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Finegrained 3D differential operators hint at the inevitability of their dual reciprocal portrayals

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ABSTRACT

Extending differential operators during transition from 3D operations to prospective 4D operations imply the need to expand their 4D range far beyond the usual set-theoretical universe from which subsets of the domain are composed. The prospective infrastructural expansion related to the attempted extending of operations virtually requires an extra dual reciprocal space and thus implies presence of a certain multispatial structure of both the mathematical- and the corresponding to it physical reality. At this point the implication is only operational for it is deduced from the attempted extension of operational domain/scope of geometric differential operator, which, in turn, demands an expansion of their range. The necessary presence of an extra space is not being postulated but emerges from comparative evaluations of differential operators. The operational necessity of presence of paired dual reciprocal spaces or quasispatial structures also generalizes contravariance for multispatiality.

***Keywords*:** Finegrained differential operators, paired dual reciprocal spaces

1. INTRODUCTION

Sir James Jeans once observed that it may be argued that everything is mathematical [1]. If so then mathematics must not be arbitrary but mathematical forms, operational procedures and the nature of the (corresponding to them) geometric structures should be discovered, not

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postulated. Yet operational procedures could be invented – or perhaps just guessed – from the operational symmetries that hint at the natural need for completeness of the proceduralized operations. This attempted operational approach, in turn, requires that the prospective development of mathematics should be guided by the experimental hints that suggest presence of some geometric or abstract quasigeometric structures that could fit the envisioned *a priori* complete operational procedures. The structures must not be arbitrary; if they actually do exist, it is because of completeness of the operations to be performed on them.

Then the discovered and abstracted mathematical structures should be matched to the operational procedures that are supposed to operate on the structures and vice versa. Then again, these – yet to be devised – more comprehensive prospective operational procedures should fit the, corresponding to them, geometric or quasigeometric structures. Neither should new mathematical ideas be encapsulated in artificially designed forms, no matter how conveniently they may simplify some – otherwise difficult to perform – calculations.

If the operational procedures are incomplete, then either the structures are broken or perhaps impossible to be implemented within the envisioned mathematical reality. In the latter case, the former paradigms underlying the reality – as it was previously envisaged – should be reevaluated afresh and altered accordingly, for existence of the reality cannot be arbitrary.

This abstract process of incremental development of mathematical theories demands new synthetic approach, because some of the previously designed structures may have features exceeding our present expectations as to our ability (or perhaps inability) to construct them. Some operational procedures may also contain (or just suggest the feasibility of) abstract operations that appear incompatible with the present interpretations of the structures to be operated on. The point is that we must not just discard structures which we cannot understand yet. Neither should we inhibit operations whose significance we cannot yet fully grasp.

On the one hand we must not tailor our new mathematical ideas to the present capabilities of our mathematical minds. On the other hand, we should never insist that all new ideas must always be directly or immediately derivable from what has already been – more or less rigorously – proved in the past, as Lagrange and Legendre did when they rejected Fourier's analysis, because the latter did not fit their preconceived ideas of mathematical derivations forming a pyramid like that of Euclid's, of their mathematical knowledge.

The main reason for proposing syntheses of various ideas rather than fighting for supremacy is that not everything developed in the past was quite correct nor was everything opposed, or just flatly rejected in the past, entirely wrong.

The doubtless success of practical applications of Fourier analysis shows that feedback from reasonings commonly used in physical sciences could be yet another source of new mathematical ideas regardless of one's predilection for pure reasonings or perhaps for their more abstract character. For more abstract does not necessarily equate to more realistic. It is common to point out that even very abstract mathematical ideas eventually find an application to reality. But it is rare to find an admission that some among the abstract ideas turned into obstacles preventing development of new theories. I am for equal opportunity, not against abstract creations, even though many of them imposed unreasonable prohibitions on the development. However, I am against curtailing inconvenient facts and suppressing truths.

The essence of the outlined synthetic approach is that neither algebraic nor geometric methods alone could suffice to devise realistic mathematics. The experimental feedback from physical applications is not meant that physics should control the development of abstract mathematics, but it should prevent arbitrary creations of artlike mathematical reasonings without relevance to realistic physical applications. If mathematics is to be discovered, the clues to what seems essential and what is merely accidental should come, or be deduced, from natural experiments. Deductions from axioms and primitive notions, whose formulation assumes presence of some kind of vision – as well as prejudices associated with the vision, no matter how close it seems to the physical reality reflected in the Nature – are inadmissible. This is not a critique of Euclid and his followers' ingenious deductive approach, who certainly did their best, but realization how far mathematical methods progressed since then. Presence of unreconciled results of curious yet definitely unbiased experiments testifies to both the progress but also to inadequacy of purely axiomatic deductions today. The development of realistic no-nonsense mathematics certainly needs feedback from physical experiments.

Some curious experimental results, though formerly quite unanticipated, suggest actual presence of an abstract multispatial structure of the mathematical reality that underlies physics as well. The fact that previously unreconciled results of curious yet unbiased experiments have been reconciled when reconsidered in multispatial framework [2-5], subtly yet persistently suggest that shifts of certain paradigms are indeed necessary in mathematics.

Also the former mathematics' failure to explain some experimentally confirmed ideas – such the wave-particle duality – demands to take a fresh look at the traditional mathematics, because even though de Broglie, for one, abandoned his theories built upon his idea of the aforesaid duality, the failure was not really his. For it was actually the former mathematics that failed him [6]. Yet to rectify the faulty traditional mathematics we need to establish new framework for the mathematics that experienced (or caused) three major conceptual failures, insofar as I am concerned. But the list of mathematical nonsenses is longer than that, which suggests that more mathematical and mathematics-imposed failures can be expected. In the present paper, however, I shall focus mainly on abstract mathematical (i.e. essentially theoretical) hints at the need for structural multispatial representations.

The apparently needed conceptual shift from the traditionally unspoken single space reality (SSR) paradigm to multispatial reality (MSR) paradigm demands very significant redefining of some algebraic as well as a few field- and differential operators.

2. WHY WE NEED REALISTIC GENERIC DIFFERENTIAL OPERATORS?

Structures invented primarily for the sake of proving (linguistically but not always realistically) that the structures conform to some lofty prejudices about what they should look like in order to appear as allegedly complying with conveniently concocted classifications, is meaningless if the structures cannot be unambiguously operated on, or cannot be constructed. The mixed (operationally structural or structurally operational) traditional definitions are rigged by already fixing the outcome of what was supposed to be discovered.

If an algebraic/operational structure is already defined as a set and an action, as it is common in pure mathematics, such definitions defy both operations and structures, because they tightly tie them together. This means that a superset containing elements incompatible with the action, is tacitly left outside the definition. Also, a superaction that could produce elements laying outside the original set would be tacitly left separated from the definition. This kind of approach is a formula for disaster, no matter how "scientific" it may appear.

Theory of categories mixes structural templates with operational actions; such a mixture can make it difficult, if not impossible, to unambiguously deploy so-defined structures in the synthetic approach to mathematics that attempts to match operational structures with the – corresponding to them – constructible geometric structures. If the "working" mathematician is investigating phenomena actually happening in the physical reality, then mixing actions, which are based on operations, with constructible structures that are supposed to be operated on, virtually derails some prospective investigations without us being aware of the fact. In a sense, categories can confuse any realistic reasonings in applied mathematics. Validity of operational procedures is easy to perceive, whereas constructability of geometric structures is not always so obvious. This is the main reason for relying on procedures involving differential operators rather than on artificial classification schemas some of which apparently have nothing to do with the actual physical reality that is revealed in curious unbiased experiments.

Perhaps neither the abstract nor the physical reality is so simple as to be encapsulated inside the abstract classification schemas apparently invented for talking or "proving" one's prejudices about the reality in question, but not really for working on realistic physical phenomena. So, let us start working instead of pretending to do some work by inventing more spaces named after their "inventors", most of which are not even worthy of the name 'space', for they are just sets whose existence is being postulated. There is just one physical space, namely the one in which birds fly, as Heisenberg reportedly once quipped, and everything else is either a set or an abstract view of a certain set of geometric or quasigeometric objects.

I am not discarding theory of categories but am showing that perhaps realistically-minded scientists should eventually stop the endless – and generally pointless – categorizing and start operating on realistic – as opposed to just "erected" in one's mind (or merely on paper) via convenient existential postulates usually disguised as definitions – structures and then also construct realistic geometric structures upon truly realistic operational procedures. The former categorized artlike mathematics can prove almost everything on paper but was apparently unable to reconcile even childishly simple yet previously quite unanticipated experimental results [2-4, 7]. To the undeniable credit of the traditional highly categorized pure mathematics I must admit that it created a lot of nonsenses, including even fake (i.e. not just false but rigged) theorems [8] while upholding such prohibitions on clear thinking as the quite unwarranted – not to say silly – prohibition of (doable [9, 10]) division by zero. This is the other reason for resorting to generic differential operators rather than to rely on the "elegant" inventions that cannot guarantee the completeness of operations not to mention their relevance to the geometric structures being operated on. I have already showed that differentials reveal complementary hidden variables at a deeper than present levels of investigation of physical phenomena [11]. Recall that the complementary hidden variables are not necessarily based on the same idea as the so-called supplementary hidden variables that were supposed to turn quantum physics into basically deterministic theory, even though the distinction between complementary and supplementary hidden variables could actually vanish in practice.

3. FIELD OPERATORS VS. DIFFERENTIAL OPERATORS

In the traditional nabla ∇ notation for a vector field W the left and right sides of the following operators are not really equal: $W \bullet \nabla \neq \nabla \bullet W$ for the left-hand side (LHS) is [all-purpose or general] scalar differential operator but RHS is scalar divergence, which is [also differential] field operator, so that $W \bullet \nabla = \sum_n W_n \frac{\partial}{\partial x^n}$ according to Chow [12, p. 22].

Nevertheless, since field operators involving differentials are actually differential field operators of the 'field operator' kind or class, as enlightening as the above Chow's remark was, it is still insufficiently precise to distinguish their usage and quite unambiguously express the character and purpose of all possible compound operations formed with the help of the nabla operator. The quest for precision is further intensified by the possibility that in a multispatial setting the attributes of interspatial operators may change in transit from one space to another. Thus, I shall not always distinguish field operators (including differential field operators such as the divergence) from prototypes/templates of general/all-purpose or compound differential operators such as W•∇. Since only homogeneous and noncurved orthogonal reference frames shall be investigated in the present paper, contravariant and covariant representations of vectors are expected to be the same, unless indicated otherwise.

4. FINEGRAINED 3D MNEMONIC DIFFERENTIAL OPERATORS

I shall consider only 3D rectilinear homogeneous orthogonal coordinates for the sake of simplicity of this presentation. Curvilinear coordinates are discussed in [13] p.23ff. Although differential operators are independent of our choice of coordinate system some of their representations can deviate from the conventionally accepted formulas. Abstract operators on algebraic varieties are concisely discussed in [14].

Homogeneous function was often defined as f(tx,ty) = tf(x,y), for example [15, 16]. Yet in reference to algebraic bases I shall use the term 'homogeneous' as equivalent to the phrase "denominated in the same units". Although the redefined operators are intended for multispatial structures, even in single space irrotational field need not always preclude the presence of circulation [17] nor can presence of nonradial twist of purely radial/center-bound scalar potential within radial force fields be denied [3, 18-20].

Summary of conventions: The subscript n = 1,2,3 runs through the usual orthogonal directions x,y,z, in the algebraic 3D Euclidean (i.e. conceived as composed of number points) space $\mathbb{R}^3$ – compare [13, pp. 65, 73]. In an algebraic or abstract geometric space $e_n = e_1,e_2,e_3$, i.e. n=1,2,3 runs through the values of the three directional unit vectors j,k,l. Whenever n is used as superscript, however, it signified just the contravariant component, not power, unless specifically mentioned. This convention is confusing and thus it should be abandoned in the future. According to the traditional convention symbols in bold font denote vectors. Note that the symbolic nabla operator ∇ standing in the above formulas does not mean that we can always multiply by the nabla operator [13], which is just as symbolic abbreviation of the formulas equipped with the algebraic summing sigma symbol Σ. It is the 3D unit radius (or pointing vector) r^n that is actually being multiplied by the functions F() operated on. The bases $e_1,e_2,e_3,\ldots,e_n$, for use in exterior linear spaces are discussed in [22, p. 49ff] and transitions between various bases are exemplified in [22, p. 125]. The 3D basis e_1,e_2,e_3 of a linear vector space is discussed in [23-26]. The wedge product $\wedge$ and the meet (or regressive) product $\vee$ shall be used when needed mainly for operations on multivectors in the framework of geometric algebra.

Let us redefine the usual field operators in the 3D Euclidean space $\mathbb{R}^3$ of number-points following Tai [13], by renaming them with qualifiers as: VGrad, SDiv, and VCurl as follows:

$$VGrad\ F(x,y,z) \coloneqq \sum_n \boldsymbol{r}^n \frac{\partial F(x)}{\partial x^n} \quad \mapsto \quad Grad\ F \Leftrightarrow \vec{\nabla} F \tag{1}$$

$$SDiv\ F(x,y,z) \coloneqq \sum_n r^n \bullet \frac{\partial F(x^n)}{\partial x^n} \quad \mapsto \quad Div\ F \Leftrightarrow \vec{\nabla} \bullet F \tag{2}$$

$$VCurl\ F(x,y,z) \coloneqq \sum_n r^n \times \frac{\partial F(x^n)}{\partial x^n} \quad \mapsto \quad Curl\ F \Leftrightarrow \vec{\nabla} \times F \tag{3}$$

which correspond to the traditional field operators: gradient, divergence and curl, respectively, of scalar and vector fields (F and F). Here the radius r^n is assumed to be unit vector represented in an orthogonal frame – compare [13, 27]. The qualifiers 'S' and 'V' preceding the mnemonic operators' names designate scalar and vector results of the actions of these operators, respectively. The main reason for the distinction is that previous usage of these differential field operators was oftentimes inconsistent and sometimes confusing, which fact was exposed and extensively discussed by Tai [13]. Additional expositions on various problems associated with the traditional nabla operator can be found in [28-33]. Vector field operators in curvilinear coordinates are discussed in [34].

The vectorial nabla operator $\vec{\nabla} = \sum_n \boldsymbol{r}^n$ signifies here the action of an abstract unit pointer vector (following Tai) even though the operator is not really a vector by itself, whereas the usual nabla ∇ could also signify scalarlike action mirroring that of the vectorial nabla in multispatial setting. Vector functions are shown in bold font and scalars in regular font, as usual. The symbol ↦ reads here: pertains; it indicates how the finegrained operators VGrad, SDiv and VCurl relate to the traditionally defined field operators Grad, Div and Curl.

The scalar divergence SDiv is similar to the traditional divergence operator Div if the unit vectors evaluate as $j^2 = k^2 = l^2 = -1$ which is geometrically necessary condition in both the complex and the quaternionic domain even though the compliance with geometry was not always respected. The bold dot • denotes the scalar product of two 3D vectors and the cross product × denotes the vector-valued product of two 3D vectors that yields axial vector perpendicular to these two. The scalar/algebraic multiplication symbol was usually omitted in the VGrad and Grad operators due to traditionally accepted algebraic convention.

The redefined mnemonic differential operators standing on the LHS correspond to three traditional 3D differential operators [13]. Here n = 1,2,3 in homogeneous orthogonal basis $e_n = e_1, e_2, e_3$ of 3D Euclidean spaces and for quasialgebraic representation of multivectors in geometric algebra. The finegrained definitions offered here are preliminary, not final.

By analogy to the operator VGrad I propose scalar gradient differential operator SGrad

$$SGrad\ F(x,y,z) \coloneqq \sum_n 1 \frac{\partial F_n(x^n)}{\partial x^n} \quad \mapsto \quad \nabla F \tag{4}$$

and similarly, I propose also an algebraic gradient differential operator AGrad defined as:

$$AGrad\ F(x,y,z) \coloneqq \sum_n [e_n \circ \frac{\partial F_n(x^n)}{\partial x^n}] = \sum_n e_n \circ SGrad\ F \quad \mapsto \quad \sum_n e_n \circ \nabla F \tag{5}$$

which are the respective scalar and algebraic counterparts of the traditional differential field operator Grad. The algebraic operator AGrad is suitable for expressions comprising the wedge and regressive products. SGrad represents the scalar differential of the function F(x,y,z) irrespective of any particular basis of the space in which the function F() is housed or hosted. It is also rendered in the compound multiplicative form like the other differential operators.

The algebraic qualifying prefix ‘A’ does specify generic scalar differential for backward compatibility with expressions used in geometric algebra. Notice that if the basis e_n would be under the nabla operator, the result would not always be the same as in (5). This is one more reason why the differential operators are redefined in the present paper. The light dot ∘ is an algebraic counterpart of the bold dot that signifies scalar product of vectors housed in a single space. Scalar multiplication is sum of products of elements in the same diagonal position: with subscripts jj, kk, ll, on the diagonal when depicted in matrix form. It can be used for scalar multiplication of vectors or quasivectors hosted in different homogeneous spaces too.

Since divergence of a vector field appears to be quite arbitrary and depend on the choice of axes in space [35] we need both an algebraic- and vector-valued operators acting just like the traditional scalar-valued divergence does. It is the schema of multiplication of diagonal components in the divergence that matters for us, not its scalar or vector envelope/depiction.

In fairly recent past it was *anathema* to suggest that energy should also be depicted vectorially like a particle – despite the experimentally confirmed wave-particle duality – and even today, after scattering of photons was observed [36], some still prefer to vehemently deny the need for enhanced operators that would make predictions of akin ideas and their consequences, feasible. Therefore, I shall introduce also vectorial and algebraic divergence operators in addition to the traditional scalar divergence in order to make that kind of depictions easier even if not always straightforward or unique.

The algebraic counterpart of the scalar divergence operator SDiv shall be called ADiv

$$ADiv\, F(x,y,z) \coloneqq \sum_n e_n \circ \frac{\partial F(x^n)}{\partial x^n} = \sum_n e_n \circ \nabla F = \sum_n e_n \circ SGrad\, F \quad \Leftrightarrow \quad AGrad\, F \quad (6)$$

which is equivalent to AGrad: ADiv⇔AGrad. It means that ADiv may be compounded under either divergence or gradient AGrad operator, if both are depicted in the same single space. There $\Sigma_n e_n$ runs through the 3D basis $e_n=e_1,e_2,e_3$ that is commonly used in geometric algebra. A vector counterpart of algebraic divergence operator VDiv can also be formed like ADiv

$$VDiv\, \boldsymbol{F}(x,y,z) \coloneqq \sum_n \boldsymbol{r}^n \circ \frac{\partial \boldsymbol{F}(x^n)}{\partial x^n} \quad \mapsto \quad VDiv\, \boldsymbol{F} = \vec{\nabla} \circ \boldsymbol{F} \Leftrightarrow VGrad\, \boldsymbol{F} \qquad (7)$$

where the $\Sigma_n r^n$ runs through the usual 3D vectorial basis (j,k,l) as before. Hence VDiv F equates to VGrad F so that: VDiv F = VGrad F which can be helpful in compounding.

To complete the basic set of mnemonic differential curl operators we need to define also the algebraic ACurl, scalar SCurl, wedge WCurl and meet MCurl, counterparts of the VCurl:

$$ACurl\, \boldsymbol{F}(x,y,z) \coloneqq \sum_n e_n \otimes \frac{\partial \boldsymbol{F}(x^n)}{\partial x^n} = \sum_n e_n \otimes SGrad\, F \qquad (8)$$

$$SCurl\, \boldsymbol{F}(x,y,z) \coloneqq \sum_n \boldsymbol{r}^n \otimes \frac{\partial \boldsymbol{F}(x^n)}{\partial x^n} = \sum_n \boldsymbol{r}^n \otimes VGrad\, F \qquad (9)$$

$$WCurl\, \boldsymbol{F}(x,y,z) \coloneqq \sum_n e_n \wedge \frac{\partial F(x^n)}{\partial x^n} = \sum_n e_n \wedge SGrad\, F \qquad (10)$$

$$MCurl\, \boldsymbol{F}(x,y,z) \coloneqq \sum_n e_n \vee \frac{\partial F(x^n)}{\partial x^n} = \sum_n e_n \vee SGrad\, F \qquad (11)$$

with the scalar counterpart ⊗ of the vector/cross product of vectors that is denoted by cross ×, which generates the axial vector VCurl. The symbol ∨ denotes the so-called regressive product of Grassmann algebra, which creates 'meet' (or intersection) whose result is dual to that of the wedge (or exterior) product ∧ which creates union [22]. Recall that union of two vectors creates bivector. The regressive product was routinely unmentioned in the former mathematics, perhaps in order to avoid the idea of multispatiality that lurks behind it.

Were it not for the fact that the unjustified SSR paradigm was reigning supreme, Grassmann might have realized that mathematical reality appears as a multispatial structure just by comparing the dual relationship of these two complementary products. Grassmann was intellectually courageous indeed, but his blind reliance on the, quite uncontested back then, SSR paradigm, presumably stupefied him just as almost everyone else until recently. What a waste of great mathematical mind endowed with enviable imagination.

5. DIFFERENTIALS INSTEAD OF FORMS OR FORMALIZED STRUCTURES

Although vectorial notation can simplify some calculations involving differentials, Tait already remarked that quaternions have slight advantage over vectors for unlike cross product of vectors quaternions retain associativity, for $j \times (k \times k) = 0 \neq (j \times k) \times k = -j$ in the (j,k,l) frame when the imaginary unit i is to be used as a symbolic operator [37]. Unlike 4D quaternions, 3D vectors are operationally incomplete oversimplifications of hypercomplex numbers, and thus neither forms nor vectors nor any other vectorlike formal structures should be used to decide the abstract operational nature of the structural objects corresponding to the hypercomplex numbers. The structural incompleteness of vectors, which implies their operational incompleteness too, prompted some authors to declare any division of vectors as meaningless [38] p.182, [39]. Therefore, differentials and differential operators are the safer choice for foolproof structural explorations and for the operational ones also.

Although it is well-known that multiplication of vectors by imaginary unit means rotation by 90° – compare [40-44, p. 61], the geometrization of imaginary number points is not quite as uncontroversial (under the SSR paradigm) as one might expected.

The main result of Poncelet's works was the introduction of imaginaries in geometry [45]. The geometrization of the – essentially algebraic – imaginaries apparently required the introduction of the inhomogeneous algebraic 4D basis [namely: 1,j,k,l in my present notation] because real quaternions form a linear associative algebra over the real numbers [46]. But the 4D quaternionic basis is not quite unique [47] and the quaternion group is of order 8 [48].

In short: these facts provide yet another theoretical hint suggesting that perhaps the abstract mathematical reality is not really confined to a nD simple space for n>4 but requires multispatial structures. It has also been argued that a quaternion manifold is a 4m-dimensional real manifold carrying an integrable almost quaternal structure [49], which is yet another indication that quaternionic and hypercomplex, in general, manifolds point to multispatiality. But such hints are simply too much to accept or digest even for the traditional mathematics that was born and raised under wings of the SSR paradigm. Physics was even more upset.

In the context of quantum mechanics, the terms 'real' and 'imaginary' can be used to describe many two-component quantities, but the terms do not refer to the 'existence' (i.e. ontology) of the components of the wave function [50], which may need different approach.

To avoid the conceptual confusion brought in by the (despised by physics) quaternions, Gibbs defined vector algebra in which nD vector is treated as tuple of n real numbers [51]. Given the resemblance of vectors and forces, calculus of vectors and then multivectors (in geometric algebra) is used in applied sciences. Thus, it was declared that 4D quaternion is an operator which changes one vector into another [52]. Yet the quaternion group of order 8 [48] also hints (indirectly) at the need of pairing of 4D heterogeneous spatial structures.

However, a [single] force is specified by its magnitude, direction and line of action; so as far as magnitude and direction are concerned, force may be specified by a vector [53]. But when it comes to line of action, which – in general – is a curve (that depends on at least the kinetics as Frenet-Serret formulas of differential geometry clearly indicate), which can also be twisted in motion, the force vector is not going to always remain the same during the realistic physical motion. It shall be shown later on that the usual reliance on the allegedly absolute tensor calculus may not always suffice. Neither multivectors of geometric algebra can supply such solutions. Recall that we can handle angular twisting of vectors but twisting due to radial forces was unheard of before.

Hence perhaps mathematics should not really be about the possibility of existence but rather about the feasibility of operations and the viability of representations of the prospective structures being operated on. The mathematics I foresee is like buying too large shoes for a little child with the expectation that the child will eventually grow up and then the big shoes will be ready for it to wear them comfortably. It may not be the wisest approach to raising children, but mathematics should learn to wear much bigger shoes, tailored for far greater scope of the abstract mathematical reality than we can imagine now. I prefer unrestricted possibilities rather than thinking in terms of shortsighted forms, often too restrictive or too simplified to be truly predictive. Mathematics that cannot support or just explain experimentally confirmed ideas is just a pointless game of often incoherent lofty words.

Recall that the scalar divergence of vector field **v** even in plane polar coordinates (r,θ) is

$$(v^r)_r + \frac{1}{r}v^2 + \frac{1}{r}(v^\theta)_\theta \tag{12}$$

compare [54]. Further elaboration of other possible variations of geometric and algebraic differential expressions shall be done elsewhere on as needed basis. It is not my intention to define all possible variants of differential operators but only to show that their evaluation subtly demands conceptual shift to the MSR paradigm. The elusive demand was not always perceived as such in the past. Abstract flows, fields and forms are discussed in [55].

Note that the redefined mnemonic differential and algebraic operators are of compound multiplicative group-theoretical character on purpose. While on the subject of algebraic and differential operators, a few other operators shall be defined as well. Some of the mnemonic differential operators are still not quite clearly defined, yet even their preliminary renditions show large variety of operations possible under the MSR paradigm.

In a 3D Euclidean space equipped with the homogeneous algebraic-geometric 3D basis $e_n = e_1,e_2,e_3$ that corresponds to the usual vectorial geometric basis comprising the three directional unit vectors j,k,l, which determine the directions of the variable coordinates x^n of the scalar function $F(x^n)$, these representations shall be distinguished for the sake of clarity and simplicity of evaluations.

6. SOME AUXILIARY NONDIFFERENTIAL OPERATORS

The nondifferential operator AFun is the result of assigning algebraic basis to a function

$$AFun\ F = \sum_n e_n \circ F_n(x^n) \tag{13}$$

which could emerge from the scalar nondifferential operator SFun of a scalar function F()

$$SFun\ F = \sum_n 1\ F_n(x^n) \tag{14}$$

which is just the given function in multiplicative compound form that can represent a 3D function, nevertheless. Notice that multiplying by multiplicative unit 1 does not alter the outcome of the operators but emphasizes their multiplicative group-theoretical character, which can make the control of possible errors during composing operators easier to discern.

The vectorial nondifferential operator VFun can also be formed by analogy as follows:

$$VFun\ F = \sum_n \boldsymbol{r}^n \circ F_n(x^n) \tag{15}$$

which is just a multiplicative vectorial representation of the given scalar function F(). The nondifferential operators are not absolutely essential, but they could be helpful in forming of various compound operators and/or multistaged operational procedures.

The auxiliary nondifferential operators and the multiplicative structure of the mnemonic operators – both differential and nondifferential – is patterned on the approach to symbolic functions, some of which may be no functions at all and even may not have definite values [56]. Theory of distributions, which deals with generalized functions, was devised under auspices of the formerly unspoken SSR paradigm and therefore is not yet immediately applicable to the multispatial reality that is proposed under the envisioned MSR paradigm.

As it seems natural to introduce the scalar product of functions as $<\phi,\psi>=\int_{-\infty}^{\infty}\phi\psi dx$ then if f(x) is an integrable function then the integral $\int_{-\infty}^{\infty} f(x)\phi(x)dx$ is definitely a continuous linear functional on the space of testing functions ϕ(). Symbolic functions can be combined algebraically as if they were ordinary functions, except that the product or quotient of two symbolic functions may not have a meaning. Yet just like the Dirac δ-function, which is a symbolic function that is neither integrable nor a regular function, the symbolic function becomes operationally viable when it is cast as linear functional [56]; compare also [57] where symbolic and infinite functions are dealt with in the complx domain $\mathbb{C}$. Derivatives of functionals (not functions) were concisely discussed in [58]. Recall that for linear functionals the conditions of continuity and boundedness are equivalent [59, p. I: 77]. Therefore I do not have objections when it comes to differentiation of even symbolic functionals.

7. GENERIC DIFFERENTIAL OPERATORS IN COMPLEX DOMAIN

The differential operator CDiff can be used to evaluate an algebraic complex differential

$$CDiff\ F(x^n) = \sum_n \{e_n \circ F(x^n)\}' = \sum_n \left\{e_n \circ \frac{\partial F(x^n)}{\partial x^n} + F(x^n) \circ de_n\right\} =$$

$$= \sum_n \left\{ e_n \circ \frac{\partial F(x^n)}{\partial x^n} + ie_n \circ F(x^n) \right\} = ADiv\, F + i\, AFun\, F = AGrad\, F + i\, AFun\, F \qquad (16)$$

in compliance with the product differentiation rule (PDR). The operator CDiff is complex-valued generic differential operator of a pseudogeometric function expressed in terms of geometric-algebra. The complex expression on the RHS arises because the 3D cyclic basis $e_n = e_1,e_2,e_3$ is rotated/cycled by the imaginary unit i so that $de_n = e_{n+1} = ie_n$ where the imaginary unit i serving as cyclic rotation operator for elements of the basis e_n with n = 1,2,3.

Although it is not absolutely necessary for grasping its conceptual meaning and role, one may compare the generic differentials to the covariant derivative/differential entertained in differential geometry [60] as well as to Lie derivative [61, 62]. The complex differential operator CDiff is a simplified compound differential due to its 3D homogeneous basis. Note that the differential operator (A×∇)×B where A and B are vector fields cannot be translated into exterior calculus but can be translated into Lie derivative [63, p. 164f]. This is yet another hint in favor of resorting to differential operators rather than relying on vector calculi.

The operator CDiff does not reveal anything new over what is already known in theory of analytic/holomorphic functions, but it confirms that a cyclic forwards-acting differential operator equates to algebraic gradient of the given function, indeed. In fact, none of these finegrained mnemonic differential operators defined above brings in any essentially new features. Nevertheless, what is conceptually new, namely that their dual evaluations virtually demand a paradigm shift from SSR to MSR, shall be shown below. Close conceptual relationship between abstract duality and ±∞ in complex domain is exhibited in [64].

The novelty of the demand for multispatial mathematics lays in that it was invisible and thus unrecognized before. Everything else is already known, even though some implications of the MSR paradigm have been tacitly avoided or even suppressed in order to fit the unspoken SSR paradigm. While avoidance – for fear of being labelled as unrealistic – is understandable, the suppression made under pretense of sticking to reality created many fictions maintained in defiance of various factual truths. This defiant sustenance of abstract fictions including fake theorems and few factually false claims is what caused deterioration of some mathematical reasonings which then are unsuspectingly used in physical sciences.

One among such former mathematical fictions is the traditional definition of divergence as exclusively scalar magnitude. Yet scalars, vectors and tensors are abstract attributes of representations, not necessarily the inherent characteristics of the physical magnitudes they are used to describe. This does not mean that scalar representation of divergence is invalid but just that when magnitudes traditionally perceived as scalars start varying while they are acted on by differential operators, their variability may not be the same in all direction, which it would be the case if they were innate scalars. Divergence is used in many physical applications [65]. Yet potential energy, which is compound notion presumed to behave like scalar just because its nature seems independent of directions in the usual space of motion, may need to be reclassified as revealing also vectorial features. Recall that the potential energy that is spent on the work done by the given force field – which is represented by scalar product – reveals directional features perceptible in multispatial representations of force fields [2] even though these features were defied in the past, presumably due to the SSR paradigm.

This double character of physical magnitudes is not unusual in multispatial framework. For prospective interspatial transitions, however, scalar, vector and algebraic representations should be available, regardless of whether or not we fully understood the transitions.

As a matter of fact, when rigorously evaluated, some of the anticipated extensions of the (expanded) traditional differential operators suggest presence of extra quasigeometric spatial structures, in order for the expanded operators to be operationally sound. It is not entirely the case that the traditional uses of differential operators were always wrong, but that they may be evaluated in several different ways – compare critiques in [13]. When the differential operators are rigorously evaluated, as it shall be done in what follows, then their necessary expansion may leave us with no other option but to adopt the MSR paradigm. This statement is not an argument but is used here to inform what to watch for. In other words: I am not presupposing that the abstract mathematical reality is to be assumed as multispatial one, but if the evaluation strictly adheres to proven rules of mathematics, then the properly evaluated and rigorously expanded differential operators virtually hint at the conceptual necessity – not just at remote possibility – of existence of quasispatial superstructure of the mathematical and consequently also the physical reality. Extension functions are defined for sets in Euclidean spaces in the traditional SSR setting [66].

8. DIFFERENTIAL OPERATORS CONTAIN TWIST

Abstract algebraic extension of the traditional differential operator Grad, which shall be called ADiff, of a 3D curve function F(x,y,z) = F(P) in the 3D space P, can now be defined as

$$\mathrm{ADiff\,F(x,y,z)} \coloneqq \sum\nolimits_{\mathrm{n}} \left[\mathrm{e_n} \circ \frac{\partial \mathrm{F_n(x^n)}}{\partial \mathrm{x^n}} + \mathrm{F_n(x^n)} \circ \frac{\partial \mathrm{e_n}}{\partial \mathrm{x^n}}\right] = \mathrm{AGrad\,F(P)} + \mathrm{ATwst\,F(P)} \qquad (17)$$

and similar vectorial extension of the traditional differential operator Grad can be defined as

$$\mathrm{VDiff\,F(x,y,z)} \coloneqq \sum\nolimits_{\mathrm{n}} \left[\boldsymbol{r}^n \circ \frac{\partial \mathrm{F_n(x^n)}}{\partial \mathrm{x^n}} + \mathrm{F_n(x^n)} \circ \frac{\partial \boldsymbol{r}^n}{\partial \mathrm{x^n}}\right] = \mathrm{VDiv\,F(P)} + \mathrm{VTwst\,F(P)} \qquad (18)$$

where the imaginary unit i is treated as coordinate-independent symbolic algebraic operator. In the sense we can write algebraically: $e_{n+1} = ie_n$ as it is common in the complex domain $\mathbb{C}$ and in geometric algebra, where it is usually treated as being coordinate-independent. That is why I denote the usual coordinate-dependent unit vectors of the orthogonal geometric basis (j,k,l) for the 3D rectangular frame (x,y,z). Yet the purely vectorial basis that corresponds to the geometric one could be interchanged with the algebraic basis (e_1,e_2,e_3). The algebraic basis can be used for vectors, bivectors and for multivectors [22]. For the time being, the superscripts denote orthogonal contravariant coordinates so that n = 1,2,3 in the 3D Euclidean space $\mathbb{R}^3$ that will also be called the primary space P. While extension pertains to extended scope of the operations (in terms of action), an expansion would expand the domain of the operators – in structural terms of the space(s) on which they operate – as well.

The left-hand side (LHS) term of the evaluation shown in eq. (17) resembles the algebraic gradient operator AGrad as it was defined in eq. (5), which can be restated with F(P) as

$$\mathrm{AGrad\,F(P)} \coloneqq \sum\nolimits_n e_n \circ \frac{\partial F_n(x^n)}{\partial \mathrm{x^n}} = \mathrm{AGrad\,F_n(x^n)} \quad \mapsto \quad \sum\nolimits_n e_n \circ \nabla F \qquad (19)$$

and ATwst is an algebraic twisting differential operator, for it acts perpendicularly to the curve represented by the 3D function F(P) = F(x,y,z) and its effective magnitude evaluates to

$$\mathrm{ATwst\ F(P)} \coloneqq \sum_{\mathrm{n}} \left[\mathrm{F_n(x^n)} \circ \frac{\partial \mathrm{e_n}}{\partial \mathrm{x^n}}\right] = \sum_{\mathrm{n}}[\mathrm{F_n(x^n)} \circ \mathrm{e_{n+1}}] = \sum_{\mathrm{n}}[\mathrm{F_n(x^n)} \circ \mathrm{ie_n}] \tag{20}$$

where the algebraic differential evaluates cyclically to: $\frac{\partial e_n}{\partial \mathrm{x^n}} = e_{n+1} = ie_n$ in real Euclidean 3D space $\mathbb{R}^3$. Complex Euclidean space is discussed in [67]. Complex-analytic symplectic spaces are presented in [68]. Similarly, the differential vector operator VTwst evaluates to

$$\mathrm{VTwst\ F(P)} \coloneqq \sum_{\mathrm{n}} \left[\mathrm{F_n(x^n)} \circ \frac{\partial \boldsymbol{r}^n}{\partial \mathrm{x^n}}\right] = \sum_{\mathrm{n}}[\mathrm{F_n(x^n)} \circ \boldsymbol{r}^{n+1}] = \sum_{\mathrm{n}}[\mathrm{F_n(x^n)} \circ i\boldsymbol{r}^n] \tag{21}$$

which is formally identical with ATwst. But their multispatial portrayals may differ.

Notice that the algebraic gradient operator AGrad is expressed in terms of the algebraic basis vectors (e_1,e_2,e_3) that are row vectors [69] (which can also be regarded as the basis of a tangent space [70]) resembles the vector operator VGrad that is usually cast in the orthogonal vectorial basis (j,k,l) of the 3D Euclidean space $\mathbb{R}^3$ that is identified with the primary space P which can also be alternatively equipped with the algebraic basis (e_1,e_2,e_3).

The fact that a twisting element arises in purely mathematical evaluation of the algebraic differential operator is not surprising to me. The detail that nonradial/twisting effects of purely radial force fields exist is a fact of differential geometry [21]. It was experimentally confirmed [2, 3], and resulted in generation of repulsive forces [18, 71-74], that depend also on density of matter (not only on density of mass) [20]. Hence the Galilei's claim that gravitational interactions are only radial and that they are independent of constitution of matter (i.e. of density of matter) is false and has already been refuted experimentally [3]. Yet faulty traditional physics still upholds and perpetuates such untenable deprecated nonsenses.

The algebraic twist ATwst was obtained from algebraic differential of direction in (17), (20), and the vectorial VTwst from differential of vectorial directions in (18), (21). This is the reason why algebraic approaches – including geometric/Clifford algebra – often fail to distinguish between their algebraized geometric representations and their analytic (though not holomorphic in the conventional complex-analytical sense of the term 'analytic') differential representations. Although algebraization of geometric and quasigeometric operations can be rightly viewed as simplifying cumbersome differential operations, the price to be paid for the oversimplification is often neglecting some terms that differentials would acquire if the differential operations would not be omitted. Such routinely neglected differential terms are complementary hidden variables [11]. I am not saying that geometric algebra, differential forms and the allegedly "absolute" tensor calculus are inherently faulty. But their scope of validity is not unrestricted. Nonetheless, I am showing here and in what follows that those celebrated methods can – and sometimes do – virtually curtail certain abstract ideas and concepts whose actual implementation seems impossible under the SSR paradigm, at least at present. Yet it is also true that some differential operators may have no solutions [75].

If considered in vectorial terms, entirely geometric evaluation of an attempted extension of the differential operator VGrad should be differentiated with respect to the arclength ds of the given trajectory curve function. Algebraic and geometric evaluations are not always equivalent, and the fact is that inconsiderate use of highly structured methods of geometric algebra can create some – often unrecognized – conceptual problems. I am not saying that methods of geometric algebra are inherently faulty or that they always lead to deficient evaluations, but only that they can tacitly conceal some subtle abstract geometric features, such as the apparently operational need for multispatiality, as shall be shown below.

The usual saying: let there be a differentiable manifold – as it is commonly practiced in the allegedly pure (even though virtually polluted by such rigged postulative assertions) former mathematics – is just a meaningless tacit supposition that postulates existence of objects that are sometimes quite impossible to construct. The tacit existential postulates (often disguised as rigorous definitions or declarations) can neither create the declared objects nor objectively imply their actual (i.e. constructive in geometric sense) existence. Traditional postulative approach to mathematics is defective conceptually and thus disastrous to some of the purported operations it allegedly performs.

9. GEOMETRIC DIFFERENTIAL OPERATORS INDUCE PAIRING OF SPACES

Vector-valued geometric differential operator GDiff that acts on a 3D scalar function F(x,y,z) = F(P) = F_p(P) = F_p in 3D space P equipped with homogeneous orthogonal vectorial and algebraic basis p=e_1,e_2,e_3 where n = 1,2,3 runs through directional unit vectors j,k,l, can be defined and – on evaluation according to product differentiation rule (PDR) – expressed as:

$$GDiff\ F(P) \coloneqq \sum_n [e_n \circ \frac{\partial F_n(x^n)}{\partial x^n} + F_n(x^n) \circ \frac{\partial e_n}{\partial x^n}] = AGrad\ F(P) + \sum_n \left[F_n(x^n) \circ \frac{\partial e_n}{ds}\frac{ds}{\partial x^n}\right] =$$

$$= AGrad\ F_p + \sum_n \left[F_n(x^n) \circ \frac{\partial e_n}{\partial s} \frac{1}{\frac{\partial x^n}{ds}}\ \right] = AGrad\ F_p + \sum_n \left\{\left[F_n(x^n) \circ \frac{\partial e_n}{\partial s}\right] \cdot V_n^{(-1)}(s)\right\} =$$

$$= AGrad\ F_p(P) + GTwst\ F_p(P) = AGrad\ F_p(P) + GTwst\ F\left\{Pdp_s \cdot\ V_n^{(-1)}(s)\right\} \tag{22}$$

which – in algebraic setting – splits into an algebraic gradient operator AGrad and an abstract yet locally geometric vectorial twisting differential operator GTwst, which can be defined as:

$$GTwst\ F(P) \coloneqq \sum_n \left\{\left[F_n(x^n) \circ \frac{\partial e_n}{\partial s}\right] \cdot V_n^{(-1)}(s)\right\} = GTwst\left\{F(P)dp_s \cdot V_n^{(-1)}(s)\right\} =$$

$$= GTwst\left\{F(P)dp_s \cdot V_n^{(-1)}(s)\right\} = GTwst\ \{FTurn\ F_p(P) \cdot VRecip(s)\} \tag{23}$$

where the twisting geometric differential operator GTwst, splits into two parts: FTurn and VRecip, even though the eq. (22) is evaluated solely under the traditional SSR paradigm.

Unlike the algebraic and vectorial differential operator ADiff, and VDiff, respectively, the geometric differential operator GDiff, which combines algebraic and vectorial features, clearly demands the presence of a reciprocal term VRecip in addition to FTurn of the other twists. It seems not impossible to simplify some of the derivations and obtain the same result in several different ways, but for the time being it is enough to show just the most persuasive way. The geometric differential operator GDiff is destined for use within the trihedron (i.e. the Frenet frame) that moves along the given trajectory curve and is codetermined also by the kinetics of the object comoving with the trihedron. The necessary presence of the reciprocal term means that the operator GDiff induces pairing of dual reciprocal spaces and thus virtually reveals the existence of a certain multispatial structure that is required to accommodate the operation performed by GDiff. Abstract dual pairings are discussed in the SSR setting in [76].

Note that while superscript conventionally indicates contravariant variable, the negative superscript (-1) in parentheses denotes here and henceforth a certain reciprocal entity (i.e. an inverse power), not contravariance.

The small dot · indicates algebraic multiplication whose operands are depicted in different bases of the same pair of dual reciprocal spaces. The term $F(P)dp_p$ is covariant for it has the form of differential in the basis p of the space P. Now I can define pseudooperator BTurn as

$$BTurn\ p = \sum_n \frac{\partial e_n}{\partial s}\ dp_s \tag{24}$$

as the base turning pseudooperator comoving with the moving trihedron of a Frenet frame.

The reciprocal pseudooperator $VRecip(s) = V_n^{(-1)}(s)$ is inverted reciprocal virtually depending on the arclength s yet apparently is shown as vector in yet another space equipped with a dual reciprocal basis, with respect to the space P with its homogeneous native orthogonal algebraic basis p. The pseudooperator VRecip shall be further explained elsewhere in the framework of its dual reciprocal space Q (equipped with its own native algebraic basis q), because both the path s and its differential dp_s are shown above in terms of the basis p of the primary space P and thus belong in the framework of the space P, not Q. The GDiff operator apparently demands specific locations of objects in spaces consistent with the objects' representations. There is no longer the allegedly single universal space that served as a mathematical universe, or a universal trash can, into which everything could be thrown. That is why the MSR paradigm with its multispatial representations needs to be adopted.

Differential operator FTurn means “function turn” turning the function F (in the primary 3D space P whose native algebraic basis is p) about the function's arclength-based path s

$$FTurn\ F(P) \coloneqq \sum_n F_p(P) \circ dp_s = \sum_n F_n(x^n) \circ \frac{\partial e_n}{\partial s} = \sum_n F_p(P) \circ\ BTurn\ p =$$

$$= AFun\ F_p(P) \circ\ BTurn\ p \tag{25}$$

which amounts to multiplication of the function by the base turn pseudooperator BTurn acting here on the native basis p of the primary space P. The frame-independent vectorial status of operators refers to their relevance to the trihedron that moves and turns with the Frenet frame. The eq. (22) indicates that the vector-valued geometric differential GDiff just cannot be contained within the single primary 3D geometric space P but demands also the deployment of a paired dual reciprocal geometric structure comprising two dual reciprocal spaces each equipped with their own homogeneous orthogonal native algebraic basis, if they are supposed to live in and thus move inside the trihedron. Pseudodifferential, differential and integral homogeneous operators were discussed in grouptheoretical setting in [77].

Concise exposition of transition between the local variables s and t is given in [78] in differential-geometric setting, which – although restricted to just single space under the traditional SSR paradigm – can be used as an analogy to the reasonings presented here. Formal operations on spaces having diverse algebraic bases were concisely discussed and exemplified by operational examples in [79], in the SSR framework, though. Since it is common to use power series expansions (as well as their inversions [80]) to evaluate functions, the eq. (22) indicates the – not so obvious under the SSR paradigm – necessity to evaluate the reciprocals within both: the native basis of the primary space and a foreign basis in an extra, distinct secondary reciprocal space that is dual to the given primary space. It does not matter whether the geometric differential operator GDiff is interpreted as splitting the operation of differentiation into terms housed in two dual reciprocal spaces or perhaps as inducing the pairing of dual reciprocal spaces. To me, the most astonishing result is that even such a simple operation as geometric differentiation cannot be performed entirely within single space, i.e. in the framework confined to auspices of the SSR paradigm.

Since vectors are mere 3D oversimplifications of 4D quaternions, there is a price to pay for the oversimplification when differentials are calculated. Former mathematics overlooked the need to pay the price and was doing everything to discredit those who realized that calling 3D vector pure quaternion does not really relate it to the 4D quaternions. This assertion is not just a matter of semantics. Former pure mathematics was trivialized by sincere scientists such as Gibbs, who intended to do good, but because he needed something more operational than 4D quaternions, whose interpretation under the SSR paradigm became somewhat unrelated to physics, codified vector calculus yet still set under the unspoken SSR paradigm.

This finegrained evaluation of the differential operator GDiff and its main consequence: the induced dual pairing of reciprocal spaces, is the new result attained in the present paper.

10. WHY GEOMETRIC DIFFERENTIALS NEED PAIRED DUAL SPACES?

What is the scoop of the derivation of the eq. (22)? Since the differential operator GDiff has been derived with respect to all relevant operational rules of differential calculus applied within a homogeneous basis, and has, nevertheless, generated undesirable contravariant terms, then there must be a reason for that. Apparently, the operator GDiff needs an extra dual reciprocal space to accommodate the geometric operations encapsulated in it. From the new synthetic point of view, my conclusion is that instead of trying to conceal the seemingly unwanted result of otherwise quite correct and legitimate operations, maybe we should expand the formerly unspoken SSR paradigm and thus admit an extra spatial structure in which the reciprocal term could be natively housed and its presence therein explained.

It is not that contravariant terms are inadmissible or that inhomogeneous bases are wrong, but the point is that the undesirable contravariant terms would destroy the heretofore maintained and desired homogeneity unless they are placed inside an extra space paired with the given primary space so that each space is equipped with its own homogeneous basis and both homogeneity and orthogonality can be preserved in each of the paired spaces. Homogeneity of geometric representations is crucial for maintaining both the orthogonality and unambiguous metric in each of the paired dual spaces. Admitting inhomogeneity would effectively turn the original geometry into a mere diagram and the spaces into mere sets/ensembles. Giving up geometry for a diagram is unacceptable trade-off; and to exchange spaces for sets, which can be agglomerations of arbitrary things, trashes almost all geometric concepts for nothing in return, regardless of whether we recognize that or not. Let me elaborate upon this conclusion in terms of covariance versus contravariance.

The set of quantities $\{W_a\}$ properly represents the components of a covariant vector W

$$W'_a = \frac{\partial x^b}{\partial x^{a'}} W_b \quad \Longrightarrow \quad W'_a \partial x^{a'} = \partial x^b W_b \tag{26}$$

if each item transforms according to the covariance rule on LHS of (26) [38] p.392, which retains the directly integrable form of differential $W'_a \partial x^{a'}$ so that it could be legitimately integrated and/or further differentiated. Yes, further – or deeper – differentiation also requires legitimately obtained and correctly formed and represented differentials on input.

The set of quantities denotes components of contravariant vector W if each transforms as

$$W'_a = \frac{\partial x^{a'}}{\partial x^b} W_b \quad \Longrightarrow \quad \frac{1}{\partial x^{a'}} W'_a = \frac{1}{\partial x^b} W_b \tag{27}$$

according to the contravariance rule on LHS of (27) [38] p.391, which implies reciprocal depiction of the contravariant vector, but not always its properly obtained (i.e. legitimately differentiated) actual differential if the depiction is being made within the same single space. For the reciprocal $\frac{1}{\partial x^b}$ on the LHS is not differential; hence: $\frac{1}{\partial x^b} \neq \frac{d}{\partial x^b}$ where the latter is a differential operator standing "in waiting" for an operand to operate on but the former is not.

Thus, the contravariant vector – or perhaps better: the vector represented in contravariant form – could not be treated as valid integrable differential if it is to be operated within the same single (and simple) 3D space. That is why some expressions obtained in tensor calculus could even inadvertently inspire often unrecognizable ambiguity and thus create conceptual confusions. Contravariant differentiation is akin to gradient in the SSR setting – see concise explanation in [81]. Yet sometimes it could inadvertently lead to ambiguous interpretations.

The vector differential operators ADiff, VDiff and GDiff comprise also intrinsic twist operators yet the apparently intrinsic mathematical feature was not always recognized in the traditional mathematics or physics. However, the presence of such a twist was confirmed in several experiments [2, 3, 18, 20, 82]. The abstract mathematical process of curving and twisting 3D curves representing trajectory paths of kinetically (i.e. not dynamically even though not necessarily regarded as passive projectiles) moving physical objects is also depicted in the Frenet-Serret formulas of differential geometry.

This fact does not defy the set-theoretical practice of defining mappings under the 3D SSR paradigm, but it hints at the possibility that if some traditional mathematical conventions are untenable in 4D spatial structures, then we should perhaps consider shifting the oversimplified SSR paradigm to the more comprehensive MSR paradigm. Although Leibniz's algebraic approach simplified operational calculations, the – often more difficult – Newton's geometric approach to differentials is geometrically and structurally more revealing, in general. Academic textbooks and papers rarely admitted that the usual tensorial line element ds in

$$ds^2 = g_{ab} dx^a dx^b \tag{28}$$

is deficient. Yet some authors honestly disclosed that there is a problem indeed, but usually tried to just sweep it under the proverbial carpet by justifying its use along the following lines: "Although ds is not a 1-form, it shares enough of the properties of a 1-form for it to be legitimate quantity for integration along a curve." [83]. It sounds just as: although foxes are not really like dogs, they share enough traits that we could put them in charge of a henhouse. After several generations of selective breeding and many hens lost, the problem could be "solved". But why not adopt shepherd dogs that can protect the hens without devouring them?

I have no problem with the justification, for sometimes we must use whatever we can, but I object to tacitly concealing inconvenient issues and to suppression of alternative approaches or virtual censorship of constructive critiques of methods based on the – allegedly absolute – tensor calculus. Inflated claims do not improve anything but only mislead and confuse.

Therefore, to merely write down a 4D operational equation on paper, or just ponder them in one's mind, does not necessarily make them operationally valid or structurally sound, even if the equation is not quite incorrect. Hence synthetic mapping of operational procedures to some corresponding to them geometric or quasigeometric structures is a must. Intuition alone is not enough. Proofs based on allegedly selfevident axioms can only delude or confuse.

11. THE ROLE OF INTUITION IN SYNTHETIC MATHEMATICS

According to Brouwer, formalists insist that mathematical exactness does exist on paper, whereas for intuitionists it exists in the human intellect [84]. Both of these views could be interpreted as having derogative connotations. Yet the main result of intuitionistic critique of negative propositions can be simply formulated as follows: For a general proposition it is in general meaningless to treat its negation as a [valid] definite proposition [85]. Perhaps that is why some authors concluded that intuitionism wanted to reconstruct mathematics on the basis of intuitively selfevident facts [86]. Maybe that is why some disillusioned pragmatists among mathematicians working in applied areas rejected intuitionism, because it became difficult, if not impossible, to find definitely selfevident theoretical facts that are nontrivial and yet still relevant to the problems at hand. In the present paper I have showed that when the informal (or less than formal from the traditional axiomatic point of view) intuitionism is employed in conjunction with the (new) synthetic approach to mathematics that I proposed and employed, these two became powerful combination for predictive reasonings, provided all the – deformalized on purpose, rather than informal, which descriptor is often identified with being notoriously lousy – derivations are rigorously and legitimately conducted. However, in this methodological mixture I did not try to decide right up front which facts may appear as inherently selfevident. Instead, I deployed mathematically unassailable operational procedures, which cannot be compromised or contradicted if they are complete, for they cannot be abolished as meaningless for no valid operational or structural reasons. As long as the most fundamental underlying paradigms remain unchanged, the operational procedures are expected to be reasonably stable. Combination of intuitionistic syntheses, under the constructive umbrella of operational procedures matched to prospective constructible structures, is evidently capable of revealing previously imperceptible operational structures, which would be normally dismissed as being not quite compatible with some mathematical prejudices inherited from past conclusions "rounded-up" in order not to disturb the past preconceptions often elevated to the status of indestructible constructions or invincible principles. Yet unlike natural principles derived from operations whose lifecycle is tied to the scope of validity of the operations, the (often arbitrary) unhinged principles invented only to support one's claims, are going to become either invalid (when their scope of validity shrinks to nothing) or obsolete and thus discarded.

12. CONCLUSIONS

The main result of the present paper is that even the fairly simple operational procedure that is encapsulated within the generic, vector-valued geometric differential operator GDiff apparently demands the split of the whole operational procedure of differentiation being performed into two distinct yet interrelated differential operations performed in two mutually dual reciprocal spaces.

Although this result was obtained under the traditional single space reality (SSR) paradigm it suggests the need for paradigm shift from the SSR to the multispatial reality (MSR) paradigm that was explained for the sake of comprehension but was not postulated. In the sense, the emerging demand to operate under the MSR paradigm has been synthesized from abstract features of differential operators rather than being postulated via a formal definition, which could be perceived as an existential postulate in disguise.

Hence the need to deploy the MSR paradigm is evidently imposed on us by the structure of differential operators, not by axiomatics invented to support some ingenious intuitions. This fact also confirms the supposition that mathematical features should be discovered rather than postulated, no matter how selfevident the prospective postulates and axioms may appear.

Since paired dual reciprocal spaces are just two distinct views of the same geometric or algebraic objects, scalar potentials of these spaces must be essentially identical, even though expressed in different native algebraic bases in which their spatial and temporal paths are represented. This supposition is justified by the fact that different spatial representations do not alter the innate character/nature of the represented objects.

When the finegrained differential and nondifferential operators are represented in abstract multiplicative formula, they can oftentimes reveal subtler operational and procedural features than the traditional representations of the same or similar operators did.

The mnemonic names of operators permit an easier grasp of the actions and the effects that result from applications of the operators, especially if they are compounded.

It has been indirectly shown that algebraic, vectorial, and scalar representations of differential and nondifferential operators do not really pertain to their inherent nature but just characterize the way in which they are depicted and perceived within the structures in which they are portrayed. The same is true of the functions these operators act upon.

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