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SHORT COMMUNICATION

## Stapp parameterization and scattering phases for coupled states

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### ABSTRACT

Variable phase approach is viewed for two-channel scattering. The search of the scattering phases and mixing parameter for coupled states  ${}^3S_1$ - ${}^3D_1$  using Stapp parameterization is carried out. The nucleon-nucleon potential in the coordinate representation is used for numerical calculations. The influence of the choice of a numerical method (method Euler and methods Runge-Kutta) on the calculations of scattering phases and the mixing parameter is taken into account.

**Keywords:** variable phase approach, shifts, deuteron, parameterization, coupled states

### 1. INTRODUCTION

Nucleon-nucleon interaction remains interesting for experimenters and theorists. For example, a comparison of nucleon-nucleon potentials was made in reviews [1, 2]. Nucleon-nucleon interaction in a chiral SU(3) quark model is described in paper [3]. Also, Weinberg's approach [4] and the method of unitary clothing transformations [5] was applied for nucleon-nucleon scattering. The features of the nucleon-nucleon interaction can be described using the

Schrödinger equation. The main known and most common methods for solving the Schrödinger equation in order to obtain the scattering phases or amplitudes are the method of successive approximations, Born approximation, variable phase approach (VPA), Brysk approximation and others. When applied to the problems of nucleon-nucleon scattering, the important and main advantage of VPA is that, this method allows us to obtain scattering phases without finding at the same time wave functions as solutions to the Schrödinger equation. The phase equation provides a direct connection between the scattering phase and the interaction potential. VPA for two-channel scattering is described in detail in reviews [6, 7]. This method was repeatedly used to solve various problems of atomic and nuclear physics. VPA has been applied to single-channel nucleon-nucleon scattering. For example, scalar amplitudes were obtained using VPA in paper [8]. In this paper, the use of VPA for two-channel scattering is considered. The search for scattering phases and the mixing parameter with the use of the nucleon-nucleon potential in explicit form in coordinate representation was performed.

## 2. THE VARIABLE PHASE APPROACH FOR TWO-CHANNEL SCATTERING

We apply VPA to a two-channel scattering problem. The system of Schrödinger equations with tensor mixing will have the following form [6, 9]:

$$\begin{cases} \frac{d^2 u_J(r)}{dr^2} + \left[ k^2 - \frac{J(J-1)}{r^2} - V_{J,J-1} \right] u_J(r) = T_J w_J(r); \\ \frac{d^2 w_J(r)}{dr^2} + \left[ k^2 - \frac{(J+2)(J+1)}{r^2} - V_{J,J+1} \right] w_J(r) = T_J u_J(r). \end{cases} \quad (1)$$

Here potentials were written in the form [6]:

$$\begin{cases} V_{J,J-1}(r) = V_C(r) - \frac{2(J-1)}{2J+1} \cdot V_T(r) + (J-1) \cdot V_{LS}(r) + (J-1)(J-2) \cdot V_{LL}(r); \\ V_{J,J+1}(r) = V_C(r) - \frac{2(J+2)}{2J+1} \cdot V_T(r) - (J+2) \cdot V_{LS}(r) + (J+3)(J+2) \cdot V_{LL}(r); \\ T_J(r) = \frac{6\sqrt{J(J+1)}}{2J+1} \cdot V_T(r). \end{cases} \quad (2)$$

In the system (1)  $u_J(r)$ ,  $w_J(r)$  – the radial wave functions;  $V_C(r)$ ,  $V_{LS}(r)$ ,  $V_{LL}(r)$ ,  $V_T(r)$  – central, spin-orbital, quadratic and tensor parts of the potential for nucleon-nucleon interaction. We introduce wave functions that are expressed in terms of four new functions:

$$\begin{cases} u_J(r) = A(r) [\cos \delta_\alpha(r) \cdot j_\alpha(kr) - \sin \delta_\alpha(r) \cdot n_\alpha(kr)], \\ w_J(r) = B(r) [\cos \delta_\beta(r) \cdot j_\beta(kr) - \sin \delta_\beta(r) \cdot n_\beta(kr)]. \end{cases} \quad (3)$$

Here  $\delta_\alpha(r)$ ,  $\delta_\beta(r)$  – the phase functions;  $A(r)$ ,  $B(r)$  – the amplitude functions of the channels  $\alpha$  and  $\beta$ . The physical meaning of the functions  $\delta_\alpha(r)$ ,  $\delta_\beta(r)$ ,  $A(r)$ ,  $B(r)$  is found out if some additional conditions are imposed on them, because instead of the two unknown functions  $u_J(r)$  and  $w_J(r)$  are introduced other four unknowns functions. Therefore, it is necessary to fulfill additional conditions for the derivatives of the sought-for wave functions  $u_J(r)$  and  $w_J(r)$ :

$$\begin{cases} u'_J = A[\cos \delta_\alpha \cdot j'_\alpha - \sin \delta_\alpha \cdot n'_\alpha]; \\ w'_J = B[\cos \delta_\beta \cdot j'_\beta - \sin \delta_\beta \cdot n'_\beta], \end{cases} \quad (4)$$

which are equivalent to the following conditions

$$\begin{cases} A'[\cos \delta_\alpha \cdot j_\alpha - \sin \delta_\alpha \cdot n_\alpha] - \delta'_\alpha A[\sin \delta_\alpha \cdot j_\alpha + \cos \delta_\alpha \cdot n_\alpha] = 0; \\ B'[\cos \delta_\beta \cdot j_\beta - \sin \delta_\beta \cdot n_\beta] - \delta'_\beta B[\sin \delta_\beta \cdot j_\beta + \cos \delta_\beta \cdot n_\beta] = 0. \end{cases} \quad (5)$$

Similarly to single-channel scattering [6], we find equations that are satisfied with phase and amplitude functions. To do this, we differentiate the system of equations (5) and substitute the obtained result together with expression (3) into the system of Schrödinger equations (1).

Then we get the following two systems of equations:

$$\begin{cases} A'[\cos \delta_\alpha \cdot j'_\alpha - \sin \delta_\alpha \cdot n'_\alpha] - \delta'_\alpha A[\sin \delta_\alpha \cdot j_\alpha + \cos \delta_\alpha \cdot n_\alpha] = 0; \\ A'[\cos \delta_\alpha \cdot j_\alpha - \sin \delta_\alpha \cdot n_\alpha] - \delta'_\alpha A[\sin \delta_\alpha \cdot j'_\alpha + \cos \delta_\alpha \cdot n'_\alpha] - V_{J,J-1}A[\cos \delta_\alpha \cdot j_\alpha - \sin \delta_\alpha \cdot n_\alpha] = \\ = T_J B[\cos \delta_\beta \cdot j_\beta - \sin \delta_\beta \cdot n_\beta]; \end{cases} \quad (6)$$

$$\begin{cases} B'[\cos \delta_\beta \cdot j'_\beta - \sin \delta_\beta \cdot n'_\beta] - \delta'_\beta B[\sin \delta_\beta \cdot j_\beta + \cos \delta_\beta \cdot n_\beta] = 0; \\ B'[\cos \delta_\beta \cdot j_\beta - \sin \delta_\beta \cdot n_\beta] - \delta'_\beta B[\sin \delta_\beta \cdot j'_\beta + \cos \delta_\beta \cdot n'_\beta] - V_{J,J+1}B[\cos \delta_\beta \cdot j_\beta - \sin \delta_\beta \cdot n_\beta] = \\ = T_J A[\cos \delta_\alpha \cdot j_\alpha - \sin \delta_\alpha \cdot n_\alpha]. \end{cases} \quad (7)$$

Taking into account the form of the vronskians  $j_{\alpha,\beta} \cdot n'_{\alpha,\beta} - n_{\alpha,\beta} \cdot j'_{\alpha,\beta} = k_{\alpha,\beta} = k$  of the solutions of the free Schrödinger equation and excluding the derivatives of phase and amplitude functions, we obtain the system, as expected, from four nonlinear coupled equations for the functions  $\delta_\alpha(r)$ ,  $\delta_\beta(r)$ ,  $A(r)$ ,  $B(r)$  in this form [10]

$$\begin{cases} \delta'_\alpha = -\frac{V_{J,J-1}}{k} P_\alpha^2 - \frac{T_J tg \varepsilon}{k} P_\alpha P_\beta; \\ \delta'_\beta = -\frac{V_{J,J+1}}{k} P_\beta^2 - \frac{T_J}{ktg \varepsilon} P_\beta P_\alpha; \\ A' = -\frac{V_{J,J-1} A}{k} P_\alpha Q_\alpha - \frac{T_J A tg \varepsilon}{k} P_\beta Q_\alpha; \\ B' = -\frac{V_{J,J+1} A tg \varepsilon}{k} P_\beta Q_\beta - \frac{T_J A}{k} P_\alpha Q_\beta; \end{cases} \quad (8)$$

where  $tg \varepsilon = B/A$  – the parameter of the ratio (mixing) of scattering amplitudes;

$$P_i = \cos \delta_i \cdot j_i - \sin \delta_i \cdot n_i; \quad Q_i = \sin \delta_i \cdot j_i + \cos \delta_i \cdot n_i; \quad i = \alpha, \beta.$$

Consequently, a system of four coupled nonlinear differential equations of the 1st order was obtained for phase and amplitude functions. The analytic methods of the solution of system (8) are the methods (parameterizations) of McHale-Thaler [11], Blatt-Biedenharn [12], Stapp [13] or Matveenko-Ponomarev-Feifman [14].

Equations for Stapp parameterization can be written in the form [6, 13]:

$$\delta'_{J,J-1} = -\frac{1}{k \cos 2\varepsilon_J} \left[ V_{J,J-1} (\cos^4 \varepsilon_J P_{J,J-1}^2 - \sin^4 \varepsilon_J Q_{J,J-1}^2) - V_{J,J+1} \sin^2 \varepsilon_J \cos^2 \varepsilon_J (P_{J,J+1}^2 - Q_{J,J+1}^2) - T_J \sin 2\varepsilon_J (\cos^2 \varepsilon_J P_{J,J-1} Q_{J,J+1} - \sin^2 \varepsilon_J P_{J,J+1} Q_{J,J-1}) \right]; \quad (9)$$

$$\delta'_{J,J+1} = -\frac{1}{k \cos 2\varepsilon_J} \left[ V_{J,J+1} (\cos^4 \varepsilon_J P_{J,J+1}^2 - \sin^4 \varepsilon_J Q_{J,J+1}^2) - V_{J,J-1} \sin^2 \varepsilon_J \cos^2 \varepsilon_J (P_{J,J-1}^2 - Q_{J,J-1}^2) - T_J \sin 2\varepsilon_J (\cos^2 \varepsilon_J P_{J,J+1} Q_{J,J-1} - \sin^2 \varepsilon_J P_{J,J-1} Q_{J,J+1}) \right]; \quad (10)$$

$$\varepsilon'_J = -\frac{1}{k} \left[ T_J (\cos^2 \varepsilon_J P_{J,J-1} P_{J,J+1} + \sin^2 \varepsilon_J Q_{J,J-1} Q_{J,J+1}) - V_{J,J-1} \sin \varepsilon_J \cos \varepsilon_J P_{J,J-1} Q_{J,J-1} - V_{J,J+1} \sin \varepsilon_J \cos \varepsilon_J P_{J,J+1} Q_{J,J+1} \right]; \quad (11)$$

where the values  $P_{J,L}$ ,  $Q_{J,L}$  (at  $L = J \mp 1$ ) are equal to:

$$\begin{cases} P_{J,L} \equiv \cos \delta_{J,L} \cdot j_L - \sin \delta_{J,L} \cdot n_L; \\ Q_{J,L} \equiv \sin \delta_{J,L} \cdot j_L + \cos \delta_{J,L} \cdot n_L. \end{cases} \quad (12)$$

Initial conditions for phase functions  $\delta_{J,J-1}$ ,  $\delta_{J,J+1}$  and mixing parameter  $\varepsilon_J$

$$\delta_{J,J-1}(0) = \delta_{J,J+1}(0) = \varepsilon_J(0) = 0. \quad (13)$$

In spite of the mathematical cumbersome, the system of equations (9)-(11) is simple and convenient for calculations, because its equations contain a small number of similar elements. These equations are suitable for use for the presence of coupled states. Solutions of equations (9)-(11) for small values of the coordinates do not turn into infinity.

### 3. CALCULATIONS AND CONCLUSIONS

At the full moment for the nucleon-nucleon system  $J=1$  in the case of coupled states  ${}^3S_1$ - ${}^3D_1$  (deuteron), potentials (2) in the system (1) are written in the form

$$\begin{cases} V_{10}(r) = V_C(r); \\ V_{12}(r) = V_C(r) - 2 \cdot V_T(r) - 3 \cdot V_{LS}(r); \\ T_1(r) = \sqrt{8} \cdot V_T(r). \end{cases} \quad (14)$$

The simplest and most visible potential in the coordinate representation is Reid68 potential. The values obtained for this potential are in good agreement with the experimental data. Reid68 potential of was fitted independently in all partial waves with  $J \leq 2$ . For channels  ${}^3S_1$ - ${}^3D_1$ , the Reid68 potential (soft core version) has the following form [15]:

$$V({}^3S_1 - {}^3D_1) = V_C + V_T \cdot S_{12} + V_{LS} \cdot (\vec{L} \cdot \vec{S}), \quad (15)$$

where such central, tensor and spin-orbital parts of the potential are used [15]

$$V_C = h \frac{e^{-x}}{x} + 105.468 \frac{e^{-2x}}{x} - 3187.8 \frac{e^{-4x}}{x} + 9924.3 \frac{e^{-6x}}{x}; \quad (16)$$

$$V_T = -h \left[ \left( \frac{1}{x} + \frac{3}{x^2} + \frac{3}{x^3} \right) e^{-x} - \left( \frac{12}{x^2} + \frac{3}{x^3} \right) e^{-4x} \right] + 351.77 \frac{e^{-4x}}{x} - 1673.5 \frac{e^{-6x}}{x}; \quad (17)$$

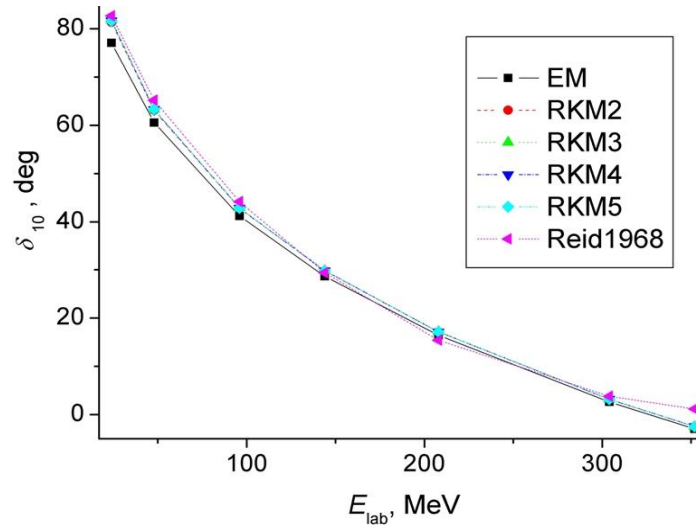
$$V_{LS} = 708.91 \frac{e^{-4x}}{x} - 2713.1 \frac{e^{-6x}}{x}. \quad (18)$$

In formulas (16)-(18), the value  $x = \mu r$ , where  $\mu = 0.7$  fm corresponds to the pion radius; here  $h = 10.463$  MeV and the value  $\hbar^2/m = 41.47$  MeV·fm<sup>2</sup> is accepted. The singularities of the type  $x^{-2}$  and  $x^{-3}$ , which are related to the tensor interaction, are excluded from the potential, so that in all channels the behavior of the potential at short distances is of the character  $x^{-1}$  [16]. In Table 1 and Figs. 1-3 the results of calculations of the values for scattering phases  $\delta_{10}$ ,  $\delta_{12}$  and mixing parameter  $\varepsilon_1$  for coupled states  ${}^3S_1$ - ${}^3D_1$  with the use of potential Reid68 (15) are presented. Stapp parameterization (9)-(11) is used to calculate these quantities.

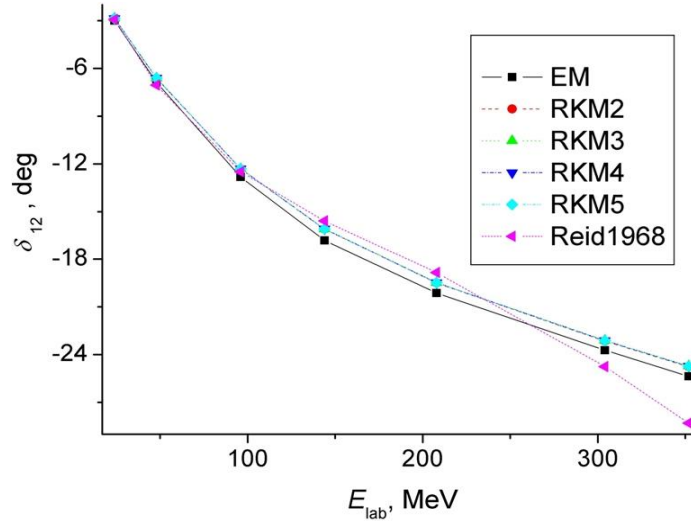
**Table 1.** Scattering phases  $\delta_{10}$ ,  $\delta_{12}$  and mixing parameter  $\varepsilon_1$

$E_{lab},$ MeV	EM	RKM2	RKM3	RKM4	RKM5
Scattering phase $\delta_{10}$					
24	77.046	81.328	81.804	81.643	81.630
48	60.566	63.108	63.325	63.286	63.300
96	41.198	42.799	42.896	42.870	42.880
144	28.663	29.763	29.835	29.796	29.796
208	16.449	17.132	17.187	17.149	17.150
304	2.637	3.201	3.225	3.197	3.199
352	-2.971	-2.462	-2.456	-2.479	-2.477
Scattering phase $\delta_{12}$					
24	-3.008	-2.863	-2.856	-2.859	-2.859
48	-6.877	-6.619	-6.612	-6.613	-6.613
96	-12.825	-12.324	-12.314	-12.314	-12.314
144	-16.812	-16.103	-16.088	-16.089	-16.089
208	-20.135	-19.500	-19.477	-19.480	-19.480
304	-23.718	-23.146	-23.108	-23.111	-23.112
352	-25.362	-24.760	-24.716	-24.720	-24.721
Mixing parameter $\varepsilon_1$					
24	2.633	1.854	1.805	1.827	1.828
48	2.948	2.351	2.330	2.336	2.334
96	3.688	3.274	3.275	3.279	3.278
144	4.647	4.355	4.364	4.371	4.372
208	6.229	5.842	5.852	5.861	5.863
304	8.172	7.762	7.805	7.818	7.819
352	8.903	8.563	8.602	8.616	8.618

The influence of the choice of a numerical method on the calculations of scattering phases and the mixing parameter is taken into account. The following numerical methods are chosen for this purpose [17-21]: the Euler method and Runge-Kutta methods of 2-, 3-, 4-, and 5th order accuracy – the designation EM, RKM2, RKM3, RKM4 and RKM5 respectively. The slight deviations between the data obtained by the Euler and Runge-Kutta methods are observed at energies  $E_{lab} = 24-208$  MeV. Significant deviations between the data obtained and the results of the original work [15] (on Figs. 1-3 is the designation as Reid1968) only at high energies. The obtained scattering phases for two-channel scattering can be used to calculate the values of scalar amplitude and total cross-section [8]. In further studies of the features of the VPA, we can use other parametrizations and modern nucleon-nucleon potentials in the coordinate representation.



**Figure 1.** Scattering phase  $\delta_{10}$



**Figure 2.** Scattering phase  $\delta_{12}$

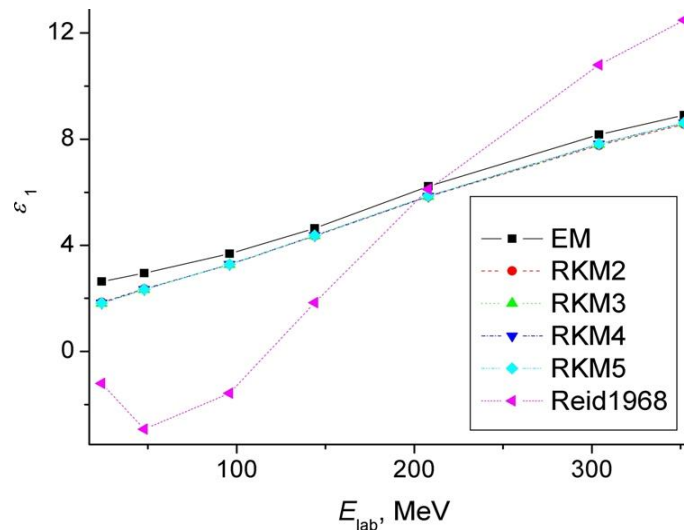


Figure 3. Mixing parameter  $\varepsilon_1$

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