{n}_1^2\check{n}_2\theta_\infty^4+15\psi^3\check{n}_1^3-5\psi^2\check{n}_1^3\check{n}_2+20\psi^2\check{n}_1^3\theta_\infty+24\psi^2\check{n}_1^2\check{n}_2\theta_\infty-51\psi\check{n}_1^3\check{n}_2^2+16\psi\check{n}_1^3\check{n}_2\theta_\infty+6\psi\check{n}_1^3\theta_\infty^2+\\
& 752\psi\check{n}_1^2\check{n}_2^2\theta_\infty-240\psi\check{n}_1^2\check{n}_2\theta_\infty^2+11\theta_{surr}^{4}\check{n}_1\check{n}_2-96\theta_{surr}^{4}\check{n}_2^2+105\check{n}_1^3\check{n}_2^3-420\check{n}_1^3\check{n}_2^2\theta_\infty+90\check{n}_1^3\check{n}_2\theta_\infty^2-1160\check{n}_1^2\check{n}_2^3\theta_\infty+1760\check{n}_1^2\check{n}_2^2\theta_\infty^2-\\
& 264\check{n}_1^2\check{n}_2\theta_\infty^3-20\psi^2\check{n}_1^3+4\psi^2\check{n}_1^2\check{n}_2-72\psi^2\check{n}_1\check{n}_2\theta_\infty-12\psi\check{n}_1^3\theta_\infty-232\psi\check{n}_1^2\check{n}_2^2+72\psi\check{n}_1^2\check{n}_2\theta_\infty+720\psi\check{n}_1\check{n}_2^2\theta_\infty-144\psi\check{n}_1\check{n}_2\theta_\infty^2+\\
& 84\check{n}_1^3\check{n}_2^2-36\check{n}_1^3\check{n}_2\theta_\infty+420\check{n}_1^2\check{n}_2^3-1384\check{n}_1^2\check{n}_2^2\theta_\infty+320\check{n}_1^2\check{n}_2\theta_\infty^2-840\check{n}_1\check{n}_2^3\theta_\infty+720\check{n}_1\check{n}_2^2\theta_\infty^2-72\check{n}_1\check{n}_2\theta_\infty^3+25\psi^2\check{n}_1^2-12\psi^2\check{n}_1\check{n}_2+\\
& 6\psi\check{n}_1^3-26\psi\check{n}_1^2\check{n}_2+28\psi\check{n}_1^2\theta_\infty-376\psi\check{n}_1\check{n}_2^2+264\psi\check{n}_1\check{n}_2\theta_\infty+6\check{n}_1^3\check{n}_2+369\check{n}_1^2\check{n}_2^2-164\check{n}_1^2\check{n}_2\theta_\infty+6\check{n}_1^2\theta_\infty^2+580\check{n}_1\check{n}_2^3-1384\check{n}_1\check{n}_2^2\theta_\infty+\\
& 276\check{n}_1\check{n}_2\theta_\infty^2+24\psi^2\check{n}_2-28\psi\check{n}_1^2-32\psi\check{n}_1\check{n}_2-240\psi\check{n}_2^2+48\psi\check{n}_2\theta_\infty+36\check{n}_1^2\check{n}_2-12\check{n}_1^2\theta_\infty+576\check{n}_1\check{n}_2^2-284\check{n}_1\check{n}_2\theta_\infty+280\check{n}_2^3-\\
& 240\check{n}_2^2\theta_\infty+24\check{n}_2\theta_\infty^2+11\psi\check{n}_1-96\psi\check{n}_2+6\check{n}_1^2+69\check{n}_1\check{n}_2+8\check{n}_1\theta_\infty+336\check{n}_2^2-96\check{n}_2\theta_\infty-8\check{n}_1+84\check{n}_2+1)/\left(\check{n}_1\theta_\infty-\check{n}_1-1\right)^7
\end{aligned}
$$

From the definition in Eq. (14), the DTM series solution may be expressed as:

$$\theta(\tau)=\theta_0+\theta_1+\theta_2+\theta_3+\theta_4+\ldots \qquad (16)$$

which is given in the appendix

3. 2. Homotopy perturbation method (HPM)

3. 2. 1. Method of solution by Homotopy perturbation method

The comparative advantages and the provision of acceptable analytical results with convenient convergence and stability coupled with total analytic procedures of homotopy perturbation method compel us to consider the method for solving the system of nonlinear differential equations in Eqs. (9).

3. 2. 2. The basic idea of homotopy perturbation method

In order to establish the basic idea behind homotopy perturbation method, consider a system of nonlinear differential equations given as

$$A(U) - f(r) = 0, \quad r \in \Omega, \tag{17}$$

with the boundary conditions

$$B\left(u, \frac{\partial u}{\partial \eta}\right) = 0, \qquad r \in \Gamma, \tag{18}$$

where *A* is a general differential operator, *B* is a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω

The operator *A* can be divided into two parts, which are *L* and *N*, where *L* is a linear operator, *N* is a non-linear operator. Eq. (16) can be therefore rewritten as follows

$$L(u) + N(u) - f(r) = 0. \tag{19}$$

by the homotopy technique, a homotopy $U(r,p): \Omega \times [0,1] \to R$ can be constructed, which satisfies

$$H(U,p) = (1-p)\left[L(U) - L(U_o)\right] + p\left[A(U) - f(r)\right] = 0, \quad p \in [0,1], \tag{20}$$

or

$$H(U,p) = L(U) - L(U_o) + pL(U_o) + p\left[N(U) - f(r)\right] = 0. \tag{21}$$

In the above Eqs. (20) and (21), $p \in [0,1]$ is an embedding parameter, u_o is an initial approximation of equation of Eq. (16), which satisfies the boundary conditions.

Also, from Eqs. (20) and Eq. (21), we will have

$$H(U,0) = L(U) - L(U_o) = 0, \tag{22}$$

or

$$H(U,0) = A(U) - f(r) = 0. \tag{23}$$

The changing process of *p* from zero to unity is just that of $U(r,p)$ from $u_o(r)$ to $u(r)$ This is referred to homotopy in topology. Using the embedding parameter *p* as a small parameter, the solution of Eqs. (20) and Eq. (21) can be assumed to be written as a power series in p as given in Eq. (24)

$$U = U_o + pU_1 + p^2U_2 + \ldots \tag{24}$$

It should be pointed out that of all the values of p between 0 and 1, $p=1$ produces the best result. Therefore, setting $p = 1$, results in the approximation solution of Eq. (9)

$$u = \lim_{p\to 1} U = U_o + U_1 + U_2 + \ldots \tag{25}$$

The basic idea expressed above is a combination of homotopy and perturbation method. Hence, the method is called homotopy perturbation method (HPM), which has eliminated the limitations of the traditional perturbation methods. On the other hand, this technique can have full advantages of the traditional perturbation techniques. The series Eq. (25) is convergent for most cases.

Let,

$$\theta(\tau) = \theta_0 + p\theta_1 + p^2\theta_2 + p^3\theta_3 + p^4\theta_4 + p^5\theta_5 + p^6\theta_6 + p^7\theta_7 \tag{26}$$

Substituting into Eq. (9) and applying the principle of Homotopy perturbation,

$$HPMEq:(1-p)\left((-\check{n}_1\theta_\infty+1)(\dot\theta_0+p\dot\theta_1+p^2\dot\theta_2+p^3\dot\theta_3+p^4\dot\theta_4+p^5\dot\theta_5+p^6\dot\theta_6+p^7\dot\theta_7)\right)+$$
$$p\begin{pmatrix}(-\check{n}_1\theta_\infty+1)(\dot\theta_0+p\dot\theta_1+p^2\dot\theta_2+p^3\dot\theta_3+p^4\dot\theta_4+p^5\dot\theta_5+p^6\dot\theta_6+p^7\dot\theta_7)+\\ \check{n}_1(\theta_0+p\theta_1+p^2\theta_2+p^3\theta_3+p^4\theta_4+p^5\theta_5+p^6\theta_6+p^7\theta_7)(\dot\theta_0+p\dot\theta_1+p^2\dot\theta_2+p^3\dot\theta_3+p^4\dot\theta_4+p^5\dot\theta_5+p^6\dot\theta_6+p^7\dot\theta_7)+\\ \check{n}_2((\theta_0+p\theta_1+p^2\theta_2+p^3\theta_3+p^4\theta_4+p^5\theta_5+p^6\theta_6+p^7\theta_7)^4-\theta_{surr}{}^4)+\theta_0+p\theta_1+p^2\theta_2+p^3\theta_3+p^4\theta_4+p^5\theta_5+p^6\theta_6+p^7\theta_7-\psi-\theta_\infty\end{pmatrix} \tag{27}$$

The resulting equations based on the power of p are,

$$p^0:(-\check{n}_1\theta_\infty+1)\dot\theta_0=0 \tag{28}$$

$$p^1:(-\check{n}_1\theta_\infty+1)\dot\theta_1+(\check{n}_1\theta_\infty-1)\dot\theta_0+(-\check{n}_1\theta_\infty+1)\dot\theta_0+\check{n}_1\theta_0\dot\theta_0+\check{n}_2\left(\theta_0{}^4-\theta_{surr}{}^4\right)+\theta_0-\psi-\theta_\infty=0 \tag{29}$$

$$p^2:(-\check{n}_1\theta_\infty+1)\dot\theta_2+(\check{n}_1\theta_\infty-1)\dot\theta_1+(-\check{n}_1\theta_\infty+1)\dot\theta_1+\check{n}_1\theta_0\dot\theta_1+\check{n}_1\theta_1\dot\theta_0+4\check{n}_2\theta_0{}^3\theta_1+\theta_1=0 \tag{30}$$

$$\begin{aligned}p^3:&(-\check{n}_1\theta_\infty+1)\dot\theta_3+(\check{n}_1\theta_\infty-1)\dot\theta_2+(-\check{n}_1\theta_\infty+1)\dot\theta_2+\check{n}_1\theta_0\dot\theta_2+\check{n}_1\theta_1\dot\theta_1+\check{n}_1\theta_2\dot\theta_0+\\ &\check{n}_2(2\theta_0{}^2\left(2\theta_0\theta_2+\theta_1{}^2\right)+4\theta_0{}^2\theta_1{}^2)+\theta_2=0\end{aligned} \tag{31}$$

$$\begin{aligned}p^4:&(-\check{n}_1\theta_\infty+1)\dot\theta_4+(\check{n}_1\theta_\infty-1)\dot\theta_3+(-\check{n}_1\theta_\infty+1)\dot\theta_3+\check{n}_1\theta_0\dot\theta_3+\check{n}_1\theta_1\dot\theta_2+\check{n}_1\theta_2\dot\theta_1+\check{n}_1\theta_3\dot\theta_0+\\ &\check{n}_2(2\theta_0{}^2\left(2\theta_0\theta_3+2\theta_1\theta_2\right)+4\theta_0\theta_1\left(2\theta_0\theta_2+\theta_1{}^2\right))+\theta_3=0\end{aligned} \tag{32}$$

$$p^5 : \left(-\check{n}_1\theta_\infty + 1\right)\dot{\theta}_5 + \left(\check{n}_1\theta_\infty - 1\right)\dot{\theta}_4 + \left(-\check{n}_1\theta_\infty + 1\right)\dot{\theta}_4 + \check{n}_1\theta_0\dot{\theta}_4 + \check{n}_1\theta_1\dot{\theta}_3 + \check{n}_1\theta_2\dot{\theta}_2 + \check{n}_1\theta_3\dot{\theta}_1 + \check{n}_1\theta_4\dot{\theta}_0 + \check{n}_2(2\theta_0{}^2\left(2\theta_0\theta_4 + 2\theta_1\theta_3 + \theta_2{}^2\right) + 4\theta_0\theta_1\left(2\theta_0\theta_3 + 2\theta_1\theta_2\right) + \left(2\theta_0\theta_2 + \theta_1{}^2\right)^2) + \theta_4 = 0 \tag{33}$$

$$p^6 : \left(-\check{n}_1\theta_\infty + 1\right)\dot{\theta}_6 + \left(\check{n}_1\theta_\infty - 1\right)\dot{\theta}_5 + \left(-\check{n}_1\theta_\infty + 1\right)\dot{\theta}_5 + \check{n}_1\theta_0\dot{\theta}_5 + \check{n}_1\theta_1\dot{\theta}_4 + \check{n}_1\theta_2\dot{\theta}_3 + \check{n}_1\theta_3\dot{\theta}_2 + \check{n}_1\theta_4\dot{\theta}_1 + \check{n}_1\theta_5\dot{\theta}_0 + \check{n}_2(2\theta_0{}^2\left(2\theta_0\theta_5 + 2\theta_1\theta_4 + 2\theta_2\theta_3\right) + 4\theta_0\theta_1\left(2\theta_0\theta_4 + 2\theta_1\theta_3 + \theta_2{}^2\right) + 2\left(2\theta_0\theta_2 + \theta_1{}^2\right)\left(2\theta_0\theta_3 + 2\theta_1\theta_2\right)) + \theta_5 = 0 \tag{34}$$

$$p^7 : \left(-\check{n}_1\theta_\infty + 1\right)\dot{\theta}_7 + \left(\check{n}_1\theta_\infty - 1\right)\dot{\theta}_6 + \left(-\check{n}_1\theta_\infty + 1\right)\dot{\theta}_6 + \check{n}_1\theta_0\dot{\theta}_6 + \check{n}_1\theta_1\dot{\theta}_5 + \check{n}_1\theta_2\dot{\theta}_4 + \check{n}_1\theta_3\dot{\theta}_3 + \check{n}_1\theta_4\dot{\theta}_2 + \check{n}_1\theta_5\dot{\theta}_1 + \check{n}_1\theta_6\dot{\theta}_0 + \check{n}_2(2\theta_0{}^2(2\theta_0\theta_6 + 2\theta_1\theta_5 + 2\theta_2\theta_4 + \theta_3{}^2) + 4\theta_0\theta_1\left(2\theta_0\theta_5 + 2\theta_1\theta_4 + 2\theta_2\theta_3\right) + 2\left(2\theta_0\theta_2 + \theta_1{}^2\right)\left(2\theta_0\theta_4 + 2\theta_1\theta_3 + \theta_2{}^2\right) + \left(2\theta_0\theta_3 + 2\theta_1\theta_2\right)^2) + \theta_6 = 0 \tag{35}$$

from the initial condition,

$$\theta_0 = 1$$

After solving the above equations, the resulting solutions are:

$$\theta_1 = -\left(\theta_{surr}{}^4\check{n}_2 + \psi - \check{n}_2 + \theta_\infty - 1\right)\tau / \left(\check{n}_1\theta_\infty - 1\right)$$

$$\theta_2 = -\left(\theta_{surr}{}^4\check{n}_2 + \psi - \check{n}_2 + \theta_\infty - 1\right)\left(2\check{n}_2\tau^2 + \left(1/2\right)\tau^2 + \check{n}_1\tau\right) / \left(\check{n}_1\theta_\infty - 1\right)^2$$

$$\theta_3 = \left(1/2\right)\left(\theta_{surr}{}^4\check{n}_2 + \psi - \check{n}_2 + \theta_\infty - 1\right)(4\tau^3\theta_{surr}{}^4\check{n}_2{}^2 + \tau^2\theta_{surr}{}^4\check{n}_1\check{n}_2 + 4\psi\tau^3\check{n}_2 - \left(28/3\right)\tau^3\check{n}_2{}^2 + 4\tau^3\check{n}_2\theta_\infty + \psi\tau^2\check{n}_1 - \left(20/3\right)\check{n}_2\tau^3 9\tau^2\check{n}_1\check{n}_2 + \tau^2\check{n}_1\theta_\infty - \left(1/3\right)\tau^3 - 3\check{n}_1\tau^2 - 2\check{n}_1{}^2\tau) / \left(\check{n}_1\theta_\infty - 1\right)^3$$

$$\theta_4 = -\left(1/6\right)\left(\theta_{surr}{}^4\check{n}_2 + \psi - \check{n}_2 + \theta_\infty - 1\right)((1/4)\tau^4 - 24\tau^4\theta_{surr}{}^4\check{n}_2{}^2 + 6\psi^2\tau^4\check{n}_2 - 60\psi\tau^4\check{n}_2{}^2 - 60\tau^4\check{n}_2{}^2\theta_\infty + 6\tau^4\check{n}_2\theta_\infty{}^2 - 24\psi\tau^4\check{n}_2 - 24\tau^4\check{n}_2\theta_\infty + 100\tau^3\check{n}_1\check{n}_2{}^2 - 4\psi\tau^3\check{n}_1 - 9\psi\tau^2\check{n}_1{}^2 + 80\tau^3\check{n}_1\check{n}_2 - 4\tau^3\check{n}_1\theta_\infty + 45\tau^2\check{n}_1{}^2\check{n}_2 - 9\tau^2\check{n}_1{}^2\theta_\infty + 6\tau^4\theta_{surr}{}^8\check{n}_2{}^3 - 60\tau^4\theta_{surr}{}^4\check{n}_2{}^3 + 70\tau^4\check{n}_2{}^3 + 84\tau^4\check{n}_2{}^2 + 21\check{n}_2\tau^4 + 7\check{n}_1\tau^3 + 18\check{n}_1{}^2\tau^2 + 6\check{n}_1{}^3\tau + 12\psi\tau^4\theta_{surr}{}^4\check{n}_2{}^2 + 12\tau^4\theta_{surr}{}^4\check{n}_2{}^2\theta_\infty - 52\tau^3\theta_{surr}{}^4\check{n}_1\check{n}_2{}^2 - 4\tau^3\theta_{surr}{}^4\check{n}_1\check{n}_2 - 9\tau^2\theta_{surr}{}^4\check{n}_1{}^2\check{n}_2 + 12\psi\tau^4\check{n}_2\theta_\infty - 52\psi\tau^3\check{n}_1\check{n}_2 - 52\tau^3\check{n}_1\check{n}_2\theta_\infty) / \left(\check{n}_1\theta_\infty - 1\right)^4 ((\left(\theta_{surr}{}^4\check{n}_2 + \psi - \check{n}_2 + \theta_\infty - 1\right)((1/4)\tau^4 - 24\tau^4\theta_{surr}{}^4\check{n}_2{}^2 + 6\psi^2\tau^4\check{n}_2 - 60\psi\tau^4\check{n}_2{}^2 - 60\tau^4\check{n}_2{}^2\theta_\infty + 6\tau^4\check{n}_2\theta_\infty{}^2 - 24\psi\tau^4\check{n}_2 - 24\tau^4\check{n}_2\theta_\infty + 100\tau^3\check{n}_1\check{n}_2{}^2 - 4\psi\tau^3\check{n}_1 - 9\psi\tau^2\check{n}_1{}^2 + 80\tau^3\check{n}_1\check{n}_2 - 4\tau^3\check{n}_1\theta_\infty + 45\tau^2\check{n}_1{}^2\check{n}_2 - 9\tau^2\check{n}_1{}^2\theta_\infty + 6\tau^4\theta_{surr}{}^8\check{n}_2{}^3 - 60\tau^4\theta_{surr}{}^4\check{n}_2{}^3 + 70\tau^4\check{n}_2{}^3 + 84\tau^4\check{n}_2{}^2 + 21\check{n}_2\tau^4 + 7\check{n}_1\tau^3 + 18\check{n}_1{}^2\tau^2 + 6\check{n}_1{}^3\tau + 12\psi\tau^4\theta_{surr}{}^4\check{n}_2{}^2 + 12\tau^4\theta_{surr}{}^4\check{n}_2{}^2\theta_\infty - 52\tau^3\theta_{surr}{}^4\check{n}_1\check{n}_2{}^2 - 4\tau^3\theta_{surr}{}^4\check{n}_1\check{n}_2 - 9\tau^2\theta_{surr}{}^4\check{n}_1{}^2\check{n}_2 + 12\psi\tau^4\check{n}_2\theta_\infty - 52\psi\tau^3\check{n}_1\check{n}_2 - 52\tau^3\check{n}_1\check{n}_2\theta_\infty))$$

Due to space for the large number of terms, the solutions of Eq. (33)-(35) are given in the appendix

From the definition in Eq. (24),

$$\theta(\tau) = \theta_0 + p\theta_1 + p^2\theta_2 + p^3\theta_3 + p^4\theta_4 + p^5\theta_5 + p^6\theta_6 + p^7\theta_7 \tag{36}$$

From the principle of HPM, *p = 1*,

The above series solution becomes

$$\theta(\tau) = \theta_0 + \theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5 + \theta_6 + \theta_7 \tag{37}$$

4. RESULTS AND DISCUSSION

Fig. (2 - 4) depict the verification of the two analytical scheme used with a numerical forth order Runge-Kutta. The schemes, DTM and HPM were first verified individual as shown in Fig. (2 - 3) and a good agreement with the numerical method was obtained. In order to visualize and determine the scheme with less error, a super-imposed plot which shows the temperature profile of a 20μm combusting iron particle is inspected as shown in Fig. 3 together with table 1. From the figure, it is evident that DTM gives a better result than HPM even though both methods are efficient for the problem in concern.

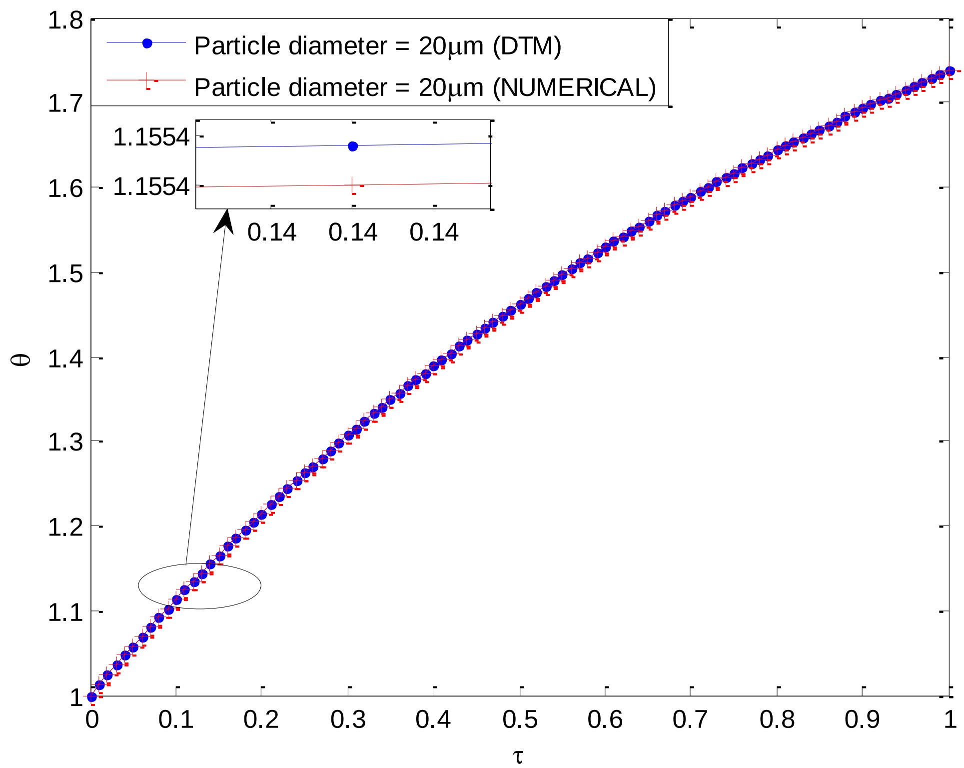

Figure 2. Verification of DTM with Numerical

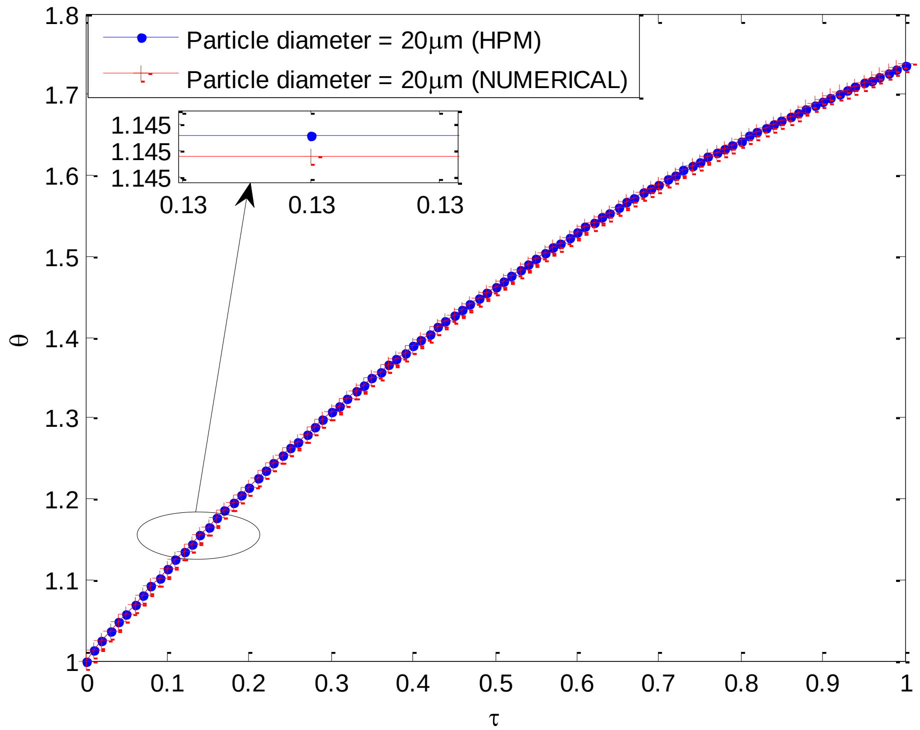

Figure 3. Verification of HPM with Numerical

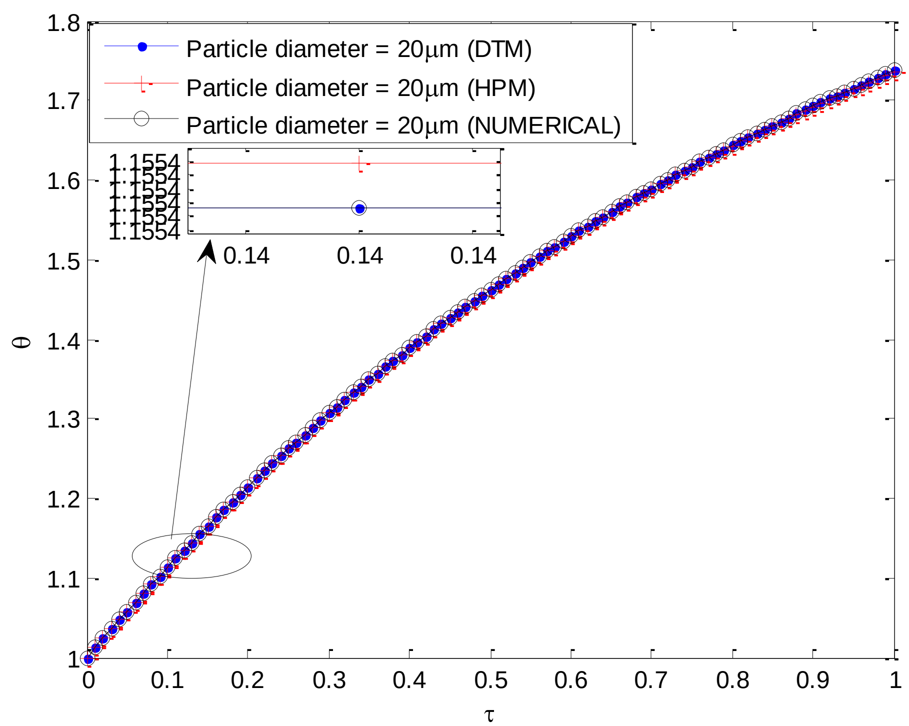

Figure 4. Verification of DTM and HPM with Numerical

4. 1. Effect of particle diameter on the temperature history

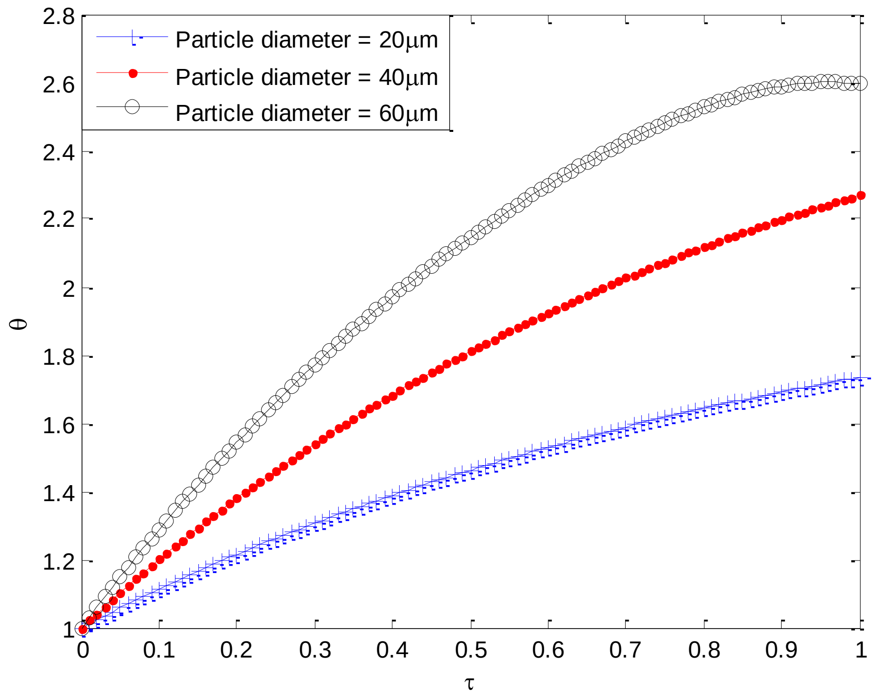

Figure 5. Effect of particle diameter on the temperature. Profile with DTM

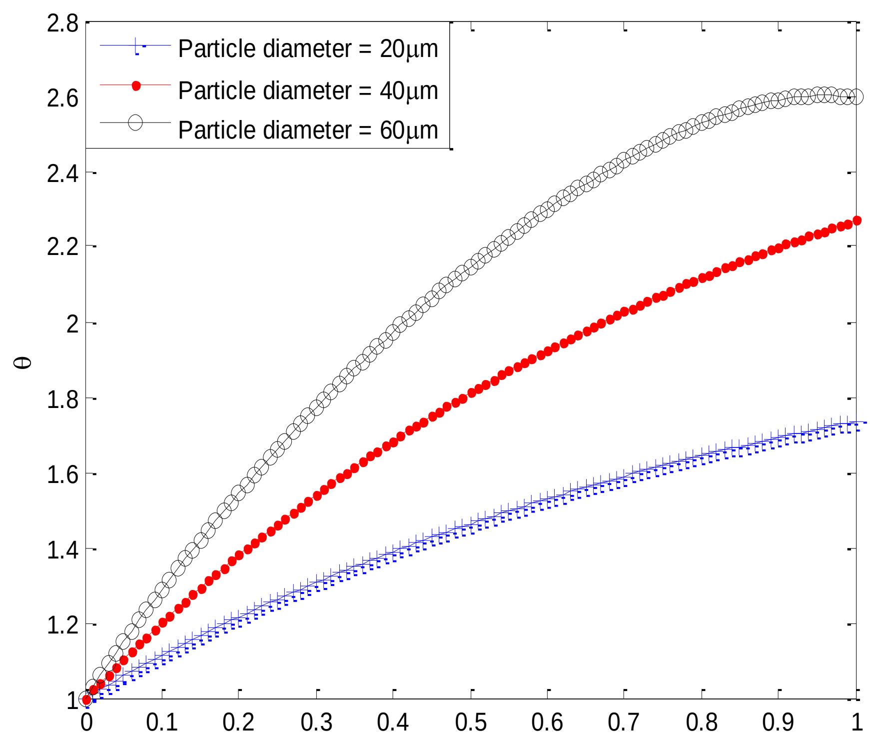

Figure 6. Effect of particle diameter on the temperature Profile with HPM

Fig. 5 and Fig. 6 depict the effect of the combusting particle diameter on temperature profile and burning rate using DTM and HPM. From the graphs, it can be easily seen that particle diameter have evident influence on the temperature profile. A particle with 60 μm diameter was observed to possess a higher temperature profile which means that an increase in the combusting particle diameter causes a corresponding increase in the temperature profile as well as the burning time. As a result of this evident impact, the particle diameter may be used as a controlling agent in reducing the hazardous effects that normally propagate from iron particle combustion.

4. 2. Effect of ε_1 and ε_2 on the temperature history

Fig. 7 and Fig. 8 depict the influence of ε_1 and ε_2 on the temperature profile. From the figures, it can be seen that increasing ε_1 and ε_2 decreases the combustion temperature with this effect more pronounced with ε_2. The decrease in combustion temperature with a corresponding increase in ε_1 and ε_2 is as a result of an increase in the radiation heat transfer term in the combustion particle.

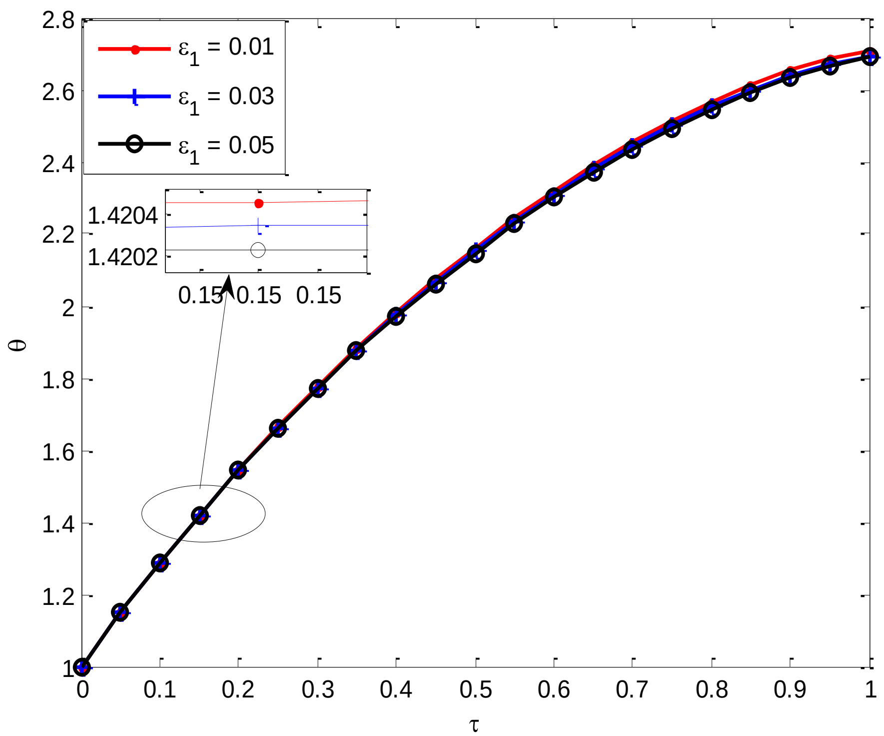

Figure 7. Effect of ε_1 on the temperature profile

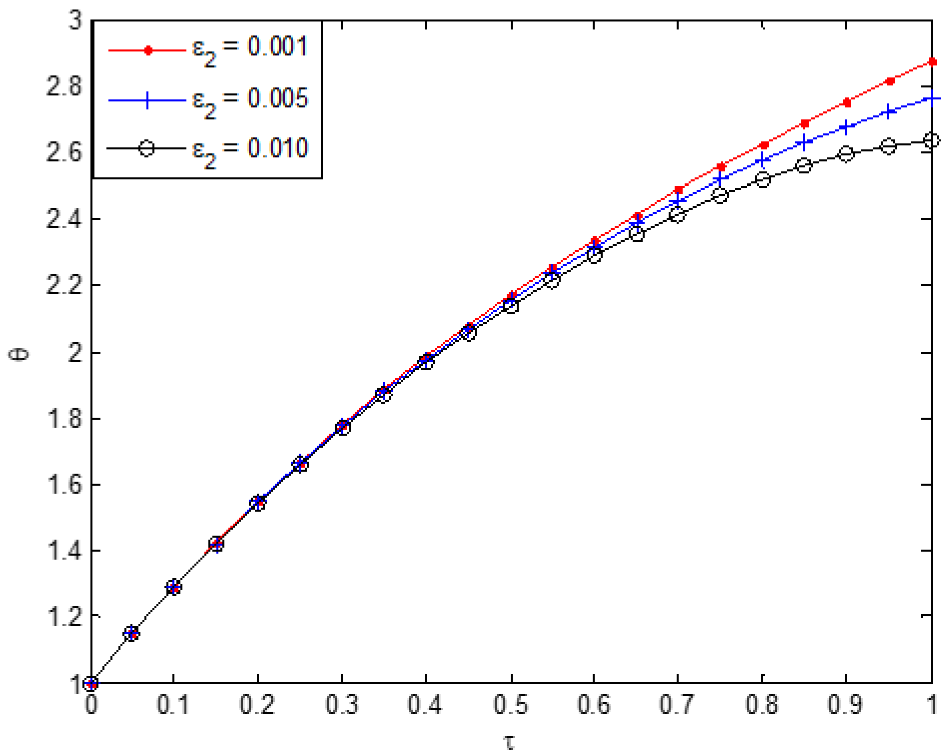

Figure 8. Effect of ε_2 on the temperature profile

4. 3. Effect of the heat realized parameter and surrounding temperature on the temperature history

Table 1. Comparism of the two analytical scheme with a numerical method for a 20 µm iron particle.

τ	θ (τ) for a Particle diameter of (µm)		
	Numerical	**HPM**	**DTM**
0.0	1.000000000000000	1.000000000000000	1.000000000000000
0.1	1.113333969181095	1.113333981177264	1.113333969181095
0.2	1.215117348008129	1.215117711150980	1.215117348008133
0.3	1.306590963764452	1.306593345739770	1.306590963764455
0.4	1.388844932815013	1.388852820269113	1.388844932815017
0.5	1.462843666959360	1.462858827578177	1.462843666959367
0.6	1.529449065177065	1.529458070391791	1.529449065177069
0.7	1.589444597054137	1.589387100510296	1.589444597054139
0.8	1.643562984179010	1.643271057270020	1.643562984179017
0.9	1.692520185796611	1.691615617727115	1.692520185796617
1.0	1.737058395009000	1.734792471017498	1.737058395009900

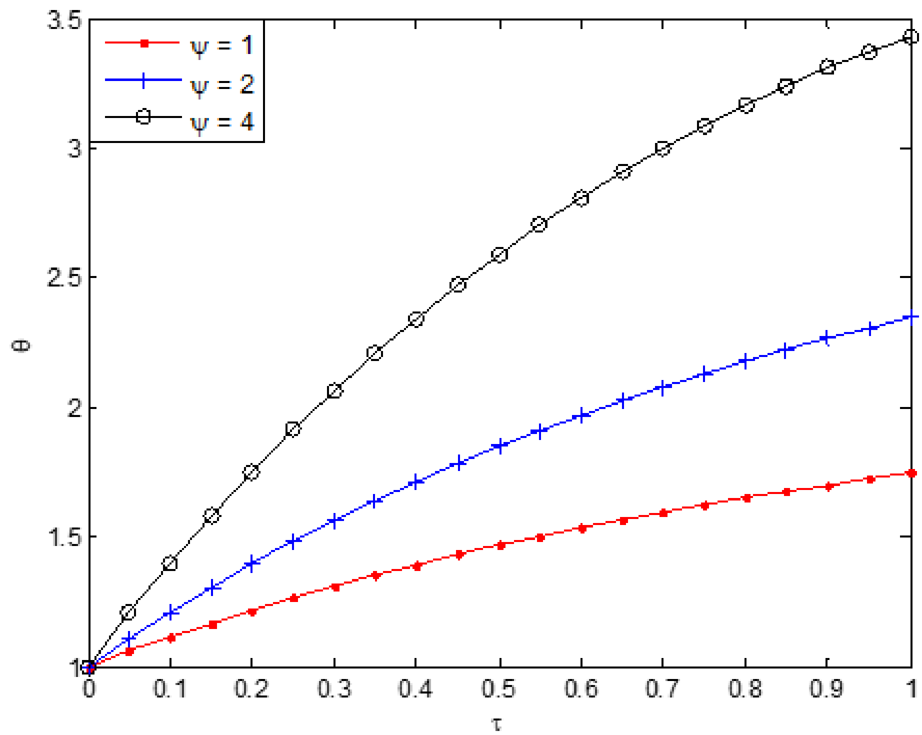

Figure 9. Effect of heat realized term on the temperature profile

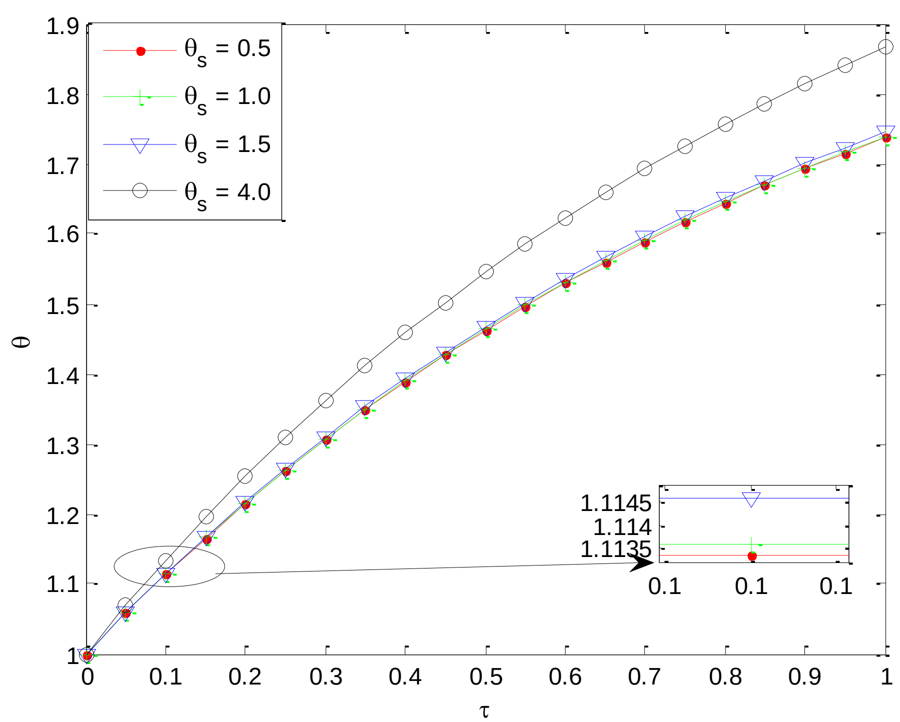

Figure 10. Effect of surrounding temperature on the temperature profile

Fig. 9 and Fig. 10 depict the influence of the heat realized parameter and the surrounding temperature on the combustion temperature. From the plots, we can conclude that increasing the heat realized parameter and the surrounding temperature increases the combustion temperature. This increase is significant for the heat realized parameter variation than that of the surrounding temperatures except for high values of surrounding temperature.

5. CONCLUSIONS

In this work, a comparative study of DTM and HPM has been carried out for the determination of the temperature history of iron particle during combustion process. The results of the DTM and HPM solutions were verified numerically. It was established that DTM gives a better result than HPM even though both schemes are efficient for the problem investigated. Also, parametric studies were performed to fully understand how the combusting particle diameter, density, radiative term, heat realized term and other parameters affect the burning time as well as the combustion temperature. The results revealed that by increasing the heat realized parameter, combustion temperature increased until a steady state was reached. It is hoped that the present study will enhance the understanding of the combustion of the particle and also obviate the challenges facing industries on combustion of metallic particles such as iron particles as well as in the determination of different particles burning time.

References

[1] Sun J.H., Dobashi R, Hirano T. Combustion behavior of iron particles suspended in air. *Combust. Sci. Technol.* 2000, 150: 99-114

[2] Haghiri A, Bidabadi M. Dynamic behavior of particles across flame propagation through micro-iron dust cloud with thermal radiation effect. *Fuel* 2011; 90: 2413–21.

[3] Hertzberg M, Zlochower I. A. and Cashdollar K. L. Metal Dust Combustion Explosion Limits, Pressures, and Temperatures. The Combustion Institute. Pittsburgh, PA, 1992; pp.1827-1835.

[4] Hatami M, Ganji DD, Jafaryar M, Farkhadnia F. Transient combustion analysis for iron-micro particles in a gaseous media by weighted residual methods. *Case Studies Therm Eng* 2014; 4: 24-31.

[5] He, J. H. Homotopy perturbation technique. *Computer Methods in Applied Mechanics and Engineering*, 1999, 178: 257-262.

[6] He, J. H.: New interpretation of homotopy perturbation method. *International Journal of Modern Physics B,* 2006, 20: 2561-2568.

[7] He, J. H.: A coupling method of homotopy technique and perturbation technique for nonlinear problems. *International Journal of Non-Linear Mechanics,* 2000: 3537-43.

[8] He, J. H.: Some asymptotic methods for strongly nonlinear equations. *International Journal of Modern Physics B*, 2006, 20: 1141-1199.

[9] He, J. H.: New perturbation technique which is also valid for large parameters. *Journal of Sound and Vibration*, 2000, 229: 1257-1263.

[10] Saedodin, S. and Shahbabaei, M. Thermal analysis of natural convection in porous fins with homotopy perturbation method (HPM). *Arabian Journal for Science and Engineering*, 2013, Volume 38, Issue 8, pp 2227–2231

[11] Darvishi, M. T., Gorla, R. S. R., Gorla, R. and Aziz, A. Thermal performance of a porous radial fin with natural convection and radiative heat losses. *Thermal Science*, 2015, 19(2): 669-678.

[12] Moradi, A., Hayat, T. and Alsaedi, A. Convective-radiative thermal analysis of triangular fins with temperature-dependent thermal conductivity by DTM. *Energy Conversion and Management*, 2014, 77: 70-77.

[13] Ha, H. Ganji, D. D. and Abbasi, M. Determination of temperature distribution for porous fin with temperature-dependent heat generation by homotopy analysis method. *Journal of Applied Mechanical Engineering*, 2005, 4(1): 1-5.

[14] Gbeninyi, M. G., Adeleye, O., Yinusa A. A. Analysis of convective-radiative porous fin with temperature-dependent internal heat generation and magnetic field using Homotopy Perturbation method. *Journal of Computational and Applied Mechanics*, 2017, Vol. 12, No. 2, pp. 127-145.

APPENDIX

The first-fifth-term solution of DTM is

$$\theta(\tau)=1+\left(-\left(\theta_{surr}^{4}\check{n}_2+\psi-\check{n}_2+\theta_\infty-1\right)/\left(\check{n}_1\theta_\infty-\check{n}_1-1\right)\right)\tau+\left(\left(1/2\right)\left(\theta_{surr}^{4}\check{n}_2+\psi-\check{n}_2+\theta_\infty-1\right)(\theta_{surr}^{4}\check{n}_1\check{n}_2-4\check{n}_1\check{n}_2\theta_\infty+\psi\check{n}_1+3\check{n}_1\check{n}_2+4\check{n}_2+1)/\left(\check{n}_1\theta_\infty-\check{n}_1-1\right)^3\right)\tau^2$$

$$\left(\begin{array}{l}
-\left(1/6\right)\left(\theta_{surr}^{4}\check{n}_2+\psi-\check{n}_2+\theta_\infty-1\right)(3\theta_{surr}^{8}\check{n}_1^{2}\check{n}_2^{2}-12\theta_{surr}^{4}\check{n}_1^{2}\check{n}_2^{2}\theta_\infty^{2}+8\theta_{surr}^{4}\check{n}_1^{2}\check{n}_2^{2}\theta_\infty+6\psi\theta_{surr}^{4}\check{n}_1^{2}\check{n}_2-2\theta_{surr}^{4}\check{n}_1^{2}\check{n}_2^{2}+2\theta_{surr}^{4}\check{n}_1^{2}\check{n}_2\theta_\infty+24\theta_{surr}^{4}\check{n}_1\check{n}_2^{2}\theta_\infty- \\
2\theta_{surr}^{4}\check{n}_1^{2}\check{n}_2-8\theta_{surr}^{4}\check{n}_1\check{n}_2^{2}-12\psi\check{n}_1^{2}\check{n}_2\theta_\infty^{2}+4\theta_{surr}^{4}\check{n}_1\check{n}_2-12\theta_{surr}^{4}\check{n}_2^{2}+28\check{n}_1^{2}\check{n}_2^{2}\theta_\infty^{2}-12\check{n}_1^{2}\check{n}_2\theta_\infty^{3}+8\psi\check{n}_1^{2}\check{n}_2\theta_\infty-40\check{n}_1^{2}\check{n}_2^{2}\theta_\infty+28\check{n}_1^{2}\check{n}_2\theta_\infty^{2}+3\psi^{2}\check{n}_1^{2}- \\
2\psi\check{n}_1^{2}\check{n}_2+2\psi\check{n}_1^{2}\theta_\infty+24\psi\check{n}_1\check{n}_2\theta_\infty+15\check{n}_1^{2}\check{n}_2^{2}-22\check{n}_1^{2}\check{n}_2\theta_\infty-56\check{n}_1\check{n}_2^{2}\theta_\infty+24\check{n}_1\check{n}_2\theta_\infty^{2}-2\psi\check{n}_1^{2}-8\psi\check{n}_1\check{n}_2+6\check{n}_1^{2}\check{n}_2+40\check{n}_1\check{n}_2^{2}-48\check{n}_1\check{n}_2\theta_\infty+4\psi\check{n}_1- \\
12\psi\check{n}_2+20\check{n}_1\check{n}_2+2\check{n}_1\theta_\infty+28\check{n}_2^{2}-12\check{n}_2\theta_\infty-2\check{n}_1+20\check{n}_2+1)/\left(\check{n}_1\theta_\infty-\check{n}_1-1\right)^5
\end{array}\right)\tau^3$$

$$\left(\begin{array}{l}
\left(1/24\right)\left(\theta_{surr}^{4}\check{n}_2+\psi-\check{n}_2+\theta_\infty-1\right)(15\theta_{surr}^{12}\check{n}_1^{3}\check{n}_2^{3}-24\theta_{surr}^{8}\check{n}_1^{3}\check{n}_2^{3}\theta_\infty^{3}-12\theta_{surr}^{8}\check{n}_1^{3}\check{n}_2^{3}\theta_\infty^{2}-4\theta_{surr}^{8}\check{n}_1^{3}\check{n}_2^{3}\theta_\infty+72\theta_{surr}^{8}\check{n}_1^{2}\check{n}_2^{3}\theta_\infty^{2}+ \\
45\psi\theta_{surr}^{8}\check{n}_1^{3}\check{n}_2^{2}-5\theta_{surr}^{8}\check{n}_1^{3}\check{n}_2^{3}+20\theta_{surr}^{8}\check{n}_1^{3}\check{n}_2^{2}\theta_\infty+24\theta_{surr}^{8}\check{n}_1^{2}\check{n}_2^{3}\theta_\infty-48\psi\theta_{surr}^{4}\check{n}_1^{3}\check{n}_2^{2}\theta_\infty^{3}-20\theta_{surr}^{8}\check{n}_1^{3}\check{n}_2^{2}+4\theta_{surr}^{8}\check{n}_1^{2}\check{n}_2^{3}- \\
72\theta_{surr}^{8}\check{n}_1\check{n}_2^{3}\theta_\infty+240\theta_{surr}^{4}\check{n}_1^{3}\check{n}_2^{3}\theta_\infty^{3}-48\theta_{surr}^{4}\check{n}_1^{3}\check{n}_2^{2}\theta_\infty^{4}-24\psi\theta_{surr}^{4}\check{n}_1^{3}\check{n}_2^{2}\theta_\infty^{2}+25\theta_{surr}^{8}\check{n}_1^{2}\check{n}_2^{2}-12\theta_{surr}^{8}\check{n}_1\check{n}_2^{3}-376\theta_{surr}^{4}\check{n}_1^{3}\check{n}_2^{3}\theta_\infty^{2}+ \\
72\theta_{surr}^{4}\check{n}_1^{3}\check{n}_2^{2}\theta_\infty^{3}-8\psi\theta_{surr}^{4}\check{n}_1^{3}\check{n}_2^{2}\theta_\infty+144\psi\theta_{surr}^{4}\check{n}_1^{2}\check{n}_2^{2}\theta_\infty^{2}+24\theta_{surr}^{8}\check{n}_2^{3}+232\theta_{surr}^{4}\check{n}_1^{3}\check{n}_2^{3}\theta_\infty-40\theta_{surr}^{4}\check{n}_1^{3}\check{n}_2^{2}\theta_\infty^{2}-720\theta_{surr}^{4}\check{n}_1^{2}\check{n}_2^{3}\theta_\infty^{2}+ \\
144\theta_{surr}^{4}\check{n}_1^{2}\check{n}_2^{2}\theta_\infty^{3}+45\psi^{2}\theta_{surr}^{4}\check{n}_1^{3}\check{n}_2-10\psi\theta_{surr}^{4}\check{n}_1^{3}\check{n}_2^{2}+40\psi\theta_{surr}^{4}\check{n}_1^{3}\check{n}_2\theta_\infty+48\psi\theta_{surr}^{4}\check{n}_1^{2}\check{n}_2^{2}\theta_\infty-51\theta_{surr}^{4}\check{n}_1^{3}\check{n}_2^{3}+16\theta_{surr}^{4}\check{n}_1^{3}\check{n}_2^{2}\theta_\infty+ \\
6\theta_{surr}^{4}\check{n}_1^{3}\check{n}_2\theta_\infty^{2}+752\theta_{surr}^{4}\check{n}_1^{2}\check{n}_2^{3}\theta_\infty-240\theta_{surr}^{4}\check{n}_1^{2}\check{n}_2^{2}\theta_\infty^{2}-24\psi^{2}\check{n}_1^{3}\check{n}_2\theta_\infty^{3}-40\psi\theta_{surr}^{4}\check{n}_1^{3}\check{n}_2+8\psi\theta_{surr}^{4}\check{n}_1^{2}\check{n}_2^{2}-144\psi\theta_{surr}^{4}\check{n}_1\check{n}_2^{2}\theta_\infty+ \\
240\psi\check{n}_1^{3}\check{n}_2^{2}\theta_\infty^{3}-48\psi\check{n}_1^{3}\check{n}_2\theta_\infty^{4}-12\theta_{surr}^{4}\check{n}_1^{3}\check{n}_2\theta_\infty-232\theta_{surr}^{4}\check{n}_1^{2}\check{n}_2^{3}+72\theta_{surr}^{4}\check{n}_1^{2}\check{n}_2^{2}\theta_\infty+720\theta_{surr}^{4}\check{n}_1\check{n}_2^{3}\theta_\infty-144\theta_{surr}^{4}\check{n}_1\check{n}_2^{2}\theta_\infty^{2}- \\
280\check{n}_1^{3}\check{n}_2^{3}\theta_\infty^{3}+240\check{n}_1^{3}\check{n}_2^{2}\theta_\infty^{4}-24\check{n}_1^{3}\check{n}_2\theta_\infty^{5}-12\psi^{2}\check{n}_1^{3}\check{n}_2\theta_\infty^{2}+50\psi\theta_{surr}^{4}\check{n}_1^{2}\check{n}_2-24\psi\theta_{surr}^{4}\check{n}_1\check{n}_2^{2}-376\psi\check{n}_1^{3}\check{n}_2^{2}\theta_\infty^{2}+72\psi\check{n}_1^{3}\check{n}_2\theta_\infty^{3}+ \\
6\theta_{surr}^{4}\check{n}_1^{3}\check{n}_2-26\theta_{surr}^{4}\check{n}_1^{2}\check{n}_2^{2}+28\theta_{surr}^{4}\check{n}_1^{2}\check{n}_2\theta_\infty-376\theta_{surr}^{4}\check{n}_1\check{n}_2^{3}+264\theta_{surr}^{4}\check{n}_1\check{n}_2^{2}\theta_\infty+580\check{n}_1^{3}\check{n}_2^{3}\theta_\infty^{2}-712\check{n}_1^{3}\check{n}_2^{2}\theta_\infty^{3}+84\check{n}_1^{3}\check{n}_2\theta_\infty^{4}- \\
4\psi^{2}\check{n}_1^{3}\check{n}_2\theta_\infty+72\psi^{2}\check{n}_1^{2}\check{n}_2\theta_\infty^{2}+48\psi\theta_{surr}^{4}\check{n}_2^{2}+232\psi\check{n}_1^{3}\check{n}_2^{2}\theta_\infty-40\psi\check{n}_1^{3}\check{n}_2\theta_\infty^{2}-720\psi\check{n}_1^{2}\check{n}_2^{2}\theta_\infty^{2}+144\psi\check{n}_1^{2}\check{n}_2\theta_\infty^{3}-28\theta_{surr}^{4}\check{n}_1^{2}\check{n}_2- \\
32\theta_{surr}^{4}\check{n}_1\check{n}_2^{2}-240\theta_{surr}^{4}\check{n}_2^{3}+48\theta_{surr}^{4}\check{n}_2^{2}\theta_\infty-48\psi\theta_{surr}^{4}\check{n}_2^{2}+232\psi\check{n}_1^{3}\check{n}_2^{2}\theta_\infty-40\psi\check{n}_1^{3}\check{n}_2\theta_\infty^{2}-720\psi\check{n}_1^{2}\check{n}_2^{2}\theta_\infty^{2}+144\psi\check{n}_1^{2}\check{n}_2\theta_\infty^{3}- \\
28\theta_{surr}^{4}\check{n}_1^{2}\check{n}_2-32\theta_{surr}^{4}\check{n}_1\check{n}_2^{2}-240\theta_{surr}^{4}\check{n}_2^{3}+48\theta_{surr}^{4}\check{n}_2^{2}\theta_\infty-420\check{n}_1^{3}\check{n}_2^{3}\theta_\infty+808\check{n}_1^{3}\check{n}_2^{2}\theta_\infty^{2}-120\check{n}_1^{3}\check{n}_2\theta_\infty^{3}+840\check{n}_1^{2}\check{n}_2^{3}\theta_\infty^{2}- \\
720\check{n}_1^{2}\check{n}_2^{2}\theta_\infty^{3}+72\check{n}_1^{2}\check{n}_2\theta_\infty^{4}+15\psi^{3}\check{n}_1^{3}-5\psi^{2}\check{n}_1^{3}\check{n}_2+20\psi^{2}\check{n}_1^{3}\theta_\infty+24\psi^{2}\check{n}_1^{2}\check{n}_2\theta_\infty-51\psi\check{n}_1^{3}\check{n}_2^{2}+16\psi\check{n}_1^{3}\check{n}_2\theta_\infty+6\psi\check{n}_1^{3}\theta_\infty^{2}+ \\
752\psi\check{n}_1^{2}\check{n}_2^{2}\theta_\infty-240\psi\check{n}_1^{2}\check{n}_2\theta_\infty^{2}+11\theta_{surr}^{4}\check{n}_1\check{n}_2-96\theta_{surr}^{4}\check{n}_2^{2}+105\check{n}_1^{3}\check{n}_2^{3}-420\check{n}_1^{3}\check{n}_2^{2}\theta_\infty+90\check{n}_1^{3}\check{n}_2\theta_\infty^{2}-1160\check{n}_1^{2}\check{n}_2^{3}\theta_\infty+1760\check{n}_1^{2}\check{n}_2^{2}\theta_\infty^{2}- \\
264\check{n}_1^{2}\check{n}_2\theta_\infty^{3}-20\psi^{2}\check{n}_1^{3}+4\psi^{2}\check{n}_1^{2}\check{n}_2-72\psi^{2}\check{n}_1\check{n}_2\theta_\infty-12\psi\check{n}_1^{3}\theta_\infty-232\psi\check{n}_1^{2}\check{n}_2^{2}+72\psi\check{n}_1^{2}\check{n}_2\theta_\infty+720\psi\check{n}_1\check{n}_2^{2}\theta_\infty-144\psi\check{n}_1\check{n}_2\theta_\infty^{2}+ \\
84\check{n}_1^{3}\check{n}_2^{2}-36\check{n}_1^{3}\check{n}_2\theta_\infty+420\check{n}_1^{2}\check{n}_2^{3}-1384\check{n}_1^{2}\check{n}_2^{2}\theta_\infty+320\check{n}_1^{2}\check{n}_2\theta_\infty^{2}-840\check{n}_1\check{n}_2^{3}\theta_\infty+720\check{n}_1\check{n}_2^{2}\theta_\infty^{2}-72\check{n}_1\check{n}_2\theta_\infty^{3}+25\psi^{2}\check{n}_1^{2}-12\psi^{2}\check{n}_1\check{n}_2+ \\
6\psi\check{n}_1^{3}-26\psi\check{n}_1^{2}\check{n}_2+28\psi\check{n}_1^{2}\theta_\infty-376\psi\check{n}_1\check{n}_2^{2}+264\psi\check{n}_1\check{n}_2\theta_\infty+6\check{n}_1^{3}\check{n}_2+369\check{n}_1^{2}\check{n}_2^{2}-164\check{n}_1^{2}\check{n}_2\theta_\infty+6\check{n}_1^{2}\theta_\infty^{2}+580\check{n}_1\check{n}_2^{3}-1384\check{n}_1\check{n}_2^{2}\theta_\infty+ \\
276\check{n}_1\check{n}_2\theta_\infty^{2}+24\psi^{2}\check{n}_2-28\psi\check{n}_1^{2}-32\psi\check{n}_1\check{n}_2-240\psi\check{n}_2^{2}+48\psi\check{n}_2\theta_\infty+36\check{n}_1^{2}\check{n}_2-12\check{n}_1^{2}\theta_\infty+576\check{n}_1\check{n}_2^{2}-284\check{n}_1\check{n}_2\theta_\infty+280\check{n}_2^{3}- \\
240\check{n}_2^{2}\theta_\infty+24\check{n}_2\theta_\infty^{2}+11\psi\check{n}_1-96\psi\check{n}_2+6\check{n}_1^{2}+69\check{n}_1\check{n}_2+8\check{n}_1\theta_\infty+336\check{n}_2^{2}-96\check{n}_2\theta_\infty-8\check{n}_1+84\check{n}_2+1)/\left(\check{n}_1\theta_\infty-\check{n}_1-1\right)^7
\end{array}\right)\tau^4$$

Solutions of Eqs. (33)-(35)

$$\begin{array}{l}
\theta_5=\left(1/24\right)\left(\theta_{surr}^{4}\check{n}_2+\psi-\check{n}_2+\theta_\infty-1\right)(-\left(1/5\right)\tau^5-519\tau^4\check{n}_1\check{n}_2+11\tau^4\check{n}_1\theta_\infty-824\tau^3\check{n}_1^{2}\check{n}_2+88\tau^3\check{n}_1^{2}\theta_\infty-264\tau^2\check{n}_1^{3}\check{n}_2+72\tau^2\check{n}_1^{3}\theta_\infty+108\tau^5\theta_{surr}^{4}\check{n}_2^{2}+ \\
\left(24/5\right)\psi^3\tau^5\check{n}_2-264\psi^2\tau^5\check{n}_2^{2}+936\psi\tau^5\check{n}_2^{3}+936\tau^5\check{n}_2^{3}\theta_\infty-264\tau^5\check{n}_2^{2}\theta_\infty^{2}+\left(24/5\right)\tau^5\check{n}_2\theta_\infty^{3}-48\psi^2\tau^5\check{n}_2+\left(4032/5\right)\psi\tau^5\check{n}_2^{2}+\left(4032/5\right)\tau^5\check{n}_2^{2}\theta_\infty-48\tau^5\check{n}_2\theta_\infty^{2}- \\
1380\tau^4\check{n}_1\check{n}_2^{3}-12\tau^3\check{n}_1^{2}\psi^2+108\psi\tau^5\check{n}_2+108\tau^5\check{n}_2\theta_\infty-1776\tau^4\check{n}_1\check{n}_2^{2}-940\tau^3\check{n}_1^{2}\check{n}_2^{2}-12\tau^3\check{n}_1^{2}\theta_\infty^{2}+11\psi\tau^4\check{n}_1+88\psi\tau^3\check{n}_1^{2}+72\psi\tau^2\check{n}_1^{3}+\left(24/5\right)\tau^5\theta_{surr}^{12}\check{n}_2^{4}-264\tau^5\theta_{surr}^{8}\check{n}_2^{4}- \\
48\tau^5\theta_{surr}^{8}\check{n}_2^{3}+936\tau^5\theta_{surr}^{4}\check{n}_2^{4}+\left(4032/5\right)\tau^5\theta_{surr}^{4}\check{n}_2^{3}+\left(144/5\right)\psi\tau^5\theta_{surr}^{4}\check{n}_2^{2}\theta_\infty-360\psi\tau^4\theta_{surr}^{4}\check{n}_1\check{n}_2^{2}-360\tau^4\theta_{surr}^{4}\check{n}_1\check{n}_2^{2}\theta_\infty-24\psi\tau^3\theta_{surr}^{4}\check{n}_1^{2}\check{n}_2- \\
24\tau^3\theta_{surr}^{4}\check{n}_1^{2}\check{n}_2\theta_\infty-360\psi\tau^4\check{n}_1\check{n}_2\theta_\infty+568\tau^3\check{n}_1^{2}\check{n}_2\theta_\infty+\left(72/5\right)\psi^2\tau^5\check{n}_2\theta_\infty-528\psi\tau^5\check{n}_2^{2}\theta_\infty+\left(72/5\right)\psi\tau^5\check{n}_2\theta_\infty^{2}-180\psi^2\tau^4\check{n}_1\check{n}_2-96\psi\tau^5\check{n}_2\theta_\infty+ \\
1304\psi\tau^4\check{n}_1\check{n}_2^{2}+1304\tau^4\check{n}_1\check{n}_2^{2}\theta_\infty-180\tau^4\check{n}_1\check{n}_2\theta_\infty^{2}+640\psi\tau^4\check{n}_1\check{n}_2+568\psi\tau^3\check{n}_1^{2}\check{n}_2-24\psi\tau^3\check{n}_1^{2}\theta_\infty+640\tau^4\check{n}_1\check{n}_2\theta_\infty+\left(72/5\right)\psi\tau^5\theta_{surr}^{8}\check{n}_2^{3}+\left(72/5\right)\tau^5\theta_{surr}^{8}\check{n}_2^{3}\theta_\infty- \\
180\tau^4\theta_{surr}^{8}\check{n}_1\check{n}_2^{3}-12\tau^3\theta_{surr}^{8}\check{n}_1^{2}\check{n}_2^{2}+\left(72/5\right)\psi^2\tau^5\theta_{surr}^{4}\check{n}_2^{2}-528\psi\tau^5\theta_{surr}^{4}\check{n}_2^{3}-528\tau^5\theta_{surr}^{4}\check{n}_2^{3}\theta_\infty+\left(72/5\right)\tau^5\theta_{surr}^{4}\theta_\infty^{2}\check{n}_2^{2}-96\psi\tau^5\theta_{surr}^{4}\check{n}_2^{2}-96\tau^5\theta_{surr}^{4}\check{n}_2^{2}\theta_\infty+ \\
1304\tau^4\theta_{surr}^{4}\check{n}_1\check{n}_2^{3}+640\tau^4\theta_{surr}^{4}\check{n}_1\check{n}_2^{2}+568\tau^3\theta_{surr}^{4}\check{n}_1^{2}\check{n}_2^{2}+11\tau^4\theta_{surr}^{4}\check{n}_1\check{n}_2+88\tau^3\theta_{surr}^{4}\check{n}_1^{2}\check{n}_2+72\tau^2\theta_{surr}^{4}\check{n}_1^{3}\check{n}_2-728\tau^5\check{n}_2^{4}-1232\tau^5\check{n}_2^{3}-588\tau^5\check{n}_2^{2}- \\
68\check{n}_2\tau^5-15\check{n}_1\tau^4-100\check{n}_1^{2}\tau^3-120\check{n}_1^{3}\tau^2-24\check{n}_1^{4}\tau)/\left(\check{n}_1\theta_\infty-1\right)^5
\end{array}$$

$$
\begin{aligned}
&\theta_6 = \left(1/120\right)\left(\theta_{surr}^4\check{n}_2 + \psi - \check{n}_2 + \theta_\infty - 1\right)(-\left(1/6\right)\tau^6 + 2304\psi\tau^5\theta_{surr}^4\check{n}_1\check{n}_2{}^2\theta_\infty + 1152\psi\tau^5\theta_{surr}^8\check{n}_1\check{n}_2{}^3 + 1152\tau^5\theta_{surr}^8\check{n}_1\check{n}_2{}^3\theta_\infty + 5376\psi\tau^6\theta_{surr}^4\check{n}_2{}^3\theta_\infty + 1152\psi^2\tau^5\theta_{surr}^4\check{n}_1\check{n}_2{}^2 + \\
&264\psi\tau^6\theta_{surr}^4\check{n}_2{}^2\theta_\infty - 20448\psi\tau^5\theta_{surr}^4\check{n}_1\check{n}_2{}^3 - 20448\tau^5\theta_{surr}^4\check{n}_1\check{n}_2{}^3\theta_\infty + 1152\tau^5\theta_{surr}^4\check{n}_1\check{n}_2{}^2\theta_\infty{}^2 - 5400\psi\tau^5\theta_{surr}^4\check{n}_1\check{n}_2{}^2 - 7600\psi\tau^4\theta_{surr}^4\check{n}_1{}^2\check{n}_2{}^2 - 5400\tau^5\theta_{surr}^4\check{n}_1\check{n}_2{}^2\theta_\infty - \\
&7600\tau^4\theta_{surr}^4\check{n}_1{}^2\check{n}_2{}^2\theta_\infty - 250\psi\tau^4\theta_{surr}^4\check{n}_1{}^2\check{n}_2 - 600\psi\tau^3\theta_{surr}^4\check{n}_1{}^3\check{n}_2 - 250\tau^4\theta_{surr}^4\check{n}_1{}^2\check{n}_2\theta_\infty - 600\tau^3\theta_{surr}^4\check{n}_1{}^3\check{n}_2\theta_\infty + 1152\psi^2\tau^5\check{n}_1\check{n}_2\theta_\infty - 20448\psi\tau^5\check{n}_1\check{n}_2{}^2\theta_\infty + \\
&1152\psi\tau^5\check{n}_1\check{n}_2\theta_\infty{}^2 - 5400\psi\tau^5\check{n}_1\check{n}_2\theta_\infty - 7600\psi\tau^4\check{n}_1{}^2\check{n}_2\theta_\infty + 384\tau^5\theta_{surr}^{12}\check{n}_1\check{n}_2{}^4 + 2688\psi\tau^6\theta_{surr}^8\check{n}_2{}^4 + 2688\tau^6\theta_{surr}^8\check{n}_2{}^4\theta_\infty + 132\psi\tau^6\theta_{surr}^8\check{n}_2{}^3 + 132\tau^6\theta_{surr}^8\check{n}_2{}^3\theta_\infty - \\
&10224\tau^5\theta_{surr}^8\check{n}_1\check{n}_2{}^4 - 2700\tau^5\theta_{surr}^8\check{n}_1\check{n}_2{}^3 - 3800\tau^4\theta_{surr}^8\check{n}_1{}^2\check{n}_2{}^3 - 125\tau^4\theta_{surr}^8\check{n}_1{}^2\check{n}_2{}^2 - 300\tau^3\theta_{surr}^8\check{n}_1{}^3\check{n}_2{}^2 + 2688\psi^2\tau^6\theta_{surr}^4\check{n}_2{}^3 - 16000\psi\tau^6\theta_{surr}^4\check{n}_2{}^4 - \\
&16000\tau^6\theta_{surr}^4\check{n}_2{}^4\theta_\infty + 2688\tau^6\theta_{surr}^4\check{n}_2{}^3\theta_\infty{}^2 + 132\psi^2\tau^6\theta_{surr}^4\check{n}_2{}^2 - 9320\psi\tau^6\theta_{surr}^4\check{n}_2{}^3 - 9320\tau^6\theta_{surr}^4\check{n}_2{}^3\theta_\infty + 132\tau^6\theta_{surr}^4\theta_\infty{}^2\check{n}_2{}^2 + 31520\tau^5\theta_{surr}^4\check{n}_1\check{n}_2{}^4 - \\
&520\psi\tau^6\theta_{surr}^4\check{n}_2{}^2 - 520\tau^6\theta_{surr}^4\check{n}_2{}^2\theta_\infty + 30072\tau^5\theta_{surr}^4\check{n}_1\check{n}_2{}^3 + 21600\tau^4\theta_{surr}^4\check{n}_1{}^2\check{n}_2{}^3 + 5220\tau^5\theta_{surr}^4\check{n}_1\check{n}_2{}^2 + 12450\tau^4\theta_{surr}^4\check{n}_1{}^2\check{n}_2{}^2 + 6200\tau^3\theta_{surr}^4\check{n}_1{}^3\check{n}_2{}^2 + \\
&2688\psi^2\tau^6\check{n}_2{}^2\theta_\infty - 16000\psi\tau^6\check{n}_2{}^3\theta_\infty + 2688\psi\tau^6\check{n}_2{}^2\theta_\infty{}^2 + 26\tau^5\theta_{surr}^4\check{n}_1\check{n}_2 + 525\tau^4\theta_{surr}^4\check{n}_1{}^2\check{n}_2 + 1400\tau^3\theta_{surr}^4\check{n}_1{}^3\check{n}_2 + 600\tau^2\theta_{surr}^4\check{n}_1{}^4\check{n}_2 + 384\psi^3\tau^5\check{n}_1\check{n}_2 + \\
&132\psi^2\tau^6\check{n}_2\theta_\infty - 10224\psi^2\tau^5\check{n}_1\check{n}_2{}^2 - 9320\psi\tau^6\check{n}_2{}^2\theta_\infty + 132\psi\tau^6\check{n}_2\theta_\infty{}^2 + 31520\psi\tau^5\check{n}_1\check{n}_2{}^3 + 31520\tau^5\check{n}_1\check{n}_2{}^3\theta_\infty - 10224\tau^5\check{n}_1\check{n}_2{}^2\theta_\infty{}^2 + 384\tau^5\check{n}_1\check{n}_2\theta_\infty{}^3 - \\
&2700\psi^2\tau^5\check{n}_1\check{n}_2 - 3800\psi^2\tau^4\check{n}_1{}^2\check{n}_2 - 520\psi\tau^6\check{n}_2\theta_\infty + 30072\psi\tau^5\check{n}_1\check{n}_2{}^2 + 21600\psi\tau^4\check{n}_1{}^2\check{n}_2{}^2 + 30072\tau^5\check{n}_1\check{n}_2{}^2\theta_\infty - 2700\tau^5\check{n}_1\check{n}_2\theta_\infty{}^2 + 21600\tau^4\check{n}_1{}^2\check{n}_2{}^2\theta_\infty - \\
&3800\tau^4\check{n}_1{}^2\check{n}_2\theta_\infty{}^2 + 5220\psi\tau^5\check{n}_1\check{n}_2 + 12450\psi\tau^4\check{n}_1{}^2\check{n}_2 - 250\psi\tau^4\check{n}_1{}^2\theta_\infty + 6200\psi\tau^3\check{n}_1{}^3\check{n}_2 - 600\psi\tau^3\check{n}_1{}^3\theta_\infty + 5220\tau^5\check{n}_1\check{n}_2\theta_\infty + 12450\tau^4\check{n}_1{}^2\check{n}_2\theta_\infty + 6200\tau^3\check{n}_1{}^3\check{n}_2\theta_\infty - \\
&29120/3)\tau^6\check{n}_2{}^5 - \left(63700/3\right)\tau^6\check{n}_2{}^4 - \left(45500/3\right)\tau^6\check{n}_2{}^3 - \left(11480/3\right)\tau^6\check{n}_2{}^2 - \left(682/3\right)\check{n}_2\tau^6 - 31\check{n}_1\tau^5 - 450\check{n}_1{}^2\tau^4 - 1300\check{n}_1{}^3\tau^3 - 900\check{n}_1{}^4\tau^2 - 120\check{n}_1{}^5\tau + \\
&896\tau^6\theta_{surr}^{12}\check{n}_2{}^5 + 44\tau^6\theta_{surr}^{12}\check{n}_2{}^4 - 8000\tau^6\theta_{surr}^8\check{n}_2{}^5 - 4660\tau^6\theta_{surr}^8\check{n}_2{}^4 - 260\tau^6\theta_{surr}^8\check{n}_2{}^3 + 16640\tau^6\theta_{surr}^4\check{n}_2{}^5 + 22308\tau^6\theta_{surr}^4\check{n}_2{}^4 + 7512\tau^6\theta_{surr}^4\check{n}_2{}^3 + 440\tau^6\theta_{surr}^4\check{n}_2{}^2 + \\
&896\psi^3\tau^6\check{n}_2{}^2 - 8000\psi^2\tau^6\check{n}_2{}^3 + 16640\psi\tau^6\check{n}_2{}^4 + 16640\tau^6\check{n}_2{}^4\theta_\infty - 8000\tau^6\check{n}_2{}^3\theta_\infty{}^2 + 896\tau^6\check{n}_2{}^2\theta_\infty{}^3 + 44\psi^3\tau^6\check{n}_2 - 4660\psi^2\tau^6\check{n}_2{}^2 + 22308\psi\tau^6\check{n}_2{}^3 + 22308\tau^6\check{n}_2{}^3\theta_\infty - \\
&4660\tau^6\check{n}_2{}^2\theta_\infty{}^2 + 44\tau^6\check{n}_2\theta_\infty{}^3 - 22960\tau^5\check{n}_1\check{n}_2{}^4 - 260\psi^2\tau^6\check{n}_2 + 7512\psi\tau^6\check{n}_2{}^2 + 7512\tau^6\check{n}_2{}^2\theta_\infty - 260\tau^6\check{n}_2\theta_\infty{}^2 - 40876\tau^5\check{n}_1\check{n}_2{}^3 - 21000\tau^4\check{n}_1{}^2\check{n}_2{}^3 - 125\tau^4\check{n}_1{}^2\psi^2 - \\
&300\psi^2\tau^3\check{n}_1{}^3 + 440\psi\tau^6\check{n}_2 + 440\tau^6\check{n}_2\theta_\infty - 21300\tau^5\check{n}_1\check{n}_2{}^2 - 28725\tau^4\check{n}_1{}^2\check{n}_2{}^2 - 125\tau^4\check{n}_1{}^2\theta_\infty{}^2 - 9100\tau^3\check{n}_1{}^3\check{n}_2{}^2 - 300\tau^3\check{n}_1{}^3\theta_\infty{}^2 + 26\psi\tau^5\check{n}_1 + 525\psi\tau^4\check{n}_1{}^2 + \\
&1400\psi\tau^3\check{n}_1{}^3 + 600\psi\tau^2\check{n}_1{}^4 - 3010\tau^5\check{n}_1\check{n}_2 + 26\tau^5\check{n}_1\theta_\infty - 9525\tau^4\check{n}_1{}^2\check{n}_2 + 525\tau^4\check{n}_1{}^2\theta_\infty - 8600\tau^3\check{n}_1{}^3\check{n}_2 + 1400\tau^3\check{n}_1{}^3\theta_\infty - 1800\tau^2\check{n}_1{}^4\check{n}_2 + 600\tau^2\check{n}_1{}^4\theta_\infty) / \left(\check{n}_1\theta_\infty - 1\right)^6
\end{aligned}
$$