μ-lacunary $\chi_{A_{uvw}}^3$-convergence defined by Musielak–Orlicz function

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ABSTRACT

We study some connections between μ-lacunary strong $\chi_{A_{uvw}}^3$ -convergence with respect to a $mnk$ sequence of Musielak–Orlicz function and μ-lacunary $\chi_{A_{uvw}}^3$-statistical convergence, where $A$ is a sequence of four dimensional matrices $A(uvw) = (a_{k_1...k_r\ell_1...\ell_s}(uvw))$ of complex numbers.

Keywords: Analytic sequence, $x^2$ space, difference sequence space, Musielak-modulus function, $p$-metric space, $mn$-sequences

2010 Mathematics Subject Classification: 40A05, 40C05, 40D05

1. INTRODUCTION

Throughout $w$, $\chi$ and $\Lambda$ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write $w^3$ for the set of all complex triple sequences ($x_{mnk}$), where $m, n, k \in \mathbb{N}$, the set of positive integers. Then, $w^3$ is a linear space under the coordinate wise addition and scalar multiplication.
We can represent triple sequences by matrix. In case of double sequences we write in the form of a square. In the case of a triple sequence it will be in the form of a box in three dimensional case.

Some initial work on double series is found in Apostol [1] and double sequence spaces is found in Hardy [9], Deepmala et al. [10, 11] and many others. Later on investigated by some initial work on triple sequence spaces is found in Esi [2], Esi et al. [3-8], Şahiner et al. [12], Subramanian et al. [13], Prakash et al. [14] and many others.

Let \((x_{m,n,k})\) be a triple sequence of real or complex numbers. Then the series \(\sum_{m,n,k=1}^{\infty} x_{m,n,k}\) is called a triple series. The triple series \(\sum_{m,n,k=1}^{\infty} x_{m,n,k}\) give one space is said to be convergent if and only if the triple sequence \((S_{m,n,k})\) is convergent, where

\[
S_{m,n,k} = \sum_{i,j,q=1}^{m,n,k} x_{ijq} \ (m,n,k = 1,2,3,\ldots).
\]

A sequence \(x = (x_{m,n,k})\) is said to be triple analytic if

\[
\sup_{m,n,k} \frac{1}{m+n+k} \left| x_{m,n,k} \right| < \infty.
\]

The vector space of all triple analytic sequences are usually denoted by \(\Lambda^3\). A sequence \(x = (x_{m,n,k})\) is called triple entire sequence if

\[
\left| x_{m,n,k} \right|^{\frac{1}{m+n+k}} \to 0 \quad \text{as} \quad m,n,k \to \infty.
\]

A sequence \(x = (x_{m,n,k})\) is called triple gai sequence if

\[
\left( (m+n+k)! \left| x_{m,n,k} \right| \right)^{\frac{1}{m+n+k}} \to 0 \quad \text{as} \quad m,n,k \to \infty.
\]

The triple gai sequences will be denoted by \(\chi^3\).

2. DEFINITIONS AND PRELIMINARIES

A triple sequence \(x = (x_{m,n,k})\) has limit 0 (denoted by \(P - \lim x = 0\)) (i.e)

\[
\left( (m+n+k)! \left| x_{m,n,k} \right| \right)^{\frac{1}{m+n+k}} \to 0 \quad \text{as} \quad m,n,k \to \infty.
\]

We shall write more briefly as \(P - \text{convergent to } 0\).

Definition 2.1 An Orlicz function (see [15]) is a function \(M: [0, \infty) \to [0, \infty)\) which is continuous, non-decreasing and convex with \(M(0) = 0\), \(M(x) > 0\), for \(x > 0\) and \(M(x) \to \infty\) as \(x \to \infty\). If convexity of Orlicz function \(M\) is replaced by \(M(x + y) \leq M(x) + M(y)\), then this function is called modulus function.

Lindenstrauss and Tzafriri (see [16]) used the idea of Orlicz function to construct Orlicz sequence space.

A sequence \(g = (g_{mn})\) defined by
\[ g_{mn}(v) = \sup\{|v|u - (f_{mnk})(u): u \geq 0\}, m, n, k = 1, 2, \ldots \]

is called the complementary function of a Musielak-Orlicz function \( f \). For a given Musielak-Orlicz function \( f \), (see [17]) the Musielak-Orlicz sequence space \( t_f \) is defined as follows

\[ t_f = \{ x \in w^3: |x_{mnk}|^{1/m+n+k} \to 0 \text{ as } m, n, k \to \infty \}, \]

where \( I_f \) is a convex modular defined by

\[ I_f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mnk}(|x_{mnk}|^{1/m+n+k}), \quad x = (x_{mnk}) \in t_f. \]

We consider \( t_f \) equipped with the Luxemburg metric

\[ d(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mnk}\left(\frac{|x_{mnk}|^{1/m+n+k}}{mnk}\right) \]

is an extended real number.

**Definition 2.2** Let \( mnk(\geq 3) \) be an integer. A function \( x:(M \times N \times K) \times (M \times N \times K) \times \cdots \times (M \times N \times K) \times (M \times N \times K) [m \times n \times k \text{ factors}] \rightarrow \mathbb{R}(\mathbb{C}) \) is called a real or complex mnk-sequence, where \( N \), \( R \) and \( C \) denote the sets of natural numbers and complex numbers respectively. Let \( m_1, m_2, \ldots m_r, n_1, n_2, \ldots n_s, k_1, k_2, \ldots, k_t \in \mathbb{N} \) and \( X \) be a real vector space of dimension \( w \), where \( m_1, m_2, \ldots m_r, n_1, n_2, \ldots n_s, k_1, k_2, \ldots, k_t \leq w \). A real valued function

\[ d_p(x_{11}, \ldots, x_{m_1m_2\ldots m_rn_1n_2\ldots n_sk_1k_2\ldots k_t}) = \| (d_1(x_{11}, 0), \ldots, d_n(x_{m_1m_2\ldots m_rn_1n_2\ldots n_sk_1k_2\ldots k_t}, 0)) \|_p \]

on \( X \) satisfying the following four conditions:

(i) \( \| (d_1(x_{11}, 0), \ldots, d_n(x_{m_1m_2\ldots m_rn_1n_2\ldots n_sk_1k_2\ldots k_t}, 0)) \|_p = 0 \) if and only if \( d_1(x_{11}, 0), \ldots, d_n(x_{m_1m_2\ldots m_rn_1n_2\ldots n_sk_1k_2\ldots k_t}, 0) \) are linearly dependent,

(ii) \( \| (d_1(x_{11}, 0), \ldots, d_m(x_{m_1m_2\ldots m_rn_1n_2\ldots n_sk_1k_2\ldots k_t}, x_{m_1m_2\ldots m_rn_1n_2\ldots n_sk_1k_2\ldots k_t}, 0)) \|_p \) is invariant under permutation,

(iii) For \( \alpha \in \mathbb{R}, \)

\[ \| (\alpha d_1(x_{11}, 0), \ldots, d_m(x_{m_1m_2\ldots m_pn_1n_2\ldots n_qk_1k_2\ldots k_t}, x_{m_1m_2\ldots m_pn_1n_2\ldots n_qk_1k_2\ldots k_t}, 0)) \|_p = |\alpha| \| (d_1(x_{11}, 0), \ldots, d_n(x_{m_1m_2\ldots m_pn_1n_2\ldots n_qk_1k_2\ldots k_t}, 0)) \|_p \]

(iv) For \( 1 \leq p < \infty, \)

\[ d_p((x_{11}, y_{11}), (x_{12}, y_{12}), \ldots, (x_{m_1m_2\ldots m_rn_1n_2\ldots n_sk_1k_2\ldots k_t}, y_{m_1m_2\ldots m_rn_1n_2\ldots n_sk_1k_2\ldots k_t})) \]
= \left( d_X(x_{11}, x_{12}, \ldots, x_m, m_2, \ldots, m_r, n_1, n_2, \ldots, n_s, k_1, k_2, \ldots, k_t) \right)^p + d_Y(y_{11}, y_{12}, \ldots, y_m, m_2, \ldots, m_p, n_1, n_2, \ldots, n_s, k_1, k_2, \ldots, k_t) \right)^{1/p}

(\text{or})

\( d((x_{11}, y_{11}), (x_{12}, y_{12}), \ldots, (x_m, m_2, \ldots, m_r, n_1, n_2, \ldots, n_s, k_1, k_2, \ldots, k_t), (y_{11}, y_{12}, \ldots, y_m, m_2, \ldots, m_p, n_1, n_2, \ldots, n_s, k_1, k_2, \ldots, k_t)) := \sup \{ d_X(x_{11}, x_{12}, \ldots, x_m, m_2, \ldots, m_r, n_1, n_2, \ldots, n_s, k_1, k_2, \ldots, k_t), \ldots, d_Y(y_{11}, y_{12}, \ldots, y_m, m_2, \ldots, m_p, n_1, n_2, \ldots, n_s, k_1, k_2, \ldots, k_t) \} \)

for \( x_{11}, x_{12}, \ldots, x_m, m_2, \ldots, m_r, n_1, n_2, \ldots, n_s, k_1, k_2, \ldots, k_t \in X, \ y_{11}, y_{12}, \ldots, y_m, m_2, \ldots, m_p, n_1, n_2, \ldots, n_s, k_1, k_2, \ldots, k_t \in Y \) is called the \( p \)-product metric of the Cartesian product of \( m_1, m_2, \ldots, m_r, n_1, n_2, \ldots, n_s, k_1, k_2, \ldots, k_t \) metric spaces is the \( p \)-norm of the \( m \times n \times k \)-vector of the norms of the \( m_1, m_2, \ldots, m_r, n_1, n_2, \ldots, n_s, k_1, k_2, \ldots, k_t \) subspaces.

**Definition 2.3** The triple sequence \( \theta_{i, \ell, j} = \{(m_i, n_\ell, k_j)\} \) is called triple lacunary if there exist three increasing sequences of integers such that

\[ m_0 = 0, \ h_i = m_i - m_{i-1} \to \infty \text{ as } i \to \infty \]

and

\[ n_0 = 0, \ h_\ell = n_\ell - n_{\ell-1} \to \infty \text{ as } \ell \to \infty, \]
\[ k_0 = 0, \ h_j = k_j - k_{j-1} \to \infty \text{ as } j \to \infty. \]

Let \( m_{i, \ell, j} = m_i n_\ell k_j, \ h_{i, \ell, j} = h_i h_\ell h_j \), and \( \theta_{i, \ell, j} \) is determine by

\[ l_{i, \ell, j} = \{(m, n, k): m_i - 1 < m < m_i \text{ and } n_{\ell-1} < n \leq n_\ell \text{ and } k_{j-1} < k \leq k_j\}, \]
\[ q_k = \frac{m_k}{m_{k-1}}, q_\ell = \frac{n_\ell}{n_{\ell-1}}, q_j = \frac{k_j}{k_{j-1}}. \]

Let \( F = (f_{mnk}) \) be a \( mnk \)-sequence of Musielak Orlicz functions such that \( \lim_{u \to 0^+} \sup_{mnk} f_{mnk}(u) = 0 \). Throughout this paper \( \chi_{Auvw}^3 \)-convergence of \( p \)-metric of \( mnk \)-sequence of Musielak Orlicz function determined by \( F \) will be denoted by \( f_{mnk} \in F \) for every \( m, n, k \in \mathbb{N} \).

The purpose of this paper is to introduce and study a concept of triple lacunary strong \( \chi_{Auvw}^3 \)-convergence of \( p \)-metric with respect to a \( mnk \)-sequence of Musielak Orlicz function.

We now introduce the generalizations of triple lacunary strongly \( \chi_{Auvw}^3 \)-convergence of \( p \)-metric with respect a \( mnk \)-sequence of Musielak Orlicz function and investigate some inclusion relations.
Let $A$ denote a sequence of the matrices $A_{uvw} = (a_{m_1 \ldots m_r \ldots m_k \ldots k_t}(uvw))$ of complex numbers. We write for any sequence $x = (x_{mnk})$,

$$y_{ij}(uv) = A_{ij}^{uvw}(x) = \sum_{m_1 \ldots m_r} \sum_{n_1 \ldots n_s} \sum_{k_1 \ldots k_t}(a_{m_1 \ldots m_r n_1 \ldots n_s k_1 \ldots k_t}(uvw)).$$

if it exits for each $ijq$ and $uvw$. We $A_{uvw}(x) = (A_{ij}^{uvw}(x))_{ijq}$, $Ax = (A_{uvw}(x))_{uvw}$.

**Definition 2.4** Let $\mu$ be a valued measure on $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ and $F = (\varepsilon_{ijq} m_{1 \ldots m_r n_1 \ldots n_s k_1 \ldots k_t})$ be a mnk-sequence of Musielak-Orlicz function, $A$ denote the sequence of four dimensional infinite matrices of complex numbers and $X$ be locally convex Hausdorff topological linear space whose topology is determined by a set of continuous semi norms $\eta$ and $(\varepsilon_{ijq}(d(x_{111}, 0), d(x_{122}, 0), \ldots, d(x_{m_1, m_2, \ldots, m_r - 1, n_1, n_2, \ldots, n_{s - 1}, k_1, k_2, \ldots, k_{t - 1}, 0}))_{ijq} p)$ be a $p$-metric space, $q = (q_{ijq})$ be triple analytic sequence of strictly positive real numbers.

By $w^3(p - X)$ we denote the space of all sequences defined over $(\varepsilon_{ijq}(d(x_{111}, 0), d(x_{122}, 0), \ldots, d(x_{m_1, m_2, \ldots, m_r - 1, n_1, n_2, \ldots, n_{s - 1}, k_1, k_2, \ldots, k_{t - 1}, 0}))_{ijq} p)^\mu$.

In the present paper we define the following sequence spaces:

$$\left[\Lambda_{A f N^\alpha_0}^{3q \eta} \left\| (d(x_{111}), d(x_{122}), \ldots, d(x_{m_1, m_2, \ldots, m_r - 1, n_1, n_2, \ldots, n_{s - 1}, k_1, k_2, \ldots, k_{t - 1}))_{ijq} p)^\mu \right\| \right]$$

$$= \mu \lim_{r \to} \left[ f_{ijq} \left( (N^\alpha_0(x), (d(x_{111}, 0), d(x_{122}, 0), \ldots, d(x_{m_1, m_2, \ldots, m_r - 1, n_1, n_2, \ldots, n_{s - 1}, k_1, k_2, \ldots, k_{t - 1}, 0)))_{ijq} p)^\mu \right] \geq \varepsilon = 0,$$

where

$$N^\alpha_0(x) = \frac{1}{h_{ijq}^{\alpha}} \sum_{t \in r} \sum_{j \in t} \sum_{q \in t} \eta A_{ij}^{uvw} \left( ((m_1 \ldots m_r + n_1 \ldots n_s + k_1, k_2, \ldots, k_t) ! x_{m_1 \ldots m_r n_1 \ldots n_s k_1 \ldots k_t})^{1/(m_1 \ldots m_r + n_1 \ldots n_s + k_1, k_2, \ldots, k_t)} \right),$$

uniformly in $u, v, w$

$$\left[ \Lambda_{A f N^\alpha_0}^{3q \eta} \left\| (d(x_{111}, 0), d(x_{122}, 0), \ldots, d(x_{m_1, m_2, \ldots, m_r - 1, n_1, n_2, \ldots, n_{s - 1}, k_1, k_2, \ldots, k_{t - 1}, 0)))_{ijq} p)^\mu \right\| \right]$$

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\[ = \mu_{rst}[f_{uvw}(\|N_\theta(x), (d(x_{111}, 0), d(x_{122}, 0), \ldots, d(x_{m_1m_2\ldots m_{r-1}n_1n_2\ldots n_{s-1}k_1k_2\ldots k_{t-1}, 0)})\|_p)]^{q_{ijq}} \geq k = 0, \]

where \( e = \left( \begin{array}{ccc} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{array} \right) \).

The main aim of this paper is to introduce the idea of summability of triple lacunary sequence spaces in \( p \)-metric spaces using a three valued measure. We also make an effort to study \( \mu \)-of lacunary triple sequences with respect to a sequence of Musielak Orlicz function in \( p \)-metric spaces and three valued measure \( \mu \). We also plan to study some topological properties and inclusion relation between these spaces.

3. MAIN RESULTS

**Proposition 3.1** Let \( \mu \) be a three valued measure,

\[
\left[ \chi_{A N^\theta}^{3q_{ijq}} \left( \| (d(x_{111}, 0), d(x_{122}, 0), \ldots, d(x_{m_1m_2\ldots m_{r-1}n_1n_2\ldots n_{s-1}k_1k_2\ldots k_{t-1}, 0)})\|_p \right)^\mu \right]
\]

and

\[
\left[ \Lambda_{A N^\theta}^{3q_{ijq}} \left( \| (d(x_{111}, 0), d(x_{122}, 0), \ldots, d(x_{m_1m_2\ldots m_{r-1}n_1n_2\ldots n_{s-1}k_1k_2\ldots k_{t-1}, 0)})\|_p \right)^\mu \right]
\]

are linear spaces.

**Proof.** It is routine verification. Therefore the proof is omitted.

The inclusion relation between

\[
\left[ \chi_{A N^\theta}^{3q_{ijq}} \left( \| (d(x_{111}, 0), d(x_{122}, 0), \ldots, d(x_{m_1m_2\ldots m_{r-1}n_1n_2\ldots n_{s-1}k_1k_2\ldots k_{t-1}, 0)})\|_p \right)^\mu \right]
\]

and

\[
\left[ \Lambda_{A N^\theta}^{3q_{ijq}} \left( \| (d(x_{111}, 0), d(x_{122}, 0), \ldots, d(x_{m_1m_2\ldots m_{r-1}n_1n_2\ldots n_{s-1}k_1k_2\ldots k_{t-1}, 0)})\|_p \right)^\mu \right]
\]
Theorem 3.1 Let $\mu$ be a three valued measure and $A$ be a mnk-sequence the four dimensional infinite matrices $A^{uv} = (a_{11,...,1t}^{1...n1,...,n1,...,nt}(uvw))$ of complex numbers and $F = (f_{mnk}^{ijq})$ be a mn-sequence of Musielak Orlicz function. If $x = (x_{mnk})$ triple lacunary strong $A_{uvw}$-convergent of orer $\alpha$ to zero then $x = (x_{mnk})$ triple lacunary strong $A_{uvw}$-convergent of order $\alpha$ to zero with respect to mnk-sequence of Musielak Orlicz function, (i.e)

$$
\left\{ \begin{array}{l}
\left| x_{mnk} \right| \leq \left( \left( d(x_{111}, 0), d(x_{122}, 0), \ldots, d(x_{m1,m2,\ldots,mr-1,n1,n2,\ldots,nr-1,k1,k2,\ldots,kr-1,0}) \right) \right)_p \bigg|_\mu \\
\left| x_{mnk} \right| \leq \left( \left( d(x_{111}, 0), d(x_{122}, 0), \ldots, d(x_{m1,m2,\ldots,mr-1,n1,n2,\ldots,nr-1,k1,k2,\ldots,kr-1,0}) \right) \right)_p \bigg|_\mu 
\end{array} \right.
$$

Proof. Let $F = (f_{mnk}^{ijq})$ be a mnk-sequence of Musielak Orlicz function and put sup$f_{mnk}^{ijq}(1) = T$. Let

$$
x = (x_{mnk}) \in \left( \left( d(x_{111}, 0), d(x_{122}, 0), \ldots, d(x_{m1,m2,\ldots,mr-1,n1,n2,\ldots,nr-1,k1,k2,\ldots,kr-1,0}) \right) \right)_p \bigg|_\mu
$$

and $\epsilon > 0$. We choose $0 < \delta < 1$ such that $f_{mnk}^{ijq}(u) < \epsilon$ for every $u$ with $0 \leq u \leq \delta$ ($i, j, q \in \mathbb{N}$). We can write

$$
\left\{ \begin{array}{l}
\left| x_{mnk} \right| \leq \left( \left( d(x_{111}, 0), d(x_{122}, 0), \ldots, d(x_{m1,m2,\ldots,mr-1,n1,n2,\ldots,nr-1,k1,k2,\ldots,kr-1,0}) \right) \right)_p \bigg|_\mu \\
\left| x_{mnk} \right| \leq \left( \left( d(x_{111}, 0), d(x_{122}, 0), \ldots, d(x_{m1,m2,\ldots,mr-1,n1,n2,\ldots,nr-1,k1,k2,\ldots,kr-1,0}) \right) \right)_p \bigg|_\mu 
\end{array} \right.
$$

where the first part is over $\leq \delta$ and second part is over $> \delta$. By definition of Musielak Orlicz function of $f_{mnk}^{ijq}$ for every $ijq$, we have

$$
\left| x_{mnk} \right| \leq \epsilon^H_2 + (3T\delta^{-1})^H_2 \left| \left( d(x_{111}, 0), d(x_{122}, 0), \ldots, d(x_{m1,m2,\ldots,mr-1,n1,n2,\ldots,nr-1,k1,k2,\ldots,kr-1,0}) \right) \right}_p \bigg|_\mu.
$$

Therefore

$$
x = (x_{mnk}) \in \left( \left( d(x_{111}, 0), d(x_{122}, 0), \ldots, d(x_{m1,m2,\ldots,mr-1,n1,n2,\ldots,nr-1,k1,k2,\ldots,kr-1,0}) \right) \right)_p \bigg|_\mu.
$$
\[ d(x_{m_1,m_2,\ldots,m_{r-1}n_1,n_2,\ldots,n_{s-1}k_1,k_2,\ldots,k_{t-1}}, 0) \|_p \]^\mu. \]

**Theorem 3.2** Let \( \mu \) be a three valued measure and \( A \) be a mnk-sequence of the four dimensional infinite matrices \( A^{uvw} = (a_{m_1\ldots m_{r-1}n_1\ldots n_{s-1}}^{uvw}) \) of complex numbers, \( q = (q_{ijq}) \) be a mnk-sequence of positive real numbers with \( 0 < \inf q_{ijq} = H_1 \leq \sup q_{ijq} = H_2 > \infty \) and \( F = (f_{mnk}) \) be a mnk-sequence of Musielak Orlicz function. If \( \lim_{u,v,w \to \infty} \inf f_{ijq}^{uvw} \frac{f_{ijq}(uvw)}{uvw} > 0 \), then

\[ \left[ \begin{array}{c} \chi_{3q^\eta}^{AfN^\alpha} \| (d(x_{111}, 0), d(x_{122}, 0), \ldots, d(x_{m_1,m_2,\ldots,m_{r-1}n_1,n_2,\ldots,n_{s-1}k_1,k_2,\ldots,k_{t-1}}, 0)) \|_p \end{array} \right]^{\mu} = \left[ \begin{array}{c} \chi_{3q^\eta}^{AN^\beta} \| (d(x_{111}, 0), d(x_{122}, 0), \ldots, d(x_{m_1,m_2,\ldots,m_{r-1}n_1,n_2,\ldots,n_{s-1}k_1,k_2,\ldots,k_{t-1}}, 0)) \|_p \end{array} \right]^{\mu}. \]

**Proof.** If \( \lim_{u,v,w \to \infty} \inf f_{ijq}^{uvw} \frac{f_{ijq}(uvw)}{uvw} > 0 \), then there exists a number \( \beta > 0 \) such that \( f_{ijq}(uvw) \geq \beta u \) for all \( u \geq 0 \) and \( i,j,q \in \mathbb{N} \). Let

\[ x = (x_{m_1, \ldots, m_r n_1, \ldots, n_s k_1, k_2, \ldots, k_t}) \in \left[ \begin{array}{c} \chi_{3q^\eta}^{AfN^\alpha} \| (d(x_{111}, 0), d(x_{122}, 0), \ldots, d(x_{m_1,m_2,\ldots,m_{r-1}n_1,n_2,\ldots,n_{s-1}k_1,k_2,\ldots,k_{t-1}}, 0)) \|_p \end{array} \right]^{\mu}. \]

Clearly

\[ \left[ \begin{array}{c} \chi_{3q^\eta}^{AfN^\alpha} \| (d(x_{111}, 0), d(x_{122}, 0), \ldots, d(x_{m_1,m_2,\ldots,m_{r-1}n_1,n_2,\ldots,n_{s-1}k_1,k_2,\ldots,k_{t-1}}, 0)) \|_p \end{array} \right]^{\mu} \geq \beta \left[ \begin{array}{c} \chi_{3q^\eta}^{AN^\beta} \| (d(x_{111}, 0), d(x_{122}, 0), \ldots, d(x_{m_1,m_2,\ldots,m_{r-1}n_1,n_2,\ldots,n_{s-1}k_1,k_2,\ldots,k_{t-1}}, 0)) \|_p \end{array} \right]^{\mu}. \]

Therefore

\[ x = (x_{m_1, \ldots, m_r n_1, \ldots, n_s k_1, k_2, \ldots, k_t}) \in \left[ \begin{array}{c} \chi_{3q^\eta}^{AN^\beta} \| (d(x_{111}, 0), d(x_{122}, 0), \ldots, d(x_{m_1,m_2,\ldots,m_{r-1}n_1,n_2,\ldots,n_{s-1}k_1,k_2,\ldots,k_{t-1}}, 0)) \|_p \end{array} \right]^{\mu}. \]

By using Theorem 3.1, the proof is complete.

We now give an example to show that

\[ \left[ \begin{array}{c} \chi_{3q^\eta}^{AfN^\alpha} \| (d(x_{111}, 0), d(x_{122}, 0), \ldots, d(x_{m_1,m_2,\ldots,m_{r-1}n_1,n_2,\ldots,n_{s-1}k_1,k_2,\ldots,k_{t-1}}, 0)) \|_p \end{array} \right]^{\mu} \neq \left[ \begin{array}{c} \chi_{3q^\eta}^{AN^\beta} \| (d(x_{111}, 0), d(x_{122}, 0), \ldots, d(x_{m_1,m_2,\ldots,m_{r-1}n_1,n_2,\ldots,n_{s-1}k_1,k_2,\ldots,k_{t-1}}, 0)) \|_p \end{array} \right]^{\mu}. \]
in the case when $\beta = 0$. Consider $A = I$, unit matrix,
\[
\eta(x) = \left( (m_1 \cdots m_r + n_1 \cdots n_s + k_1, k_2, \ldots, k_t!) \right)\]
\[
|x_{m_1 \cdots m_r n_1 \cdots n_s k_1 k_2 \cdots k_t}|^{1/m_1 \cdots m_r + n_1 \cdots n_s + k_1 k_2 \cdots k_t},
\]
$q_{ijq} = 1$
for every $i, j, q \in \mathbb{N}$ and
\[
f_{mnk}^{ijq}(x) = \left| x_{m_1 \cdots m_r n_1 \cdots n_s k_1 k_2 \cdots k_t} \right|^{1/((m_1 \cdots m_r + n_1 \cdots n_s + k_1, k_2, \ldots, k_t!)^{(i+1)(j+1)(q+1)})}
\]
\[
((m_1 \cdots m_r + n_1 \cdots n_s + k_1, k_2, \ldots, k_t!)^{1/m_1 \cdots m_r + n_1 \cdots n_s + k_1, k_2, \ldots, k_t}
\]
\[
(i, j, q \geq 1, x > 0)
\]
in the case $\beta > 0$. Now we define $x_{ijq} = h_{rst}^{q}$ if $i, j, q = m_r n_s k_t$ for some $r, s, t \geq 1$ and $x_{ijq} = 0$ otherwise. Then we have,
\[
\left[ \chi_{A_3 q}^{\alpha q} \left( \left\| (d(x_{111}, 0), d(x_{122}, 0), \ldots, d(x_{m_1, m_2 \cdots m_r - 1 n_1, n_2 \cdots n_s - 1 k_1, k_2 \cdots k_t - 1, 0)) \right) \right\|_p \right]^\alpha \rightarrow 1
\]
as $r, s, t \rightarrow \infty$
and so
\[
x = (x_{m_1 \cdots m_r n_1 \cdots n_s k_1 k_2 \cdots k_t}) \notin \left[ \chi_{A_3 q}^{\alpha q} \left( \left\| (d(x_{111}, 0), d(x_{122}, 0), \ldots, d(x_{m_1, m_2 \cdots m_r - 1 n_1, n_2 \cdots n_s - 1 k_1, k_2 \cdots k_t - 1, 0)) \right) \right\|_p \right]^\alpha.
\]

In this section we introduce natural relationship between $\mu$ be a three valued measure of triple lacunary $A_{uvw}$-statistical convergence of order $\alpha$ and $\mu$ be a three valued measure of triple lacunary strong $A_{uvw}$-convergence of order $\alpha$ with respect to $mnk$-sequence of Musielak Orlicz function.

**Definition 3.1** Let $\mu$ be a three valued measure and $\theta$ be a triple lacunary $mnk$-sequence. Then a $mnk$-sequence $x = (x_{m_1 \cdots m_r n_1 \cdots n_s k_1 k_2 \cdots k_t})$ is said to be $\mu$-lacunary statistically convergent of order $\alpha$ to a number zero if for every $\epsilon > 0$, $\mu(\lim_{rst \rightarrow \infty} h_{rst}^{\alpha}|K_{\theta}(\epsilon)|) = 0$, where $|K_{\theta}(\epsilon)|$ denotes the number of elements in
\[
K_{\theta}(\epsilon) = \mu \left\{ (i, j, q) \in I_{rst} : (m_1 \cdots m_r + n_1 \cdots n_s + k_1, k_2, \ldots, k_t!) \cdot
\]
\[
|x_{m_1 \cdots m_r n_1 \cdots n_s k_1 k_2 \cdots k_t}|^{1/m_1 \cdots m_r + n_1 \cdots n_s + k_1, k_2, \ldots, k_t} \geq \epsilon = 0 \right\}.
\]

The set of all triple lacunary statistical convergent of order $\alpha$ of $mnk$-sequences is denoted by $(S_{\theta}^{\alpha})^\mu$. 

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Let $\mu$ be a three valued measure and $A_{uvw} = \left( a_{m_1 \cdots m_r n_1 \cdots n_s k_1 k_2 \cdots k_t} (uvw) \right)$ be an four dimensional infinite matrix of complex numbers. Then a $mnk$-sequence $x = (x_{m_1 \cdots m_r n_1 \cdots n_s k_1, k_2, \ldots, k_t})$ is said to be $\mu$-triple lacunary $A$-statistically convergent of order $\alpha$ to a number zero if for every $\epsilon > 0$, $\mu( \lim_{rst \to \infty} h_{rst}^{-\alpha} |KA_\theta(\epsilon)| ) = 0$, where $|KA_\theta(\epsilon)|$ denotes the number of elements in

$$KA_\theta(\epsilon) = \mu \left\{ (i, j, q) \in I_{rst} : \left( (m_1 \cdots m_r + n_1 \cdots n_s + k_1, k_2, \ldots, k_t)! \cdot \left| x_{m_1 \cdots m_r n_1 \cdots n_s k_1, k_2, \ldots, k_t} \right|^{1/(m_1 \cdots m_r + n_1 \cdots n_s + k_1, k_2, \ldots, k_t)} \geq \epsilon = 0 \right\}.$$ 

The set of all triple lacunary $A$-statistical convergent of order $\alpha$ of $mnk$-sequences is denoted by $(S_\alpha^\mu(A))$. 

**Definition 3.2** Let $\mu$ be a three valued measure and $A$ be a $mnk$-sequence of the four dimensional infinite matrices $A_{uvw} = \left( a_{m_1 \cdots m_r n_1 \cdots n_s k_1, k_2 \cdots k_t} (uvw) \right)$ of complex numbers and let $q = (q_{ijl})$ be a $mnk$-sequence of positive real numbers with $0 < \inf q_{ijl} = H_1 \leq \sup q_{ijl} = H_2 < \infty$. Then a $mnk$-sequence $x = (x_{m_1 \cdots m_r n_1 \cdots n_s k_1, k_2, \ldots, k_t})$ is said to be $\mu$-lacunary $A_{uvw}$-statistically convergent of order $\alpha$ to a number zero if for every $\epsilon > 0$, $\mu( \lim_{rst \to \infty} h_{rst}^{-\alpha} |KA_{\theta}(\epsilon)| ) = 0$, where $|KA_{\theta}(\epsilon)|$ denotes the number of elements in

$$KA_{\theta}(\epsilon) = \mu \left\{ (i, j, q) \in I_{rst} : \left( (m_1 \cdots m_r + n_1 \cdots n_s + k_1, k_2, \ldots, k_t)! \cdot \left| x_{m_1 \cdots m_r n_1 \cdots n_s k_1, k_2, \ldots, k_t} \right|^{1/(m_1 \cdots m_r + n_1 \cdots n_s + k_1, k_2, \ldots, k_t)} \geq \epsilon = 0 \right\}.$$ 

The set of all $\mu$-lacunary $A_{\eta}$-statistical convergent of order $\alpha$ of $mnk$-sequences is denoted by $(S_\alpha^\mu(A, \eta))$. 

The following theorems give the relations between $\mu$-lacunary $A_{uvw}$-statistical convergence of order $\alpha$ and $\mu$-lacunary strong $A_{uvw}$-convergence of order $\alpha$ with respect to a $mnk$-sequence of Musielak Orlicz function. 

**Theorem 3.3** Let $\mu$ be a three valued measure and $F = (f_{ijq})$ be a $mnk$-sequence of Musielak Orlicz function. Then

$$\left[ \chi_{A_{f_{ijq}}}^{\forall q}, \left\| (d(x_{111}, 0), d(x_{122}, 0), \ldots, d(x_{m_1m_2\cdots m_{r-1}n_1n_2\cdots n_{s-1}k_1k_2\cdots k_t-1}, 0)) \right\|_p \right]^\mu \leq \left[ \chi_{A_{f_{ijq}}}^{\forall q}, \left\| (d(x_{111}, 0), d(x_{122}, 0), \ldots, d(x_{m_1m_2\cdots m_{r-1}n_1n_2\cdots n_{s-1}k_1k_2\cdots k_t-1}, 0)) \right\|_p \right]^\mu$$

if and only if $\mu \left( \lim_{ij=\infty} f_{ij}(u) \right) > 0$, $(u > 0)$. 

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Proof. Let $\epsilon > 0$ and $x = (x_{m_1\cdots m_r n_1\cdots n_s k_1,\ldots, k_t}) \in \left[ \chi_{\mathcal{AF} N^\theta_p}, \|(d(x_{111}, 0), d(x_{122}, 0), \ldots, d(x_{m_1, m_2, \ldots, m_r n_1, n_2, \ldots, n_s k_1, k_2, \ldots, k_t-1, 0})\right]_p^\mu$. If $\mu \left( \lim_{i,j,q} f_{ijq}(u) \right) > 0$. (u > 0), then there exists a number $d > 0$ such that $f_{ijq}(\epsilon) > d$ for $u > \epsilon$ and $i, j, q \in \mathbb{N}$. Let

$$\left[ \chi_{\mathcal{AF} N^\theta_p}, \|(d(x_{111}, 0), d(x_{122}, 0), \ldots, d(x_{m_1, m_2, \ldots, m_r n_1, n_2, \ldots, n_s k_1, k_2, \ldots, k_t-1, 0})\right]_p^\mu \geq h_{\eta}^{-\epsilon} d^n K \theta^\mu(\epsilon).$$

It follows that

$$\left[ \chi_{\mathcal{AF} N^\theta_p}, \|(d(x_{111}, 0), d(x_{122}, 0), \ldots, d(x_{m_1, m_2, \ldots, m_r n_1, n_2, \ldots, n_s k_1, k_2, \ldots, k_t-1, 0})\right]_p^\mu.$$

Conversely, suppose that $\mu \left( \lim_{i,j,q} f_{ijq}(u) \right) > 0$ does not hold, then there is a number $t > 0$ such that $\mu \left( \lim_{i,j,q} f_{ijq}(t) \right) = 0$. We can select a lacunary $mn$-sequence $\theta = (m_1 \cdots m_r n_1 \cdots n_s k_1, k_2, \ldots, k_t)$ such that $f_{ijq}(t) < 3^{-rst}$ for any $i > m_1 \cdots m_r, j > n_1 \cdots n_s, k > k_1, k_2, \ldots, k_t$. Let $A = I$, unit matrix, define the $mnk$-sequence $x$ by putting $x_{ijq} = t$ if

$$m_1, m_2, \cdots, m_r n_1, n_2, \cdots, n_s - k_1, k_2, \cdots, k_t < i, j, q < \frac{m_1, m_2, \cdots, m_r n_1, n_2, \cdots, n_s k_1, k_2, \cdots, k_t + m_1, m_2, \cdots, m_r n_1, n_2, \cdots, n_s - k_1, k_2, \cdots, k_t}{2}$$

and $x_{ijq} = 0$ if

$$\frac{m_1, m_2, \cdots, m_r n_1, n_2, \cdots, n_s k_1, k_2, \cdots, k_t + m_1, m_2, \cdots, m_r n_1, n_2, \cdots, n_s - k_1, k_2, \cdots, k_t}{2} \leq i, j, q \leq m_1, m_2, \cdots, m_r n_1, n_2, \cdots, n_s k_1, k_2, \cdots, k_t.$$

We have

$$x = (x_{m_1 \cdots m_r n_1 \cdots n_s k_1, k_2, \ldots, k_t}) \in \left[ \chi_{\mathcal{AF} N^\theta_p}, \|(d(x_{111}, 0), d(x_{122}, 0), \ldots, d(x_{m_1, m_2, \ldots, m_r n_1, n_2, \ldots, n_s k_1, k_2, \ldots, k_t-1, 0})\right]_p^\mu.$$

but $x \notin \left[ \chi_{\mathcal{AF} N^\theta_p}, \|(d(x_{111}, 0), d(x_{122}, 0), \ldots, d(x_{m_1, m_2, \ldots, m_r n_1, n_2, \ldots, n_s - k_1, k_2, \ldots, k_t-1, 0})\right]_p^\mu.$$

Theorem 3.4 Let $\mu$ be a three valued measure and $F = (f_{ijq})$ be a $mnk$-sequence of Musielak-Orlicz function. Then
\[
\begin{aligned}
&\left[ \chi_{AFN^\theta_0}^{3q\eta} \right]^{\mu} \left[ (d(x_{111},0),d(x_{122},0),\ldots,d(x_{m_1,m_2,\ldots,m_r-1,n_1,n_2,\ldots,n_s-1,k_1,k_2,\ldots,k_{t-1},0}) \right]_p \\
\geq &\left[ \chi_{A^\theta_0}^{\eta} \right]^{\mu} \left[ (d(x_{111},0),d(x_{122},0),\ldots,d(x_{m_1,m_2,\ldots,m_r-1,n_1,n_2,\ldots,n_s-1,k_1,k_2,\ldots,k_{t-1},0}) \right]_p 
\end{aligned}
\]

if and only if \( \mu \left( \sup_u \sup_{ijq} f_{ijq}(u) \right) < \infty \).

**Proof.** Let

\[
\begin{aligned}
x &\in \left[ \chi_{AFN^\theta_0}^{3q\eta} \right]^{\mu} \left[ (d(x_{111},0),d(x_{122},0),\ldots,d(x_{m_1,m_2,\ldots,m_r-1,n_1,n_2,\ldots,n_s-1,k_1,k_2,\ldots,k_{t-1},0}) \right]_p \\
&\leq h^{H_2} h^{-a} \left| KA^\theta_\eta(\epsilon) \right| + |h(\epsilon)|^{H_2}.
\end{aligned}
\]

It follows from \( \epsilon \to 0 \) that

\[
\begin{aligned}
x &\in \left[ \chi_{AFN^\theta_0}^{3q\eta} \right]^{\mu} \left[ (d(x_{111},0),d(x_{122},0),\ldots,d(x_{m_1,m_2,\ldots,m_r-1,n_1,n_2,\ldots,n_s-1,k_1,k_2,\ldots,k_{t-1},0}) \right]_p.
\end{aligned}
\]

Conversely, suppose that \( \mu \left( \sup_u \sup_{ijq} f_{ijq}(u) \right) = \infty \). Then we have

\[
0 < u_{111} < \cdots < u_{r-1s-1t-1} < u_{rst} < \cdots, \text{ such that } f_{ijq}(u_{rst}) \geq h^{\eta}_{rst} \text{ for } r,s,t \geq 1. \]

Let \( A = I \), unit matrix, define the \( mnk \)-sequence \( x \) by putting \( x_{ijq} = u_{rst} \) if \( i,j,q = m_1 m_2 \cdots m_r n_1 n_2 \cdots n_s \) for some \( r,s,t = 1,2,\ldots \) and \( x_{ijq} = 0 \) otherwise. Then we have

\[
\begin{aligned}
x &\in \left[ \chi_{AFN^\theta_0}^{3q\eta} \right]^{\mu} \left[ (d(x_{111},0),d(x_{122},0),\ldots,d(x_{m_1,m_2,\ldots,m_r-1,n_1,n_2,\ldots,n_s-1,k_1,k_2,\ldots,k_{t-1},0}) \right]_p \\
&\leq h^{H_2} h^{-a} \left| KA^\theta_\eta(\epsilon) \right| + |h(\epsilon)|^{H_2}.
\end{aligned}
\]

but

\[
\begin{aligned}
x &\notin \left[ \chi_{AFN^\theta_0}^{3q\eta} \right]^{\mu} \left[ (d(x_{111},0),d(x_{122},0),\ldots,d(x_{m_1,m_2,\ldots,m_r-1,n_1,n_2,\ldots,n_s-1,k_1,k_2,\ldots,k_{t-1},0}) \right]_p.
\end{aligned}
\]

4. CONCLUSION

In this paper we have studied some connections between \( \mu \)-lacunary strong \( \chi_{A_{uvw}}^3 \)-convergence with respect to a \( mnk \) sequence of Musielak Orlicz function and \( \mu \)-lacunary \( \chi_{A_{uvw}}^3 \)-statistical convergence, where \( A \) is a sequence of four dimensional matrices \( A(uvw) = \)
The results of this paper are more general than earlier results.

References


