Available online at www.worldscientificnews.com

World Scientific News

An International Scientific Journal

WSN 137 (2019) 229-236 EISSN 2392-2192

SHORT COMMUNICATION

Solution of linear systems of differential equations with singular constant coefficients by the Drazin inverse of matrices

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ABSTRACT

Let $A\,,B$ be $n \times n$ matrices of complex numbers. Let G a vector-valued function of the real variable t. A and B may both be singular, rank(A) = 1, and the trac of A is not equal zero. The linear system of differential equations $Ax'(t) + Bx(t) = G(t)$ is studied using the Drazin inverse A^D of A, and a new matrix $K \in \mathbb{C}^{n \times n}$. In this paper, we obtain a new closed form for the general solution of the differential system when the system is tractable.

***Keywords*:** Singular differential equations, Index, Drazin inverse

1. INTRODUCTION

The aim of this paper is to introduce a new form for the solution of linear systems of differential equations with singular constant coefficients

(Received 12 October 2019; Accepted 02 November 2019; Date of Publication 07 November 2019)

$$Ax'(t) + Bx(t) = G(t)\,, \qquad t \in \mathbb{R} \tag{1}$$

with rank of A equal to one, in the case when the system is tractable, that is, when the initial value problem

$$Ax'(t) + Bx(t) = G(t), \qquad x(t_0) = \boldsymbol{c} \tag{2}$$

has a unique solution for each consistent initial vector $\boldsymbol{c}$ associated with $t_0 \in \mathbb{R}$.

The vector $\boldsymbol{c} \in \mathbb{C}^{\boldsymbol{n}}$ is said to be a consistent initial vector for the linear system if that system has at least one solution for a particular initial condition [1]. To construction closed form solutions for linear systems of differential equations with singular constant coefficients [2, 3] we will use the Drazin inverse A^D of the coefficients matrix A [4, 5] when rank(A) =1 and trac of A is not equal zero ($Tr(A) \neq 0$), also will use a new matrix $K \in \mathbb{C}^{\boldsymbol{n}\times\boldsymbol{n}}$, where $K = AA^D - I$, and I is the identity matrix.

When the coefficients matrix A is singular, there exist things can happen. For example, the homogeneous initial value problem may be inconsistent, that means, there may not exist a solution, or there are infinitely many solutions even if $AB = BA$. But if we put condition (or conditions) on the linear system we will get the unique solution for each consistent initial vector $\boldsymbol{c}$ associated with t_0. This system is called "tractable" [1].

This paper is organized as follows. For the sake of convenience, some preliminaries are given in section 2. Section 3 gives the main results. Section 4 gives application to nth order linear system with singular constant coefficients. Finally section 5 gives concluding remarks.

2. PRELIMINARIES

Definition 1. Let $A \in \mathbb{C}^{\boldsymbol{m}\times\boldsymbol{n}}$, and B is in the reduced graded row form and row equivalent to A, then the number of non-zero rows of B is the rank of A, and denoted by rank(A).

Definition 2. Let $A \in \mathbb{C}^{\boldsymbol{n}\times\boldsymbol{n}}$, then the smallest non-negative integer k such that rank(A^k) = rank (A^{k+1}) is called the index of A, and denoted by Ind(A).
Note that Ind(**0**) = 1, where **0** is the zero matrix, and if Ind(A) = 0 then A is invertible.

Definition 3. Let $A \in \mathbb{C}^{\boldsymbol{n}\times\boldsymbol{n}}$ with Ind(A) = k , then the Drazin inverse of A is defined to be the unique matrix A^D such that

(i) $A^D A A^D = A^D$,
(ii) $AA^D = A^D A$, and
(iii) $A^{k+1} A^D = A^k$.

The Drazin inverse studied in [1, 6].

Definition 4. Let $A \in \mathbb{C}^{\boldsymbol{n}\times\boldsymbol{n}}$**,** the trace of A is given by

$$Tr(A) = \sum_{i=1}^{n} a_{ii}\,,$$

where $A = (a_{ij})$ $\quad (i = 1, \dots, n ; j = 1, \dots, n)$.

Definition 5. [1] A matrix $H \in \mathbb{C}^{n \times n}$**,** is said to be in hermite echelon form if its elements h_{ij} satisfies the following conditions

(i) H is upper triangular (i.e. $h_{ij} = 0$ where $i > j$).
(ii) h_{ii} is either 0 or 1.
(iii) If $h_{ii} = 0$, then $h_{il} = 0$ for every $l, 1 \leq l \leq n$.
(iv) If $h_{ii} = 1$, then $h_{li} = 0$ for every $l \neq i$.

Algorithm 1. [1] Computation of A^D where $A \in \mathbb{C}^{n \times n}$ and $\text{ind}(A) = k$.

(I) Let p be an integer such that $p \geq k$. (p can always be taken to equal to n if no smaller value can be determined). If $A^p = \mathbf{0}$, then $A^D = 0$. Thus we assume $A^p \neq \mathbf{0}$.

(II) Row reduce A^p to its Hermite echelon form H_{A^p}. The sequence of reducing matrices need not be saved.

(III) By noting the position of the non-zero diagonal elements in H_{A^p}, select the distinguished columns from A^p and call them v_1,v_2,…, v_r.

(IV) Form the matrix $I - H_{A^p}$ and save its non-zero columns. Call them v_{r+1},v_{r+2},…,v_n.

(V) Construct the non-singular matrix $P = [v_1| \dots |v_r|v_{r+1}| \dots |v_n]$.

(VI) Compute P^{-1}.

(VII) From the product $P^{-1}AP$. This matrix will be in the form $P^{-1}AP = \begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix}$ where C is non-singular and N is nilpotent.

(VIII) Compute C^{-1}.

(IX) Compute A^D by forming the product $A^D = P \begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}$.

Theorem 1. [1] If $A \in \mathbb{C}^{n \times n}$ is such that $\text{Ind}(A) = k > 0$, then there exists a non-singular matrix P such that

$$A = P \begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix} P^{-1},$$

where C is non-singular, and N is nilpotent of index k.

Furthermore, if P, C and N are any matrices satisfying the above conditions, then

$$A^D = P \begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}.$$

Theorem 2. [1] If $A \in \mathbb{C}^{n \times n}$ is such that $rank(A) = 1$, then $A^D = \frac{1}{(Tr(A))^2} A$ when $Tr(A) \neq 0$ and $A^D = 0$ when $Tr(A) = 0$.

Theorem 3. [1] For $A, B \in \mathbb{C}^{n\times n}$, the homogeneous differential equation

$$Ax'(t) + Bx(t) = 0$$

is tractable if and only if there exists a scalar $\lambda \in \mathbb{C}$ such that $(\lambda A + B)^{-1}$ exists.

3. THE MAIN RESULTS

In this section we introduce the general solution for the homogeneous system, and the particular solution for the nonhomogeneous systems.

Lemma 1. If $A \in \mathbb{C}^{n\times n}$ is such that rank$(A) = 1$ and $A^D = \frac{1}{[Tr(A)]^2} A$ where $Tr(A) \neq 0$, then $AA^D - K = I$, where

$$K = \begin{bmatrix} \frac{-(Tr\,(A)-a_{11})}{Tr(A)} & \frac{a_{12}}{Tr(A)} & \cdots & \frac{a_{1n}}{Tr(A)} \\ \frac{a_{21}}{Tr(A)} & \frac{-(Tr(A)-a_{22})}{Tr(A)} & \cdots & \frac{a_{2n}}{Tr(A)} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{a_{n1}}{Tr(A)} & \cdots & \cdots & \frac{-(Tr(A)-a_{nn})}{Tr(A)} \end{bmatrix} \in \mathbb{C}^{n\times n}.$$

Proof. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},$$

Such that rank$(A) = 1$ and $Tr(A) \neq 0$, where $Tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$.
So

$$A^D = \frac{1}{[Tr(A)]^2} A.$$

$$AA^D = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} \frac{a_{11}}{[Tr(A)]^2} & \frac{a_{12}}{[Tr(A)]^2} & \cdots & \frac{a_{1n}}{[Tr(A)]^2} \\ \frac{a_{21}}{[Tr(A)]^2} & \frac{a_{22}}{[Tr(A)]^2} & \cdots & \frac{a_{2n}}{[Tr(A)]^2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{a_{n1}}{[Tr(A)]^2} & \frac{a_{n2}}{[Tr(A)]^2} & \cdots & \frac{a_{nn}}{[Tr(A)]^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sum_{j=1}^{n} a_{1j}a_{j1}}{[Tr(A)]^2} & \frac{\sum_{j=1}^{n} a_{1j}a_{j2}}{[Tr(A)]^2} & \cdots & \frac{\sum_{j=1}^{n} a_{1j}a_{jn}}{[Tr(A)]^2} \\ \frac{\sum_{j=1}^{n} a_{2j}a_{j1}}{[Tr(A)]^2} & \frac{\sum_{j=1}^{n} a_{2j}a_{j2}}{[Tr(A)]^2} & \cdots & \frac{\sum_{j=1}^{n} a_{2j}a_{jn}}{[Tr(A)]^2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\sum_{j=1}^{n} a_{nj}a_{j1}}{[Tr(A)]^2} & \frac{\sum_{j=1}^{n} a_{nj}a_{j2}}{[Tr(A)]^2} & \cdots & \frac{\sum_{j=1}^{n} a_{nj}a_{jn}}{[Tr(A)]^2} \end{bmatrix}.$$

$$AA^D - K =$$

$$\begin{bmatrix} \frac{\sum_{j=1}^{n} a_{1j}a_{j1}+(Tr(A))^2-\sum_{j=1}^{n} a_{11}a_{jj}}{[Tr(A)]^2} & \frac{\sum_{j=1}^{n} a_{1j}a_{j2}-\sum_{j=1}^{n} a_{12}a_{jj}}{[Tr(A)]^2} & \cdots & \frac{\sum_{j=1}^{n} a_{1j}a_{jn}-\sum_{j=1}^{n} a_{1n}a_{jj}}{[Tr(A)]^2} \\ \frac{\sum_{j=1}^{n} a_{2j}a_{j1}-\sum_{j=1}^{n} a_{21}a_{jj}}{[Tr(A)]^2} & \frac{\sum_{j=1}^{n} a_{2j}a_{j2}+(Tr(A))^2-\sum_{j=1}^{n} a_{22}a_{jj}}{[Tr(A)]^2} & \cdots & \frac{\sum_{j=1}^{n} a_{2j}a_{jn}-\sum_{j=1}^{n} a_{2n}a_{jj}}{[Tr(A)]^2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\sum_{j=1}^{n} a_{nj}a_{j1}-\sum_{j=1}^{n} a_{nj}a_{jj}}{[Tr(A)]^2} & \frac{\sum_{j=1}^{n} a_{nj}a_{j2}-\sum_{j=1}^{n} a_{n2}a_{jj}}{[Tr(A)]^2} & \cdots & \frac{\sum_{j=1}^{n} a_{nj}a_{jn}+(Tr(A))^2-\sum_{j=1}^{n} a_{nn}a_{jj}}{[Tr(A)]^2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I.$$

Example 1. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}.$$

$$Tr(A) = 1 + 4 + 3 = 8,\ A^D = \frac{1}{8^2}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}, AA^D = \begin{bmatrix} \frac{1}{8} & \frac{1}{4} & \frac{3}{8} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \\ \frac{1}{8} & \frac{1}{4} & \frac{3}{8} \end{bmatrix},$$

$$AA^D - K = \begin{bmatrix} \frac{1}{8} & \frac{1}{4} & \frac{3}{8} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \\ \frac{1}{8} & \frac{1}{4} & \frac{3}{8} \end{bmatrix} - \begin{bmatrix} \frac{-7}{8} & \frac{2}{8} & \frac{3}{8} \\ \frac{2}{8} & \frac{-4}{8} & \frac{6}{8} \\ \frac{1}{8} & \frac{2}{8} & \frac{-5}{8} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

To simplify the formula for the solution of the linear differential system, we need to introduce the following notation. Let

$$\hat{A}_\lambda = (\lambda A + B)^{-1}A,\ \hat{B}_\lambda = (\lambda A + B)^{-1}B,\ \hat{G} = (\lambda A + B)^{-1}G,$$

where $A, B \in \mathbb{C}^{n\times n}$, $\boldsymbol{\lambda} \in \mathbb{C}$ such that $(\lambda A + B)^{-1}$ exists.

Theorem 4. Suppose $Ax'(t)+Bx(t)=0$ is tractable with $rank(\hat{A})=1$. Then the general solution is given by

$$x(t)=e^{-\hat{A}^D\hat{B}^D}(I+K)q, \qquad q\in\mathbb{C}^n.$$

Proof. Suppose that $Ax'(t)+Bx(t)=0$ is tractable, so there exists $\boldsymbol{\lambda}\in\mathbb{C}$ such that $(\lambda A+B)^{-1}$ exists. Note that $Ax'(t)+Bx(t)=0$ is tractable if and only if $\hat{A}_{\lambda}x'(t)+\hat{B}_{\lambda}x(t)=0$ is.

Now, we have $\hat{A}_{\lambda}x'(t)+\hat{B}_{\lambda}x(t)=0$. Let $y=e^{-\hat{A}^D\hat{B}^D}(I+K)q$. Then

$$\hat{A}y'=-\hat{A}(I{+}K)\hat{A}^D\hat{B}e^{-\hat{A}^D\hat{B}t}q$$

$$=-\hat{A}\hat{A}\hat{A}^D\hat{A}^D\hat{B}e^{-\hat{A}^D\hat{B}t}q$$

$$=-\hat{A}\hat{A}^D\hat{A}\hat{A}^D\hat{B}e^{-\hat{A}^D\hat{B}t}q$$

$$=-\hat{A}\hat{A}^D\hat{B}e^{-\hat{A}^D\hat{B}t}q$$

$$=-(I+K)\hat{B}e^{-\hat{A}^D\hat{B}t}q$$

$$=-\hat{B}e^{-\hat{A}^D\hat{B}t}(I+K)q=-\hat{B}y.$$

Theorem 5. Suppose $Ax'(t)+Bx(t)=0$ is tractable. Let $Ind(\hat{A})=k$. If $G(t)$ is a vector-valued function and k-times continuously differentiable around t_0. Then the equation

$$Ax'(t)+Bx(t)=G(t) \quad ,$$

always possesses solutions and a particular solution is given by

$$x(t)=\hat{A}^De^{-\hat{A}^D\hat{B}t}\int_{t_0}^{t}e^{\hat{A}^D\hat{B}s}\,\hat{G}(s)ds+(-K)\sum_{n=0}^{k-1}(-1)^n\,(\hat{A}\hat{B}^D)^n\hat{B}^DG^{(n)}(t),$$

where t_0 is arbitrary.

Proof. Let

$$x_1(t)=\hat{A}^De^{-\hat{A}^D\hat{B}t}\int_{t_0}^{t}e^{\hat{A}^D\hat{B}s}\,\hat{G}(s)ds,$$

$$x_2(t)=(-K)\sum_{n=0}^{k-1}(-1)^n\,\hat{A}^n(\hat{B}^D)^{n+1}\hat{G}^{(n)}(t),$$

we shall show that

$$\hat{A}x_1'(t)+\hat{B}x_1(t)=(I+K)G(t) \tag{3}$$

and

$$\hat{A}x_2'(t) + \hat{B}x_2(t) = (-K)G(t) \tag{4}$$

So that $x(t) = x_1(t) + x_2(t)$ is a solution for Eq. (1). To verify (3), note that

$$\hat{A}x_1'(t) = \hat{A}(\hat{A}^D G(t) - \hat{A}^D \hat{B} x_1(t))$$
$$= -\hat{A}\hat{A}^D \hat{B} x_1(t) + \hat{A}\hat{A}^D G(t)$$
$$= -\hat{B}x_1(t) + (I + K)G(t),$$

as desired. We now verify (4).

$$\hat{A}x_2'(t) = \hat{A}(-K)\sum_{n=0}^{k-1}(-1)^n \hat{A}^n(\hat{B}^D)^{n+1}\hat{G}^{(n+1)}(t)$$
$$= (-K)\sum_{n=0}^{k-1}(-1)^n (\hat{A}\hat{B}^D)^{n+1}\hat{G}^{(n+1)}(t)$$
$$= (-K)\sum_{n=1}^{k-1}(-1)^{n-1} (\hat{A}\hat{B}^D)^{n}\hat{G}^{(n)}(t)$$
$$= (-K)\hat{B}\hat{B}^D\sum_{n=1}^{k-1}(-1)^{n-1} (\hat{A}\hat{B}^D)^{n}\hat{G}^{(n)}(t)$$
$$= (-K)\hat{B}\sum_{n=1}^{k-1}(-1)^{n-1} \hat{A}^n(\hat{B}^D)^{n+1}\hat{G}^{(n)}(t)$$
$$= (-K)\hat{B}(-x_2(t) + \hat{B}^D\hat{G}^{(n)}(t))$$
$$= -\hat{B}x_2(t) + (-K)G(t)$$

as desired. So (4) holds.

If we combine theorems 4 and 5 we get the following theorem.

Theorem 6. Suppose $Ax'(t) + Bx(t) = G(t)$ is tractable. Then the general solution of $Ax'(t) + Bx(t) = G(t)$ is given by

$$x(t) = e^{-\hat{A}^D\hat{B}t}(I + K)q + \hat{A}^D e^{-\hat{A}^D\hat{B}t}\int_{t_0}^{t} e^{\hat{A}^D\hat{B}s}\,\hat{G}(s)ds$$
$$+(-K)\sum_{n=0}^{k-1}(-1)^n (\hat{A}\hat{B}^D)^n\hat{B}^D G^{(n)}(t),$$

where q is an arbitrary constant vector, t_0 is arbitrary, and $q = c = x(t_0)$ for the initial value problem (2).

4. APPLICATION

We can apply our results of this paper to nth order linear systems of differential equations with singular constant coefficients.

If we have the following system

$$\sum_{i=0}^{n} A_i \left(\frac{d}{dt}\right)^i x(t) = G(t),$$

we can reduce this system to a first order system using the substitution

$$x_0 = x,\ \ x_1 = \frac{dx_0}{dt},\quad \dots,\ \ x_{n-1} = \frac{dx_{n-2}}{dt},\quad x_n = \frac{dx_{n-1}}{dt},$$

in which case the system takes the form

$$A_n \frac{dx_{n-1}}{dt} + A_{n-1}x_{n-1} + \cdots + A_1 x_1 + A_0 x_0 = G(t).$$

Now, you can apply our results easily.

5. CONCLUDING REMARKS

It is our hope that this paper will be useful for further study of the Drazin inverse and its applications [7]. For example, maybe the ideas of this paper can be applied to systems with nonconstant coefficients.

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