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Lossless transmission lines terminated by crystal oscillator circuit

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ABSTRACT

The present paper deals with TEM propagation of waves along lossless transmission lines terminated by a crystal oscillator circuit. The crystal oscillator circuit generates nonlinear boundary conditions derived by the Kirchhoff's law. Then the mixed problem for the hyperbolic transmission line system to an initial value problem for neutral system on the boundary is reduced. The main purpose of the present paper is to show an existence-uniqueness of a periodic solution of the neutral system. This is achieved by introducing a suitable operator acting on a space of periodic functions. Its fixed point is a periodic solution of the neutral system. The advantages of the presented method on numerical example are demonstrated. The solution can be obtained by successive approximations.

***Keywords*:** Kirchhoff's law, Lossless transmission line, Mixed problem for hyperbolic system, Neutral equation, Periodic solution, Fixed point theorem

1. INTRODUCTION

Quartz and ceramic crystals are used in oscillator circuits for additional stability of frequency [1]. They provide a fairly high Q value that shows a small drift with temperature. Various applications in [2-21] are given. A simplified electrical equivalent circuit of a crystal oscillator is shown in Figure 1.

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The main purpose of the present paper is to consider lossless transmission lines terminated by a crystal oscillator circuit. From mathematical point of view the lossless transmission line system is of hyperbolic type. The general method for investigation of the mixed problem for lossless transmission line system is proposed in [22] and [23]. The same method to various problems (cf. [24-26]) is applied. Here we following this method derive boundary conditions on the base of the Kirchhoff's law and formulate the mixed problems for the lossless transmission line system. Then we reduce the mixed problem to an initial value problem on the boundary (cf. [18]). The system of equations on the boundary includes neutral type delay differential equations. Using operator formulation of the periodic problem we prove an existence-uniqueness of periodic solution as a fixed point of the operator in question.

The results are presented in six subsections. In 2.1 the boundary conditions are derived and the mixed problem for transmission line system is formulated. In 2.2 estimates of the arising nonlinearities are made. In 2.3 the mixed problem to an initial value problem on the boundary is reduced. In 2.4 an operator presentation of the periodic problem is given and some preliminary assertions are proved. In 2.5 the main result is proved - existence-uniqueness theorem for a periodic solution of the neutral system on the boundary. In 2.6 a numerical example is given to demonstrate the advantages of the method proposed. In the Conclusion it is shown of how to obtain a sequence of successive apprximations tending to the solution.

2. RESULTS

2. 1. Derivation of the boundary conditions and formulation of the mixed problem

In accordance with Figure 1 the Kirchhoff's laws to the left end of the line imply

$$E(t)-u(0,t)-Ri(0,t)=0,\ t\geq 0.$$

For the voltage at the right end $u_{right\ end}$ we obtain

$$u_{right\ end}=u_{R_1}+u_{C_1}+u_{\Psi_1}=R_1(i(\Lambda,t))+u_{C_1}+\frac{d\Psi_1}{dt}.$$

We use usually accepted denotations $C_0(u), R_1(i)$,$C_1(u), L_1(i)$ for characteristics of the nonlinear elements.

To calculate the voltage of the condenser $C_p(p=(0,1)$ we proceed from the relation assuming $u_{C_p}((x,T))\equiv u((x,T))=0$ (T will be prescribed below):

$$i_{C_p}=\frac{dq_{C_p}}{dt}=\frac{dC_p(u)}{dt}\Rightarrow\int_T^t i(x,\tau)d\tau=C_p\left(u(x,t)\right)-C_p\left(u(x,T)\right)=C_p\left(u(x,t)\right)-C_p\left(0\right),$$

$$u_{C_0}=u(\Lambda,t)=C_0^{-1}\left(C_0\left(0\right)+\int_T^t i(\Lambda,\tau)d\tau\right);\ u_{C_1}=C_1^{-1}\left(C_1\left(0\right)+\int_T^t i(\Lambda,\tau)d\tau\right).$$

In order to calculate the voltage of the inductor we proceed from the relation

$$u_{\Psi_1} = \frac{d\Psi_1}{dt} = \frac{dL_1(i)}{dt} = \frac{dL_1(i)}{di}\frac{di}{dt}.$$

Applying again the Kirchhoff's laws to the right end of the line we get ($x = \Lambda$)

$$u(\Lambda,t) = R_1(i(\Lambda,t)) + C_1^{-1}\left(C_1(0) + \int_T^t i(\Lambda,\tau)d\tau\right) + \frac{dL_1(i(\Lambda,t))}{di}\frac{di(\Lambda,t)}{dt}$$

or

$$C_0^{-1}\left(C_0(0) + \int_T^t i(\Lambda,\tau)d\tau\right) = R_1(i(\Lambda,t)) + C_1^{-1}\left(C_1(0) + \int_T^t i(\Lambda,\tau)d\tau\right) + \frac{dL_1(i(\Lambda,t))}{di}\frac{di(\Lambda,t)}{dt}.$$

We consider the lossless transmission line system of equations

$$\frac{\partial u(x,t)}{\partial x} + L\frac{\partial i(x,t)}{\partial t} = 0, \quad \frac{\partial i(x,t)}{\partial x} + C\frac{\partial u(x,t)}{\partial t} = 0 \qquad (1)$$

where the unknown functions are voltage $u(x,t)$ and current $i(x,t)$ along the line, L is per unit-length inductance, C – per unit-length capacitance, Λ – the length of the transmission line and $v = 1/\sqrt{LC}$ – the speed of propagation of the waves.

This system is of hyperbolic type and we can formulate a mixed problem: to find a solution $(u(x,t), i(x,t))$ of (1) for $(x,t) \in \Pi = \{(x,t) \in R^2 : 0 \le x \le \Lambda,\ t \ge 0\}$, satisfying the initial conditions

$$u(x,0) = u_0(x),\ i(x,0) = i_0(x) \ \text{ for } \ x \in [0,\Lambda],$$

where $u_0(x),\ i_0(x)$ are prescribed functions. The boundary conditions are already derived for $x = 0$

$$E(t) - u(0,t) - R\,i(0,t) = 0,\ t \ge 0$$

and for $x = \Lambda$

$$C_0^{-1}\left(C_0(0) + \int_T^t i(\Lambda,\tau)d\tau\right) = R_1(i(\Lambda,t)) + C_1^{-1}\left(C_1(0) + \int_T^t i(\Lambda,\tau)d\tau\right) + \frac{dL_1(i(\Lambda,t))}{di}\frac{di(\Lambda,t)}{dt}.$$

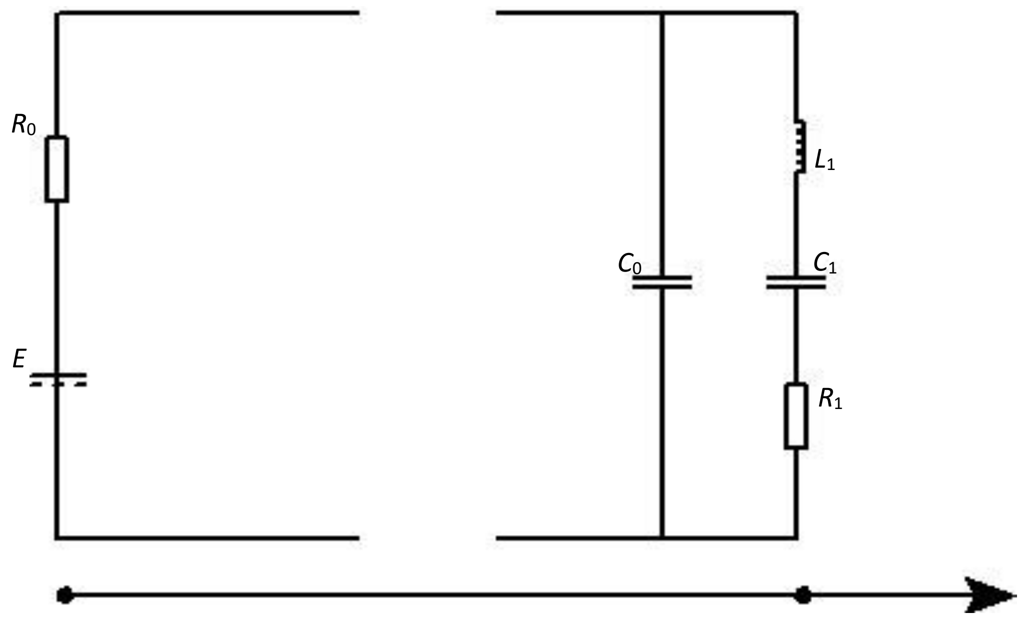

Figure 1. Lossless transmission line terminated by crystal oscillator circuit.

2. 2. Estimates of arising nonlinearities

Denote by $T=\Lambda/v=\Lambda\sqrt{LC}$. Here we consider nonlinear capacitive elements with characteristics $C_p(u)=\frac{c_p}{\sqrt[h]{1-(u/\Phi_p)}}$, $(p=0,1)$, where $c_p>0,\Phi_p>0,\ h\in[2,3]$ are constants and assume $|u|\le\phi_0<\min\{\Phi_0,\Phi_1\}$.

For the derivatives we get

$$\frac{dC_p(u)}{du}=c_p\left(-\frac{1}{h}\right)\left(1-(u/\Phi_p)\right)^{-\frac{1}{h}-1}\left(-\frac{1}{\Phi_p}\right)=\frac{c_p}{h\Phi_p\left(1-(u/\Phi_p)\right)^{\frac{1}{h}+1}}\ge\frac{c_p}{h\Phi_p\left(1+(\phi_0/\Phi_p)\right)^{\frac{1}{h}+1}}=\widehat{C}_p^1>0 .$$

Therefore $C_p^{-1}(u)$ exists and since

$$C_p(u):[-\phi_0,\phi_0]\to\left[\frac{c_p}{\sqrt[h]{1+(\phi_0/\Phi_p)}},\frac{c_p}{\sqrt[h]{1-(\phi_0/\Phi_p)}}\right]\Rightarrow$$

$$C_p^{-1}(u):\left[\frac{c_p}{\sqrt[h]{1+(\phi_0/\Phi_p)}},\frac{c_p}{\sqrt[h]{1-(\phi_0/\Phi_p)}}\right]\to[-\phi_0,\phi_0].$$

For the *I-V* and *I-L* characteristics we take polynomial $R_1(i)=\sum_{n=1}^{3}r_ni^n$ and $L_1(i)=\sum_{n=1}^{3}l_ni^n$

Assumption (L): $\left|i\right| \le i_0 \;\Rightarrow\; \dfrac{dL_1(i)}{di} = \sum_{n=1}^{3}(n+1)l_n i^n \ge \hat{L}_1 > 0$.

2. 3. Reducing the mixed problem to an initial value problem on the boundary

Proceeding as in [1] we transform (1) in a diagonal form

$$\begin{pmatrix} \dfrac{\partial U}{\partial t} \\ \dfrac{\partial I}{\partial t} \end{pmatrix} + \begin{pmatrix} \dfrac{1}{\sqrt{CL}} & 0 \\ 0 & -\dfrac{1}{\sqrt{CL}} \end{pmatrix} \begin{pmatrix} \dfrac{\partial U}{\partial x} \\ \dfrac{\partial I}{\partial x} \end{pmatrix} = 0$$

by the formulas

$$u(x,t) = \frac{1}{2}U(x,t) + \frac{1}{2}I(x,t),\;\; i(x,t) = \frac{1}{2Z_0}U(x,t) - \frac{1}{2Z_0}I(x,t).$$

Integration along the characteristics yields $U(\Lambda,t) = U(0,t-T)$, $I(\Lambda,t-T) = I(0,T)$. We assume that the unknown functions are $U(0,t) \equiv U(t), I(\Lambda,t) \equiv I(t)$ and then in view of $U(\Lambda,t) = U(0,t-T) = U(t-T)$ and $I(0,t) = I(\Lambda,t-T) = I(t-T)$ we obtain

$$u(0,t) = \frac{U(0,t)+I(0,t)}{2} = \frac{U(t)+I(t-T)}{2};\; u(\Lambda,t) = \frac{U(\Lambda,t)+I(\Lambda,t)}{2} = \frac{U(t-T)+I(t)}{2};$$

$$i(0,t) = \frac{U(0,t)-I(0,t)}{2Z_0} = \frac{U(t)-I(t-T)}{2Z_0};\;\; i(\Lambda,t) = \frac{U(t-T)-I(t)}{2Z_0}.$$

Now we are able to formulate a mixed problem for the last system with new initial conditions

$$U(x,0) = u(x,0) + Z_0\, i(x,0) = u_0(x) + Z_0 i_0(x),$$
$$I(x,0) = u(x,0) - Z_0\, i(x,0) = u_0(x) - Z_0 i_0(x).$$

Using the formulas

$$u(0,t) = \frac{1}{2}U(0,t) + \frac{1}{2}I(0,t),\;\; i(0,t) = \frac{1}{2Z_0}U(0,t) - \frac{1}{2Z_0}I(0,t),$$
$$u(\Lambda,t) = \frac{1}{2}U(\Lambda,t) + \frac{1}{2}I(\Lambda,t),\;\; i(\Lambda,t) = \frac{1}{2Z_0}U(\Lambda,t) - \frac{1}{2Z_0}I(\Lambda,t)$$

we obtain the new boundary conditions

$$E(t)-\frac{U(t)+I(t-T)}{2}-R\frac{U(t)-I(t-T)}{2Z_0}=0,\ t\in[T,2T];$$

$$C_0^{-1}\left(C_0(0)+\int_T^t\frac{U(s-T)-I(s)}{2Z_0}ds\right)=R_1\left(\frac{U(t-T)-I(t)}{2Z_0}\right)+$$
$$+C_1^{-1}\left(C_1(0)+\int_T^t\frac{U(s-T)-I(s)}{2Z_0}ds\right)+L_1'\left(\frac{U(t-T)-I(t)}{2Z_0}\right).\frac{d}{dt}\frac{U(t-T)-I(t)}{2Z_0},\ t\in[T,2T]$$

for $\Pi=\{(x,t)\in R^2: x\in[0,\Lambda],\ t\in[T,2T]\}$. The initial functions should be defined on $[0,T]$.

We transform the above boundary conditions in the form

$$U(t)=\frac{2Z_0E(t)}{R_0+Z_0}+\frac{R_0-Z_0}{R_0+Z_0}I(t-T),\ t\in[T,2T], \tag{2}$$

$$\frac{dI(t)}{dt}=\frac{dU(t-T)}{dt}+\frac{2Z_0}{L_1'\left(\frac{U(t-T)-I(t)}{2Z_0}\right)}R_1\left(\frac{U(t-T)-I(t)}{2Z_0}\right)+$$
$$+\frac{2Z_0}{L_1'\left(\frac{U(t-T)-I(t)}{2Z_0}\right)}\left[C_1^{-1}\left(C_1(0)+\int_T^t\frac{U(s-T)-I(s)}{2Z_0}ds\right)-C_0^{-1}\left(C_0(0)+\int_T^t\frac{U(s-T)-I(s)}{2Z_0}ds\right)\right],$$
$$t\in[T,2T]. \tag{3}$$

It is known (cf. [20]) that the above mixed problem is equivalent to the system (2), (3) of delay differential equations. Our main goal of the present paper is to find a T_0-periodic solution for (2), (3) with T_0-periodic initial conditions $U_0(.),\ I_0(.)\in C_{T_0}[0,T]$.

To obtain initial functions on $[0,T]$ one can shift the initial function of the mixed problem from the interval $[0,\Lambda]$ along the characteristic to the interval $[0,T]$. (cf.[20]).

2. 4. Operator presentation of the periodic problem and some preliminary assertions

Assuming $T=mT_0$ for some positive integer *m*, we introduce the sets of the unknown functions $U(t),\ I(t)$:

$$M_U=\left\{U\in C_{T_0}[T,2T]:\ |U(t)|\le U_0e^{\mu(t-T-kT_0)},t\in[T+kT_0,T+(k+1)T_0]\right\},(k=0,1,2,\ldots,m-1),$$

$$M_I=\left\{I\in C_{T_0}[T,2T]:\ \int_{T+kT_0}^{T+(k+1)T_0}I(t)dt=0\ \wedge\ |I(t)|\le I_0e^{\mu(t-T-kT_0)},t\in[T+kT_0,T+(k+1)T_0]\right\},$$
$$(k=0,1,2,\ldots,m-1)$$

where $C^1_{T_0}[T,2T]$ is the set of all continuous T_0-periodic functions and U_0, I_0, T_0, μ are prescribed positive constants and $\mu T_0 = \mu_0 = \text{const.}$

Introduce a suitable family of metrics we turn the set $M_U \times M_I$ into a metric space

$$\rho_k(U,\bar{U}) = \sup\left\{ e^{-\mu(t-T-kT_0)} \left| U(t) - \bar{U}(t) \right| : t \in [T+kT_0, T+(k+1)T_0] \right\}, (k = 0,1,2,\ldots,m-1),$$

$$\rho_k(I,\bar{I}) = \sup\left\{ e^{-\mu(t-T-kT_0)} \left| I(t) - \bar{I}(t) \right| \ : \ t \in [T+kT_0, T+(k+1)T_0] \right\}, (k = 0,1,2,\ldots,m-1),$$

$$\rho\left((U,I),(\bar{U},\bar{I})\right) = \max\left\{\rho_k(U,\bar{U}), \rho_k(I,\bar{I}) : k = 0,1,2,\ldots,m-1\right\}.$$

Now we formulate the main problem: to find a T_0-periodic solution $\left(U(t), I(t)\right)$ of (2) on $[T,2T]$ continuing T_0-periodic initial functions.

We call a generalized solution of the periodic problem the fixed point of the operator B defined below. Indeed, assuming that $E(t)$ is a T_0-periodic function we define an operator with components $B = (B_U(t), B_I(t))$ on $[T,2T]$ by the formulas:

$$B_U(U,I)(t) := \frac{2Z_0}{Z_0 + R_0} E(t) + \frac{R_0 - Z_0}{R_0 + Z_0} I(t-T) \ , t \in [T,2T]$$

$$B_I^{(k)}(U,I)(t) := \int_{T+kT_0}^{t} J(U,I)(s)ds - \left(\frac{t - T - kT_0}{T_0} - \frac{1}{2} \right) \int_{T+kT_0}^{T+(k+1)T_0} J(U,I)(s)ds - \frac{1}{T_0} \int_{T+kT_0}^{T+(k+1)T_0} \int_{T+kT_0}^{t} J(U,I)(s)dsdt$$

on every interval $[T+kT_0, T+(k+1)T_0]$ ($k = 0,1,2,\ldots,m-1$), $T = mT_0$, where

$$J(U,I) = \frac{dU(t-T)}{dt} + \frac{2Z_0}{L_1'\left(\left(U(t-T) - I(t)\right)/\left(2Z_0\right)\right)} R_1\left(\frac{U(t-T) - I(t)}{2Z_0} \right) +$$

$$+ \frac{2Z_0}{L_1'\left(\left(U(t-T) - I(t)\right)/\left(2Z_0\right)\right)} \left[C_1^{-1}\left(C_1(0) + \int_{T+kT_0}^{t} \frac{U(s-T) - I(s)}{2Z_0} ds \right) - C_0^{-1}\left(C_0(0) + \int_{T+kT_0}^{t} \frac{U(s-T) - I(s)}{2Z_0} ds \right) \right],$$

$t \in [T+kT_0, T+(k+1)T_0]$.

We notice that $U(t-T), I(t-T)$ are substituted by the initial functions and assume that $U(T) = I(T) = 0 \Rightarrow U(T+kT_0) = I(T+kT_0) = 0$.

Lemma 1. [20] If $U(.) \in M_U$ $\left(\text{resp. } I(.) \in M_I\right)$, then $F(t) = \int_{T+kT_0}^{t} U(\tau)d\tau$ (resp. $G(t) = \int_{T+kT_0}^{t} I(\tau)d\tau$) is a T_0-periodic function.

Proof: Indeed, we have

$$F(t+T_0)=\int_{T+kT_0}^{t+T_0}U(\tau)d\tau=\int_{T+kT_0}^{t}U(\tau)d\tau+\int_{t}^{t+T_0}U(\tau)d\tau=F(t)+\int_{t}^{T+kT_0}U(\tau)d\tau+\int_{T+kT_0}^{T+(k+1)T_0}U(\tau)d\tau+\int_{T+(k+1)T_0}^{t+T_0}U(\tau)d\tau=$$
$$=F(t)+\int_{t}^{T+kT_0}U(\tau)d\tau+\int_{T+kT_0}^{t}U(\theta)d\theta=F(t).$$

Lemma 1 is thus proved.

Lemma 2. If the initial functions and the source function $E(t)$ are T_0-periodic ones and $(U,I)\in M_U\times M_I$, then $V(U,I)(t),\ J(U,I)(t)$ are T_0-periodic functions.

The proof is straightforward, based on Lemma 1.

Lemma 3. [22] For every $\left(U(.),I(.)\right)\in M_U\times M_I$ it follows

$$\int_{T+kT_0}^{T+(k+1)T_0}\int_{T+kT_0}^{s}J(U,I)(\theta)d\theta ds=\int_{T+(k+1)T_0}^{T+(k+2)T_0}\int_{T+(k+1)T_0}^{s}J(U,I)(\theta)d\theta ds\ \ (k=0,1,2,\ldots).$$

Proof: Changing the variable $\xi=s-T_0$ we have

$$\int_{T+(k+1)T_0}^{T+(k+2)T_0}\int_{T+(k+1)T_0}^{s}J(U,I)(\theta)d\theta ds=\int_{T+kT_0}^{T+(k+1)T_0}\int_{T+kT_0}^{\xi+T_0}J(U,I)(\theta)d\theta d\xi=$$
$$=\int_{T+kT_0}^{T+(k+1)T_0}\left(\int_{T+kT_0}^{\xi}J(U,I)(\theta)d\theta+\int_{\xi}^{\xi+T_0}J(U,I)(\theta)d\theta\right)d\xi=\int_{T+kT_0}^{T+(k+1)T_0}\left(\int_{T+kT_0}^{\xi}J(U,I)(\theta)d\theta\right)d\xi.$$

Lemma 3. is thus proved.

Lemma 4. [22] The function $B_U(U,I)(t)$ belongs to M_U and $B_I(U,I)(t)$ $-$ to M_I.

Proof: Since $J(U,I)(t)$ is T_0-periodic we have

$$\int_{t}^{t-T_0}J(U,I)(s))ds+\int_{T+kT_0}^{T+(k+1)T_0}J(U,I)(s))ds=0\ \text{ and }\int_{T+(k+1)T_0}^{T+(k+2)T_0}J(U,I)(s))ds=\int_{T+kT_0}^{T+(k+1)T_0}J(U,I)(s))ds.$$

Besides if $t\in[T+kT_0,T+(k+1)T_0]\Rightarrow t+T_0\in[T+(k+1)T_0,T+(k+2)T_0]$. Then we have to show that $B_I^{(k)}(U,I)(t-T_0)=B_I^{(k+1)}(U,I)(t),\ t\in[T+(k+1)T_0,T+(k+2)T_0]$. Indeed, in view of Lemma 3 we obtain

$$B_I^{(k)}(U,I)(t-T_0)=\int_{T+kT_0}^{t-T_0}J(U,I)(s)ds-\left(\frac{t-T_0-T-kT_0}{T_0}-\frac{1}{2}\right)\int_{T+kT_0}^{T+(k+1)T_0}J(U,I)(s)ds-\frac{1}{T_0}\int_{T+kT_0}^{T+(k+1)T_0}\int_{T+kT_0}^{t}J(U,I)(s)dsdt=$$

$$= \int\limits_{T+kT_0}^{T+(k+1)T_0} J(U,I)(s)ds + \int\limits_{T+(k+1)T_0}^{t} J(U,I)(s)ds + \int\limits_{t}^{t-T_0} J(U,I)(s)ds - \left(\frac{t-T-(k+1)T_0}{T_0}-\frac{1}{2}\right)\int\limits_{T+(k+1)T_0}^{T+(k+2)T_0} J(U,I)(s)ds -$$

$$-\frac{1}{T_0}\int\limits_{T+kT_0}^{T+(k+1)T_0}\int\limits_{T+kT_0}^{t} J(U,I)(s)dsdt =$$

$$= \int\limits_{T+(k+1)T_0}^{t} J(U,I)(s)ds - \left(\frac{t-T-(k+1)T_0}{T_0}-\frac{1}{2}\right)\int\limits_{T+(k+1)T_0}^{T+(k+2)T_0} J(U,I)(s)ds -$$

$$-\frac{1}{T_0}\int\limits_{T+(k+1)T_0}^{T+(k+2)T_0}\int\limits_{T+(k+1)T_0}^{t} J(U,I)(s)dsdt = B_I^{(k+1)}(U,I)(t).$$

One can show in the same way that $B_I^{(k)}(U,I)(t-T_0) = B_I^{(k+1)}(U,I)(t)$.

Let us show that $\int\limits_{T+kT_0}^{T+(k+1)T_0} B_I^{(k)}(U,I)(t)\,dt = 0$. Indeed, since $\int\limits_{T+kT_0}^{T+(k+1)T_0}\left(\frac{t-T-kT_0}{T_0}-\frac{1}{2}\right)dt = 0$ we have

$$\int\limits_{T+kT_0}^{T+(k+1)T_0} B_I^{(k)}(U,I)(t)\,dt = \int\limits_{T+kT_0}^{T+(k+1)T}\int\limits_{T+kT_0}^{t} J(U,I)(s)dsdt -$$

$$-\int\limits_{T+kT_0}^{T+(k+1)T_0}\left(\frac{t-T-kT_0}{T_0}-\frac{1}{2}\right)dt \int\limits_{T+kT_0}^{T+(k+1)T_0} J(U,I)(s)ds - \int\limits_{T+kT_0}^{T+(k+1)T_0}\int\limits_{T+kT_0}^{t} J(U,I)(s)dsdt = 0.$$

We prove that $B_U(U,I)(t),\ B_I(U,I)(t)$ are continuous functions.
For $B_I(U,I)(t)$ we get

$$B_I^{(k)}(U,I)(T+(k+1)T_0) = \int\limits_{T+kT_0}^{T+(k+1)T_0} J(U,I)(s)ds - \left(\frac{T+(k+1)T_0-T-kT_0}{T_0}-\frac{1}{2}\right)\int\limits_{T+kT_0}^{T+(k+1)T_0} J(U,I)(s)ds -$$

$$-\frac{1}{T_0}\int\limits_{T+kT_0}^{T+(k+1)T_0}\int\limits_{T+kT_0}^{s} J(U,I)(\theta)d\theta ds = \frac{1}{2}\int\limits_{T+kT_0}^{T+(k+1)T_0} J(U,I)(s)ds - \frac{1}{T_0}\int\limits_{T+kT_0}^{T+(k+1)T_0}\int\limits_{T+kT_0}^{s} J(U,I)(\theta)d\theta ds.$$

On the other hand

$$B_I^{(k+1)}(U,I)(T+(k+1)T_0) := \int\limits_{T+(k+1)T_0}^{T+(k+1)T_0} J(U,I)(s)ds - \left(\frac{T+(k+1)T_0-T-(k+1)T_0}{T_0}-\frac{1}{2}\right)\int\limits_{T+(k+1)T_0}^{T+(k+2)T_0} J(U,I)(s)ds -$$

$$-\frac{1}{T_0}\int\limits_{T+(k+1)T_0}^{T+(k+2)T_0}\int\limits_{T+(k+1)T_0}^{s} J(U,I)(\theta)d\theta ds = \frac{1}{2}\int\limits_{T+(k+1)T_0}^{T+(k+2)T_0} J(U,I)(s)ds - \frac{1}{T_0}\int\limits_{T+(k+1)T_0}^{T+(k+2)T_0}\int\limits_{T+(k+1)T_0}^{s} J(U,I)(\theta)d\theta ds.$$

Since $J(U,I)(s)$ is T_0-periodic function then $\int\limits_{T+kT_0}^{T+(k+1)T_0} J(U,I)(s)ds = \int\limits_{T+(k+1)T_0}^{T+(k+2)T_0} J(U,I)(s)ds.$

In view of Lemma 3 $\int\limits_{T+kT_0}^{T+(k+1)T_0}\int\limits_{T+kT_0}^{s} J(U,I)(\theta)d\theta ds = \int\limits_{T+(k+1)T_0}^{T+(k+2)T_0}\int\limits_{T+(k+1)T_0}^{s} J(U,I)(\theta)d\theta ds.$

Therefore $B_I^{(k)}(U,I)(T+(k+1)T_0)=B_I^{(k+1)}(U,I)(T+(k+1)T_0)$.
Lemma 4 is thus proved.

Remark 1. Since $C_p^{-1}(u):\left[\dfrac{c_p}{\sqrt[h]{1+(\phi_0/\Phi_p)}},\dfrac{c_p}{\sqrt[h]{1-(\phi_0/\Phi_p)}}\right]\to[-\phi_0,\phi_0]$ we get

$$0\le\left|C_0(0)+\int_0^s\frac{U(\theta-T)-I(\theta)}{2Z_0}d\theta\right|\le c_0+\frac{U_0e^{-\mu T}+I_0}{2Z_0}\frac{e^{\mu T_0}}{\mu}\le c_0+\frac{c_0}{\sqrt[h]{1-(\phi_0/\Phi_0)}}.$$

It follows $\left|C_p^{-1}(u)\right|\le\phi_0$.

Lemma 5. The periodic problem (2), (3) has a unique solution $(U(.),I(.))\in M_U\times M_I$ iff the operator B has a fixed point $(U(.),I(.))\in M_U\times M_I$, that is, $U=B_U(U,I),\quad I=B_I(U,I)$.

Proof: Let $(U(.),I(.))\in M_U\times M_I$ be a solution of the periodic problem:

$$\begin{aligned}&\frac{dU(t)}{dt}=V(U,I)(t),t\in[T,2T]\\&U(t)=U_0(t),\frac{dU(t)}{dt}=\frac{dU_0(t)}{dt},t\in[0,T]\end{aligned},\qquad\begin{aligned}&\frac{dI(t)}{dt}=J(U,I)(t),t\in[T,2T]\\&I(t)=I_0(t),\frac{dI(t)}{dt}=\frac{dI_0(t)}{dt},t\in[0,T]\end{aligned}.$$

After integration under condition $I(T)=0$ we get $I(t)=\int_{T+kT_0}^{t}J(U,I)(t)dt$ and substituting in the last equalities $t=T+(k+1)T_0$ we obtain

$$I(T+(k+1)T_0)=\int_{T+kT_0}^{T+(k+1)T_0}J(U,I)(t)dt\Rightarrow\int_{T+kT_0}^{T+(k+1)T_0}J(U,I)(t)dt=0.$$

We have $B_I^{(k)}(U,I)(t)=\int_{T+kT_0}^{t}J(U,I)(s)ds-\dfrac{1}{T_0}\int_{T+kT_0}^{T+(k+1)T_0}\int_{T+kT_0}^{t}J(U,I)(s)dsdt$.

Changing the order of integration we get

$$\int_{T+kT_0}^{T+(k+1)T_0}\int_{T+kT_0}^{t}J(U,I)(s)dsdt=(T+(k+1)T_0)\int_{T+kT_0}^{T+(k+1)T_0}J(U,I)(s)ds-\int_{T+kT_0}^{T+(k+1)T_0}s\,J(U,I)(s)ds=$$

$$=(T+(k+1)T_0)\int_{T+kT_0}^{T+(k+1)T_0}\frac{dJ(s)}{ds}ds-\int_{T+kT_0}^{T+(k+1)T_0}sJ(U,I)(s)ds=-\int_{T+kT_0}^{T+(k+1)T_0}s\,J(U,I)(s)ds.$$

but

$$\int_{T+kT_0}^{T+(k+1)T_0} sJ(U,I)(s)ds = \int_{T+kT_0}^{T+(k+1)T_0} s\frac{dI(s)}{ds}ds = sJ(s)\Big|_{T+kT_0}^{T+(k+1)T_0} - \int_{T+kT_0}^{T+(k+1)T_0} J(s)ds = 0-0=0.$$

Therefore, $B_I^{(k)}(U,I)(t) = \int_{T+kT_0}^{t} J(U,I)(s)ds$ that is, $U = B_U(U,I)$, $I = B_I(U,I)$.

Conversely, let $B=(B_U,B_I)$ have a fixed point $(U(.),I(.)) \in M_U \times M_I$, that is, $U(t)=B_U(U,I)(t)$ and $I(t)=B_I^{(k)}(U,I)(t)$ for $t\in[T+kT_0,T+(k+1)T_0]$. Therefore $I(T+kT_0)=B_I^{(k)}(U,I)(T+(k+1)T_0)$

or

$$0 = J(T+kT_0) = B_I^{(k)}(U,I)(T+(k+1)T_0) = \int_{T+kT_0}^{T+(k+1)T_0} J(U,I)(s)ds - \frac{T+(k+1)T_0-T-kT_0}{T_0}\int_{T+kT_0}^{T+(k+1)T_0} J(U,I)(s)ds -$$
$$-\frac{1}{T_0}\int_{T+kT_0}^{T+(k+1)T_0}\int_{T+kT_0}^{s} J(U,I)(\theta)d\theta ds + \frac{1}{2}\int_{T+kT_0}^{T+(k+1)T_0} J(U,I)(s)ds = -\frac{1}{T_0}\int_{T+kT_0}^{T+(k+1)T_0}\int_{T+kT_0}^{s} J(U,I)(\theta)d\theta ds + \frac{1}{2}\int_{T+kT_0}^{T+(k+1)T_0} J(U,I)(s)ds.$$

It follows $\frac{1}{T_0}\int_{T+kT_0}^{T+(k+1)T_0}\int_{T+kT_0}^{s} J(U,I)(\theta)d\theta ds = \frac{1}{2}\int_{T+kT_0}^{T+(k+1)T_0} J(U,I)(s)ds.$

We show that $\int_{T+kT_0}^{T+(k+1)T_0} J(U,I)(s)ds = 0$. Indeed, we get

$$\left|\int_{T+kT_0}^{T+(k+1)T_0} J(U,I)(s)ds\right| \le \left|\int_{T+kT_0}^{T+(k+1)T_0} \frac{dU(s-T)}{ds}ds\right| + \frac{2Z_0}{\hat{L}_1}\left|\int_{T+kT_0}^{T+(k+1)T_0} R_1\left(\frac{U(t-T)-I(t)}{2Z_0}\right)ds\right| +$$
$$+\frac{2Z_0}{\hat{L}_1}\left[\left|\int_{T+kT_0}^{T+(k+1)T_0} C_1^{-1}\left(C_1(0)+\int_{T+kT_0}^{s}\frac{U(\theta-T)-I(\theta)}{2Z_0}d\theta\right)ds\right| + \left|\int_{T+kT_0}^{T+(k+1)T_0} C_0^{-1}\left(C_0(0)+\int_{T+kT_0}^{s}\frac{U(\theta-T)-I(\theta)}{2Z_0}d\theta\right)ds\right|\right] \le$$
$$\le \frac{4Z_0\phi_0}{\widehat{L}_1^{'}}\left|\int_{T+kT_0}^{T+(k+1)T_0} e^{\mu(s-T-kT_0)}ds\right| + \frac{2Z_0}{\widehat{L}_1^{'}}\int_{T+kT_0}^{T+(k+1)T_0}\left(|r_1|\left|\frac{U(t-T)-I(t)}{2Z_0}\right| + |r_2|\left|\frac{U(t-T)-I(t)}{2Z_0}\right|^2 + |r_3|\left|\frac{U(t-T)-I(t)}{2Z_0}\right|^3\right)ds \le$$
$$\le \frac{e^{\mu T_0}-1}{\mu}\frac{2Z_0}{\widehat{L}_1^{'}}\left[2\phi_0 + |r_1|\frac{U_0+I_0}{2Z_0} + |r_2|\left(\frac{U_0+I_0}{2Z_0}\right)^2 + |r_3|\left(\frac{U_0+I_0}{2Z_0}\right)^3\right] = M(\mu).$$

Since $\mu T_0 = \mu_0 = \text{const.}$ then $\lim_{\mu\to\infty} M(\mu) = 0$ which implies $\left|\int_{T+kT_0}^{T+(k+1)T_0} J(U,I)(s)ds\right| = 0$.

Consequently $B_I^{(k)}(U,I)(t)=\int_{T+kT_0}^{t} J(U,I)(s)ds$. Differentiating the last equality we obtain that the fixed point of the operator is a periodic solution of (2).

Lemma 6 is thus proved.

2. 5. Existence-uniqueness theorem for periodic problem

Assumption (IN): $U_0(.),\ I_0(.)\in C^1_{T_0}[0,T],\ |I_0(t)|\le I_0 e^{\mu(t-kT_0)},\ |U_0(t)|\le U_0 e^{\mu(t-kT_0)}, t\in[0,T]$ $(k=0,1,2,\dots,m-1)$.

It follows

$$|U(t-T)|\le U_0 e^{\mu(t-T-kT_0)}, |I(t-T)|\le I_0 e^{\mu(t-T-kT_0)} (k=0,1,2,\dots,m-1), t\in[T+kT_0, T+(k+1)T_0].$$

Assumption (U): $e^{\mu T_0}(U_0+I_0)/2\le\phi_0$.

It follows

$$|u(0,t)|\le(|U(t)|+|I(t-T)|)/2\le e^{\mu(t-T-kT_0)}(U_0+I_0)/2\le e^{\mu T_0}(U_0+I_0)/2\le\phi_0;$$
$$|u(\Lambda,t)|\le(|U(t-T)|+|I(t)|)/2\le e^{\mu(t-T-kT_0)}(U_0+I_0)/2\le e^{\mu T_0}(U_0+I_0)/2\le\phi_0.$$

The main existence-uniqueness result is:

Theorem 1. Let the assumptions (C), (L), (R), (IN) and (U) be satisfied and the following inequalities be fulfilled:

$$\frac{2Z_0}{Z_0+R}E_0+\frac{|R-Z_0|}{R+Z_0}I_0 e^{-\mu T}\le U_0;\ K_U=\frac{|R-Z_0|}{R+Z_0}e^{-\mu T}<1;$$

$$\frac{1}{\mu}\left(e^{\mu_0}+\frac{e^{\mu_0}-1}{\mu_0}\right)\frac{2Z_0}{\widehat{L}_1^{'}}\left[2\phi_0+|r_1|\frac{U_0+I_0}{2Z_0}+|r_2|\left(\frac{U_0+I_0}{2Z_0}\right)^2+|r_3|\left(\frac{U_0+I_0}{2Z_0}\right)^3\right]\le I_0;$$

$$K_I=e^{-\mu T}+\frac{1}{\mu\hat{L}_1}\left(\frac{1}{\mu\widehat{C}_0^1}+|r_1|+\frac{|r_2|(U_0e^{-\mu T}+I_0)e^{\mu_0}}{2Z_0}+\frac{|r_3|(U_0e^{-\mu T}+I_0)^2e^{2\mu_0}}{(2Z_0)^2}+\frac{1}{\mu\widehat{C}_1^1}\right)<1.$$

Then there exists a unique T_0-periodic solution of (2), (3).

Proof: First we show that $(B_U,B_I)\in M_U\times M_I$.

Using the inequalities $\left|\frac{t-T-kT_0}{T_0}-\frac{1}{2}\right|\le\frac{1}{2}, t\in[T+kT_0, T+(k+1)T_0]$ we get

$$\left|B_U(U,I)(t)\right| \le \frac{2Z_0}{Z_0+R}\left|E(t)\right| + \frac{\left|R-Z_0\right|}{R+Z_0}\left|I(t-T)\right| \le \frac{2Z_0}{Z_0+R}E_0 e^{\mu(t-T-kT_0)} + \frac{\left|R-Z_0\right|}{R+Z_0}I_0 e^{\mu(t-T-kT_0)}e^{-\mu T} \le U_0 e^{\mu(t-T-kT_0)}.$$

For the second component of the operator B we have

$$\left|B_I^{(k)}(U,I)(t)\right| \le \left|\int_{T+kT_0}^{t} J(U,I)(s)ds\right| + \frac{1}{2}\left|\int_{T+kT_0}^{T+(k+1)T_0} J(U,I)(s)ds\right| + \frac{1}{T_0}\left|\int_{T+kT_0}^{T+(k+1)T_0}\int_{T+kT_0}^{t} J(U,I)(s)dsdt\right| \equiv J_1 + J_2 + J_3.$$

but

$$J_1 \le \left|\int_{T+kT_0}^{t} J(U,I)(s)ds\right| \le \left|\int_{T+kT_0}^{t}\frac{dU(s-T)}{ds}ds\right| + \frac{2Z_0}{\hat{L}_1}\left|\int_{T+kT_0}^{t} C_0^{-1}\left(C_0(0)+\int_0^s \frac{U(\theta-T)-I(\theta)}{2Z_0}d\theta\right)ds\right| +$$

$$+\frac{2Z_0}{\hat{L}_1}\left|\int_{T+kT_0}^{t} R_1\left(\frac{U(s-T)-I(s)}{2Z_0}\right)ds\right| + \frac{2Z_0}{\hat{L}_1}\left|\int_{T+kT_0}^{t} C_1^{-1}\left(C_1(0)+\int_0^s \frac{U(\theta-T)-I(\theta)}{2Z_0}d\theta\right)ds\right| \le$$

$$\le \frac{4Z_0\phi_0}{\widehat{L}_1^{'}}\left|\int_{T+kT_0}^{t} e^{\mu(s-T-kT_0)}ds\right| + \frac{2Z_0}{\widehat{L}_1^{'}}\int_{T+kT_0}^{t}\left(|r_1|\left|\frac{U(s-T)-I(s)}{2Z_0}\right| + |r_2|\left|\frac{U(s-T)-I(s)}{2Z_0}\right|^2 + |r_3|\left|\frac{U(s-T)-I(s)}{2Z_0}\right|^3\right)ds \le$$

$$\le \frac{e^{\mu(t-T-kT_0)}-1}{\mu}\frac{2Z_0}{\widehat{L}_1^{'}}\left[2\phi_0 + |r_1|\frac{U_0+I_0}{2Z_0} + |r_2|\left(\frac{U_0+I_0}{2Z_0}\right)^2 + |r_3|\left(\frac{U_0+I_0}{2Z_0}\right)^3\right];$$

$$J_2 \le \frac{1}{2}\left|\int_{T+kT_0}^{T+(k+1)T_0} J(U,I)(s)ds\right| \le \frac{e^{\mu T_0}-1}{\mu}\frac{2Z_0}{\widehat{L}_1^{'}}\left[2\phi_0 + |r_1|\frac{U_0+I_0}{2Z_0} + |r_2|\left(\frac{U_0+I_0}{2Z_0}\right)^2 + |r_3|\left(\frac{U_0+I_0}{2Z_0}\right)^3\right];$$

$$J_3 = \frac{1}{T_0}\left|\int_{T+kT_0}^{T+(k+1)T_0}\int_{T+kT_0}^{t} J(U,I)(s)dsdt\right| \le$$

$$\le \frac{1}{T_0}\left|\int_{T+kT_0}^{T+(k+1)T_0}\frac{e^{\mu(t-T-kT_0)}-1}{\mu}\left[\frac{4Z_0\phi_0}{\widehat{L}_1^{'}} + \frac{2Z_0}{\widehat{L}_1^{'}}\left(|r_1|\frac{U_0+I_0}{2Z_0} + |r_2|\left(\frac{U_0+I_0}{2Z_0}\right)^2 + |r_3|\left(\frac{U_0+I_0}{2Z_0}\right)^3\right)\right]dt\right| \le$$

$$\le \frac{1}{\mu T_0}\left[\frac{4Z_0\phi_0}{\widehat{L}_1^{'}} + \frac{2Z_0}{\widehat{L}_1^{'}}\left(|r_1|\frac{U_0+I_0}{2Z_0} + |r_2|\left(\frac{U_0+I_0}{2Z_0}\right)^2 + |r_3|\left(\frac{U_0+I_0}{2Z_0}\right)^3\right)\right]\left|\int_{T+kT_0}^{T+(k+1)T_0} e^{\mu(t-T-kT_0)}dt\right| \le$$

$$\le \frac{1}{\mu T_0}\frac{2Z_0}{\widehat{L}_1^{'}}\left[2\phi_0 + |r_1|\frac{U_0+I_0}{2Z_0} + |r_2|\left(\frac{U_0+I_0}{2Z_0}\right)^2 + |r_3|\left(\frac{U_0+I_0}{2Z_0}\right)^3\right]\frac{e^{\mu T_0}-1}{\mu}.$$

Then

$$\left|B_I^{(k)}(U,I)(t)\right| \le \frac{e^{\mu(t-T-kT_0)}}{\mu}\frac{2Z_0}{\widehat{L}_1^{'}}\left[2\phi_0 + |r_1|\frac{U_0+I_0}{2Z_0} + |r_2|\left(\frac{U_0+I_0}{2Z_0}\right)^2 + |r_3|\left(\frac{U_0+I_0}{2Z_0}\right)^3\right] +$$

$$+\frac{e^{\mu T_0}-1}{\mu}\frac{2Z_0}{\widehat{L}_1^{\,'}}\left[2\phi_0+|r_1|\frac{U_0+I_0}{2Z_0}+|r_2|\left(\frac{U_0+I_0}{2Z_0}\right)^2+|r_3|\left(\frac{U_0+I_0}{2Z_0}\right)^3\right]+$$

$$+\frac{1}{\mu}\frac{e^{\mu T_0}-1}{\mu T_0}\frac{2Z_0}{\widehat{L}_1^{\,'}}\left[2\phi_0+|r_1|\frac{U_0+I_0}{2Z_0}+|r_2|\left(\frac{U_0+I_0}{2Z_0}\right)^2+|r_3|\left(\frac{U_0+I_0}{2Z_0}\right)^3\right]\le$$

$$\le e^{\mu(t-T-kT_0)}\frac{1}{\mu}\left(e^{\mu 0}+\frac{e^{\mu 0}-1}{\mu T_0}\right)\frac{2Z_0}{\widehat{L}_1^{\,'}}\left[2\phi_0+|r_1|\frac{U_0+I_0}{2Z_0}+|r_2|\left(\frac{U_0+I_0}{2Z_0}\right)^2+|r_3|\left(\frac{U_0+I_0}{2Z_0}\right)^3\right]\le I_0 e^{\mu(t-T-kT_0)}.$$

It remains to show that the operator $B=(B_U,B_I)$ is contractive one. Indeed for $t\in[T+kT_0,T+(k+1)T_0]$ we have

$$\left|B_U(U,I)(t)-B_U(\overline{U},\bar{I})(t)\right|\le\frac{|R_0-Z_0|}{R_0+Z_0}\left|I(t-T)-\bar{I}(t-T)\right|\le\frac{|R_0-Z_0|}{R_0+Z_0}e^{-\mu T}\rho_k(I,\bar{I})e^{\mu(t-T-kT_0)}.$$

It follows for $k=0,1,2,\dots,m-1$

$$\rho_k\left(B_U(U,I),B_U(\overline{U},\bar{I})\right)\le\frac{|R_0-Z_0|}{R_0+Z_0}e^{-\mu T}\rho_k(I,\bar{I})\le\frac{|R_0-Z_0|}{R_0+Z_0}e^{-\mu T}\rho\left((U,I),(\overline{U},\bar{I})\right).$$

For the second component we have

$$\left|B_I^{(k)}(U,I)(t)-B_I^{(k)}(\overline{U},\bar{I})(t)\right|\le$$

$$\le\left|\int_{T+kT_0}^{t}\left(J(U,I)(s)-J(\overline{U},\bar{I})(s)\right)ds\right|+\frac{1}{2}\left|\int_{T+kT_0}^{T+(k+1)T_0}\left(J(U,I)(s)-J(\overline{U},\bar{I})(s)\right)ds\right|+\frac{1}{T_0}\left|\int_{T+kT_0}^{T+(k+1)T_0}\int_{T+kT_0}^{t}\left(J(U,I)(s)-J(\overline{U},\bar{I})(s)\right)dsdt\right|\equiv$$

$$\equiv J_1+J_2+J_3;$$

$$J_1\le\left|\int_{T+kT_0}^{t}\left(\frac{dU(s-T)}{ds}-\frac{d\overline{U}(s-T)}{ds}\right)ds\right|+$$

$$+\frac{2Z_0}{\hat{L}_1}\left|\int_{T+kT_0}^{t}\left(C_0^{-1}\left(C_0(0)+\int_{T+kT_0}^{s}\frac{U(\theta-T)-I(\theta)}{2Z_0}d\theta\right)-C_0^{-1}\left(C_0(0)+\int_{T+kT_0}^{s}\frac{\overline{U}(\theta-T)-\bar{I}(\theta)}{2Z_0}d\theta\right)\right)ds\right|+$$

$$+\frac{2Z_0}{\hat{L}_1}\left|\int_{T+kT_0}^{t}\left(R_1\left(\frac{U(s-T)-I(s)}{2Z_0}\right)-R_1\left(\frac{\overline{U}(s-T)-\bar{I}(s)}{2Z_0}\right)\right)ds\right|+$$

$$+\frac{2Z_0}{\hat{L}_1}\left|\int_{T+kT_0}^{t}\left(C_1^{-1}\left(C_1(0)+\int_{T+kT_0}^{s}\frac{U(\theta-T)-I(\theta)}{2Z_0}d\theta\right)-C_1^{-1}\left(C_1(0)+\int_{T+kT_0}^{s}\frac{\overline{U}(\theta-T)-\bar{I}(\theta)}{2Z_0}d\theta\right)\right)ds\right|\le$$

$$\le \rho_k(U,\bar{U})e^{-\mu T}e^{\mu(t-T-kT_0)} + \frac{2Z_0}{\hat{L}_1\hat{C}_0^1}\left|\int_{T+kT_0}^{t}\left(\int_{T+kT_0}^{s}\frac{U(\theta-T)-I(\theta)}{2Z_0}d\theta - \int_{T+kT_0}^{s}\frac{\bar{U}(\theta-T)-\bar{I}(\theta)}{2Z_0}d\theta\right)ds\right| +$$

$$+\frac{2Z_0}{\hat{L}_1\hat{C}_1^1}\left|\int_{T+kT_0}^{t}\left(\int_{T+kT_0}^{s}\frac{U(\theta-T)-I(\theta)}{2Z_0}d\theta - \int_{T+kT_0}^{s}\frac{\bar{U}(\theta-T)-\bar{I}(\theta)}{2Z_0}d\theta\right)ds\right| +$$

$$+\frac{2Z_0|r_1|}{\hat{L}_1}\left|\int_{T+kT_0}^{t}\left(\frac{U(s-T)-I(s)}{2Z_0}-\frac{\bar{U}(s-T)-\bar{I}(s)}{2Z_0}\right)ds\right| +$$

$$+\frac{2Z_0|r_2|}{\hat{L}_1}\left|\int_{T+kT_0}^{t}\left(\frac{U(s-T)-I(s)}{2Z_0}+\frac{\bar{U}(s-T)-\bar{I}(s)}{2Z_0}\right)\left(\frac{U(s-T)-I(s)}{2Z_0}-\frac{\bar{U}(s-T)-\bar{I}(s)}{2Z_0}\right)ds\right| +$$

$$+\frac{2Z_0|r_3|}{\hat{L}_1}3\left|\int_{T+kT_0}^{t}\left(\frac{U(s-T)+I(s)}{2Z_0}\right)^2\left(\frac{U(s-T)-I(s)}{2Z_0}-\frac{\bar{U}(s-T)-\bar{I}(s)}{2Z_0}\right)ds\right| \le$$

$$\le e^{\mu(t-T-kT_0)}\rho_k(U,\bar{U})e^{-\mu T} + \frac{2Z_0\left(\rho_k(U,\bar{U})e^{-\mu T}+\rho_k(I,\bar{I})\right)}{2Z_0\hat{L}_1\hat{C}_0^1}\left|\int_{T+kT_0}^{t}\int_{T+kT_0}^{s}e^{\mu(\theta-T-kT_0)}d\theta ds\right| +$$

$$+\frac{2Z_0\left(\rho_k(U,\bar{U})e^{-\mu T}+\rho_k(I,\bar{I})\right)}{2Z_0\hat{L}_1\hat{C}_1^1}\left|\int_{T+kT_0}^{t}\int_{T+kT_0}^{s}e^{\mu(\theta-T-kT_0)}d\theta ds\right| +$$

$$+|r_1|\frac{2Z_0\left(\rho_k(U,\bar{U})e^{-\mu T}+\rho_k(I,\bar{I})\right)}{2Z_0\hat{L}_1}\left|\int_{T+kT_0}^{t}e^{\mu(s-T-kT_0)}ds\right| + |r_2|\frac{2Z_0\left(U_0e^{-\mu T}+I_0\right)\left(\rho_k(U,\bar{U})e^{-\mu T}+\rho_k(I,\bar{I})\right)}{2Z_0Z_0\hat{L}_1}\left|\int_{T+kT_0}^{t}e^{2\mu(s-T-kT_0)}ds\right| +$$

$$+|r_3|\frac{6Z_0\left(U_0e^{-\mu T}+I_0\right)^2\left(\rho_k(U,\bar{U})e^{-\mu T}+\rho_k(I,\bar{I})\right)}{(2Z_0)^3\hat{L}_1}\left|\int_{T+kT_0}^{t}e^{3\mu(s-T-kT_0)}ds\right| \le$$

$$\le e^{\mu(t-T-kT_0)}\rho_k(U,\bar{U})e^{-\mu T} + \frac{\rho_k(U,\bar{U})e^{-\mu T}+\rho_k(I,\bar{I})}{\hat{L}_1\hat{C}_1^1}\frac{1}{\mu}\frac{e^{\mu(t-T-kT_0)}}{\mu} + \frac{\rho_k(U,\bar{U})e^{-\mu T}+\rho_k(I,\bar{I})}{\hat{L}_1\hat{C}_0^1}\frac{1}{\mu}\frac{e^{\mu(t-T-kT_0)}}{\mu} +$$

$$+\frac{|r_1|\left(\rho_k(U,\bar{U})e^{-\mu T}+\rho_k(I,\bar{I})\right)}{\hat{L}_1}\frac{e^{\mu(t-T-kT_0)}}{\mu} + \frac{|r_2|\left(U_0e^{-\mu T}+I_0\right)\left(\rho_k(U,\bar{U})e^{-\mu T}+\rho_k(I,\bar{I})\right)}{Z_0\hat{L}_1}\frac{e^{2\mu(t-T-kT_0)}}{2\mu} +$$

$$+\frac{3|r_3|\left(U_0e^{-\mu T}+I_0\right)^2\left(\rho_k(U,\bar{U})e^{-\mu T}+\rho_k(I,\bar{I})\right)}{4(Z_0)^2\hat{L}_1}\frac{e^{3\mu(t-T-kT_0)}}{3\mu} \le$$

$$\le e^{\mu(t-T-kT_0)}\rho_k\left((U,I),(\bar{U},\bar{I})\right)\left[e^{-\mu T} + \frac{1}{\mu\hat{L}_1}\left(\frac{1}{\mu\hat{C}_0^1}+\frac{1}{\mu\hat{C}_1^1}+|r_1|+|r_2|\frac{\left(U_0e^{-\mu T}+I_0\right)e^{\mu_0}}{2Z_0}+|r_3|\frac{\left(U_0e^{-\mu T}+I_0\right)^2e^{2\mu_0}}{(2Z_0)^2}\right)\right].$$

It follows

$$\rho_k\left(B_I^{(k)}(U,I),B_I^{(k)}(\bar{U},\bar{I})\right)\le$$

$$\le\left[e^{-\mu T} + \frac{1}{\mu\hat{L}_1}\left(\frac{1}{\mu\hat{C}_0^1}+\frac{1}{\mu\hat{C}_1^1}+|r_1|+\frac{|r_2|\left(U_0e^{-\mu T}+I_0\right)e^{\mu_0}}{2Z_0}+\frac{|r_3|\left(U_0e^{-\mu T}+I_0\right)^2e^{2\mu_0}}{(2Z_0)^2}\right)\right]\rho\left((U,I),(\bar{U},\bar{I})\right).$$

Denote by

$$K = \max\left\{ \frac{|R_0 - Z_0|}{R_0 + Z_0} e^{-\mu T},\ e^{-\mu T} + \frac{1}{\mu \hat{L}_1}\left(\frac{1}{\mu \widehat{C}_1^1} + \frac{1}{\mu \widehat{C}_0^1} + \sum_{n=1}^{3} |r_n| \left(\frac{\left(U_0 e^{-\mu T} + I_0\right) e^{\mu_0}}{2Z_0} \right)^{n-1} \right) \right\}.$$

Consequently

$$\rho\left(\left(B_U(U,I), B_I(U,I)\right), \left(B_U(\bar{U},\bar{I}), B_I(\bar{U},\bar{I})\right)\right) \le K\ \rho\left((U,I),(\bar{U},\bar{I})\right).$$

The fixed point of B is the periodic solution of (2).
Theorem 1 is thus proved

2. 6. Numerical example

For a transmission line with length $\Lambda = 10\, m$; $L = 0{,}2\, \mu H/m$; $C = 80\ pF/m$;

$$v = 1/\sqrt{LC} = 1/\left(4.10^{-9}\right);\ Z_0 = \sqrt{L/C} = 50\ \ \Omega;\ R_0 = 70\Omega;\ T = \Lambda\sqrt{LC} = 4.10^{-8}\ \text{sec}.$$

Let us consider a propagation of waves with $\lambda_0 = \frac{1}{6.10^2} m$. We have $f_0 = \frac{1}{\lambda_0 \sqrt{LC}} = \frac{6.10^2}{4.10^{-9}} = 1{,}5.10^{11}\ Hz \Rightarrow T_0 = \frac{1}{f_0} = \frac{1}{1{,}5.10^{11}} = 6{,}7.10^{-12}$. Choose $\mu = 10^8$ and then $\mu T_0 = \mu_0 = 6{,}7$; $\mu T = 10^8.4.10^{-8} = 4$. Therefore $e^{-4} = e^{-4} \approx 0{,}018$.

We take the values from [1] for the elements with linear characteristics $R_0(i) = R_S = 15\Omega$, $L_0(i) = L_S = 58\ mH$, $C_0(u) = C_S = 0{,}054\ pF$, $C_1(u) = C_P = 29\ pF$. We verify the inequalities from Theorem 1 assuming $U_0 = I_0 = E_0 = \phi_0$:

$$\frac{2.50}{50+70} + \frac{20}{50+70} 0{,}018 \le 1;\ \frac{2e-1}{10^8.58.10^{-3}}\left(4.50 + 2.15\right) \le 1;\ K_U = \frac{|70-50|}{70+50} 0{,}018 \approx 0{,}003 < 1;$$

$$K_I = e^{-\mu T} + \frac{1}{\mu \hat{L}_1}\left(\frac{1}{\mu \widehat{C}_0^1} + 15 + \frac{1}{\mu \widehat{C}_1^1} \right) = 0{,}018 + \frac{1}{10^8.58.10^{-3}}\left(18{,}52.10^4 + 15 + 0{,}035.10^4\right) \approx 0{,}018 + \frac{19}{10.58} = 0{,}05.$$

3. CONCLUSIONS

The successive approximations of the solution we obtain from the operator equations

$$U_{n+1}(t) = \frac{2Z_0}{Z_0 + R_0} E(t) + \frac{R_0 - Z_0}{R_0 + Z_0} I_n(t - T)\ ,\ t \in [T, 2T]$$

$$I_{n+1}(t):=\int_{T+kT_0}^{t}J(U_n,I_n)(s)ds-\left(\frac{t-T-kT_0}{T_0}-\frac{1}{2}\right)\int_{T+kT_0}^{T+(k+1)T_0}J(U_n,I_n)(s)ds-\frac{1}{T_0}\int_{T+kT_0}^{T+(k+1)T_0}\int_{T+kT_0}^{t}J(U_n,I_n)(s)dsdt$$

$t\in[T+kT_0,\ T+(k+1)T_0]$.

Biography

Vasil Angelov was born in Sofia, Bulgaria. He educated at Sofia University, Faculty of Mathematics and Mechanics (1968-1973); PhD in Mathematics - 1981; Assoc. Professor - 1985; Doctor of Technical Sciences - 2004; Professor 2007; Chair of Department of Mathematics, University of Mining and Geology "St. I. Rilski"- Sofia - 1993. Research subjects: Delay differential equations, Fixed point theory, Classical electrodynamics, Electrodynamics and radio-technical devices, Transmission line theory and Nonlinear circuit theory. Monographs: 1) V.G. Angelov - Fixed Points in Uniform Spaces and Applications. Cluj University Press, Romania, 2009; 2) V.G. Angelov - A Method for Analysis of Transmission Lines Terminated by Nonlinear Loads. Nova Science, New York, 2014 and 130 research papers. Awards and Honors: 1) Who's Who in the World - from 1992; 2) Gold Medal of the American Biographical Institute – Ghost of the World Forum University of Cambridge – 2010; 3) International Peace Prize, United Cultural Convention, USA-2012, USA; 4) Scientist of the Year 2015, Summit of Leaders, EBA, 5) Expert of International Achievements Research Centre, Chicago, 2016. 6) Winner of World Championship – 2019 - Electrodynamics.

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