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Convergences Analysis from the Solution Function of the Riccati Fractional Differential Equation by Using Modified Homotopy Perturbation Method

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ABSTRACT

A Fractional differential equation is an equation that contains derivatives with an order of fractional numbers. Same with natural number order differential equations, this type of equation is divided into linear and nonlinear fractional differential equations. One of the equations that include nonlinear fractional differential equation is Riccati fractional differential equation (RFDE). Various methods have been applied to find solutions for fractional differential equations Riccati, one of them is the Modified Homotopy Perturbation Method (MHPM) which is a modification of the Homotopy Perturbation Method by Zaid Odibat and Shaher Momani. In this study, the MHPM was used to find solutions for fractional differential equations Riccati, which were then used to analyze the convergence of the function sequences of the solution. The result shows us that the order sequence of Riccati fractional differential equation which converges to a number causes the solution function sequence of Riccati fractional differential equation will converge to the function of the solution of Riccati fractional differential equation with the order of this number.

***Keywords*:** Differential Equation, Fractional, Riccati, Modified Homotopy Perturbation Method, Convergent

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1. INTRODUCTION

Fractional Calculus is a branch of mathematics that investigates the properties of derivatives and integrals with order of rational numbers. In particular, these disciplines include the ideas and methods of solving differential equations involving fractional derivatives of arbitrary functions that called fractional differential equations.

According to Loverro [28], it was mentioned that the concept of fractional calculus first appeared in L'Hopital's letter to Leibniz in 1695 which questioned how the derivative form of a function with an order of rational numbers. Since then, fractional calculus has continued to be developed by many famous mathematicians, such as Liouville, Grunwald, Riemann, Euler, Lagrange, Heaviside, Fourier, Abel, and others [22].

The difficulties of fractional calculus especially fractional derivatives make this field unpopular. One possible explanation of this unpopularity is that there are many non-equivalent definitions of fractional derivatives [36]. However, in recent years, fractional calculus has begun to steal attention.

This is indicated by the number of models, especially interdisciplinary applications that are easily modeled using help from fractional derivatives. Some applications, namely nonlinear oscillations of earthquakes [17], fluid-dynamic flow models [16], viscoelasticity [24, 25, 31], dynamic system control theory [5, 26], electricity networks [12,32], opportunities and statistics [27], Finance [15, 38], electrochemical corrosion [11, 30], optics and signal processing [21, 35, 39], control engineering [3,10], biosciences [29], fluid mechanics [23], electrochemistry [34], diffusion processes [13, 19], Optimal control [7,8], hydrologic model [6], pharmacokinetics [37].

Various methods have been developed to get a solution of fractional differential equations so that it is easy to apply. He [17, 18] using Homotopy Perturbation Method (HPM) to solve nonlinear fractional differential equation, Ghazanfari & Sepahvandzadeh [14] solved fractional Bratu-Type equation using Adomian Decomposition Method (ADM), Jafari et al [20] used Laplace Decompotition Method for solving linear and nonlinear fractional diffusion-wave equation, Abbasbandy [1] compared HPM and ADM for solving quadratic Riccati differential equation, Das et al [9] using Homotopy Analysis Method (HAM) to solve nonlinear fractional differential equation, Odibat & Momani [33] using *Modified Homotopy Perturbation Method* (MHPM) which is a modification from HPM to solve fractional Riccati differential equation.

Bai et al [4] solved the fractional differential equation using a monotone iterative method. Bekir et al [2] used the first integral method to determine the exact solution of the nonlinear fractional differential equation.

Odibat and Momani [33] have reviewed numerical and analytical solutions for Riccati fractional differential equations (RFDE) with the coefficient are a constant function using MHPM. The modification of the method shows rapidly convergences from the series of the solution.

The result showed the solution values for the order of fractional numbers. However, the study did not examine the sequences function of the solution convergences. In this study, the MHPM method is used to determine the solution of Riccati fractional differential equations with one of the coefficients in the form of a polynomial function and equipped with a study of the convergences of RFDE's solution function with an order sequence converging towards a number to the function of RFDE's solution with the order is these number.

2. MATERIALS AND METHODS

The Riccati fractional differential equation that used in this study has the following general forms:

$$\frac{d^{\alpha}u}{dt^{\alpha}} = A(t) + B(t)u + C(t)u^2, \qquad t > 0, \quad m-1 < \alpha \le m,$$

with the initial condition $y^{j}(0) = c_{j,}$ $j = 0,1,\ldots,m-1.$
$A(t), B(t)$ and $C(t)$ are the function in t which is the coefficient, c_j is an arbitrary constant and α is a fractional derivative order.

2. 1. Modified Homotopy Perturbation Method

In this section, we discuss the Modified Homotopy Perturbation Method to find the solutions of Riccati fractional differential equation. This method involves fractional integrals and Caputo fractional derivatives with the definition as follows:

Definition 1. Fractional Integral [28]

The fractional integral defined by Riemann-Liouville is a popular definition in fractional calculus, the definition is as follows:

Suppose α is real number, fractional integral with α as an order of function $f(x)$ is

$$J^{\alpha} f(x) = D^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \quad \text{with } \alpha > 0.$$

The fractional integral with an α orde of simple function in the form of $f(x) = x^m$ according to Riemann-Liouville is expressed in the form of multiplying Gamma functions with polynomial functions which can be expressed in the following theorem:

Theorem 2 [21]

$$D^{-\alpha}x^m = \frac{\Gamma(m+1)}{\Gamma(m+\alpha+1)} x^{m+\alpha}, \qquad \text{for } \alpha > 0, \quad m > -1, \quad x > 0.$$

Proof:

For $x_0 = 0$ and $f(x) = x^m$ obtained

$$J^{\alpha} f(x) := D^{-\alpha}x^m = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^m dt,$$

so that it can be calculated $D^{-\alpha}x^m$, where $\alpha > 0, \quad m > -1$ as follows

$$D^{-\alpha}x^{m}=\frac{1}{\Gamma(\alpha)}\int_{0}^{x}(x-t)^{\alpha-1}t^{m}dt=\frac{1}{\Gamma(\alpha)}\int_{0}^{x}(1-\frac{t}{x})^{\alpha-1}x^{\alpha-1}t^{m}dt.$$

Suppose that $t=xu$ then obtained $dt=xdu$. If $t=0$ then $u=0$ and if $t=x$ then $u=1$, so that $D^{-\alpha}x^{m}$ became

$$D^{-\alpha}x^{m}=\frac{1}{\Gamma(\alpha)}\int_{0}^{1}(1-u)^{\alpha-1}x^{\alpha-1}(xu)^{m}xdu=\frac{1}{\Gamma(\alpha)}x^{m+\alpha}\int_{0}^{1}(1-u)^{\alpha-1}u^{m}du$$

$$=\frac{1}{\Gamma(\alpha)}x^{m+\alpha}\beta(m+1,\alpha)=\frac{\Gamma(m+1)}{\Gamma(m+\alpha+1)}x^{m+\alpha}.$$

Definition 3. Caputo Fractional Derivative [22]

Suppose α is a real number, and $n-1<\alpha\le n$, n is a natural number, fractional derivative order α of $f(x)$ with respect to t is

$$D^{\alpha}f(x)=J^{n-\alpha}f(x)=\frac{1}{\Gamma(n-\alpha)}\int_{0}^{x}(x-t)^{n-\alpha-1}f^{(n)}(t)dt.$$

Modified Homotopy Perturbation Method

Suppose that the nonlinear fractional differential equation is given as follows:

$$D^{\alpha}u(t)+L(u(t))+N(u(t))=r(t),\qquad t>0,\quad m-1<\alpha\le m, \tag{1}$$

where L is the linear operator, N is the nonlinear operator, r is an analytic function, and D^{α} is Caputo fractional derivative order α, with initial condition

$$u^{k}(0)=c_{k},\qquad k=0,1,2,\dots,m-1. \tag{2}$$

Then, based on MHPM, it can obtain the homotopy equation for (1) as follows

$$u^{(m)}+L(u)-r(t)=p[u^{(m)}-N(u)-D^{\alpha}u],\qquad p\in[0,1] \tag{3}$$

or $$u^{(m)}-r(t)=p[u^{(m)}-L(u)-N(u)-D^{\alpha}u],\qquad p\in[0,1] \tag{4}$$

Basic assumptions for the solution of the equation (3) and (4) can be formed in power series p, that is

$$u=u_{0}+pu_{1}+p^{2}u_{2}+p^{3}u_{3}+\cdots \tag{5}$$

by substituting (5) to (4), it can be obtained

$$u_0^{(m)} - r(t) + pu_1^{(m)} + p^2 u_2^{(m)} + p^3 u_3^{(m)} + \cdots$$

$$= p[u_0^{(m)} - L(u_0) - N(u_0) - D^\alpha u_0] + p^2[u_1^{(m)} - L(u_1) - N(u_1) - D^\alpha u_1] +$$

$$p^3[u_2^{(m)} - L(u_2) - N(u_2) - D^\alpha u_2] + \cdots. \tag{6}$$

By comparing the identical power of p in both sections on (6), then obtained

$$\frac{d^m u_o}{dt^m} = r(t), \qquad u^k(0) = c_k,$$

$$\frac{d^m u_1}{dt^m} = \frac{d^m u_0}{dt^m} - L_0(u_0) - N_0(u_0) - D^\alpha(u_0), \qquad u^k(0) = 0,$$

$$\frac{d^m u_2}{dt^m} = \frac{d^m u_1}{dt^m} - L_1(u_0, u_1) - N_1(u_0, u_1) - D^\alpha(u_1), \qquad u^k(0) = 0,$$

$$\frac{d^m u_3}{dt^m} = \frac{d^m u_2}{dt^m} - L_2(u_0, u_1, u_2) - N_1(u_0, u_1, u_2) - D^\alpha(u_2), \qquad u^k(0) = 0, \tag{7}$$

$$\vdots$$

where $L_0, L_1, L_2, \ldots$ and $N_0, N_1, N_2, \ldots$ fulfill the equation:

$$L(u_0 + pu_1 + p^2 u_2 + \ldots) = L_0(u_0) + pL_1(u_0, u_1) + p^2 L_2(u_0, u_1, u_2) + \ldots$$

$$N(u_0 + pu_1 + p^2 u_2 + \ldots) = N_0(u_0) + pN_1(u_0, u_1) + p^2 N_2(u_0, u_1, u_2) + \ldots,$$

Then, by integrating the (7) obtained $u_0, u_1, u_2, \ldots$.

Determine $p = 1$ for (5) then obtained the approach solution using MHPM, as follows:

$$u(t) = \sum_{n=0}^{\infty} u_n(t) \tag{8}$$

3. RESULT

3. 1. Solution of Fractional Differential Equations Riccati (RFDE) α order

The Riccati fractional differential equation is given as follows:

$$\frac{d^\alpha u}{dt^\alpha} = t^n + u(t) - u^2(t), \qquad t > 0, \qquad 0 < \alpha \le 1 \tag{9}$$

with initial condition $u(0)=0.$ (10)

Based on MHPM, homotopy equation for (9) is

$$u'-t^{n} = p[u'+u-u^{2}-D^{\alpha}u]. \tag{11}$$

By substituting (5) and initial condition (10) to (11), obtained

$$
\begin{aligned}
&u_0' = t^n\,, && u_0(0)=0\\
&u_1' = u_0'+u_0-u_0^{\,2}-D^{\alpha}u_0,\,, && u_1(0)=0\\
&u_2' = u_1'+u_1-2u_0u_1-D^{\alpha}u_1,\,, && u_2(0)=0\\
&u_3' = u_2'+u_2-2u_0u_2-u_1^{\,2}-D^{\alpha}u_2,\,, && u_3(0)=0\\
&\vdots
\end{aligned}
$$

then, by integrating (12) it can obtained $u_0, u_1, u_2, \ldots$ that are

$$u_0=\frac{t^{n+1}}{n+1}\,,$$

$$u_1=\frac{t^{n+1}}{n+1}+\frac{t^{n+2}}{(n+1)(n+2)}-\frac{t^{2n+3}}{(n+1)^2(2n+3)}-\frac{\Gamma(n+2)t^{n-\alpha+2}}{\Gamma(n-\alpha+3)(n+1)}\,,$$

$$u_2=\frac{t^{n+1}}{n+1}+\frac{2t^{n+2}}{(n+1)(n+2)}-\frac{t^{n+3}}{(n+1)(n+2)(n+3)}-\frac{3t^{2n+3}}{(n+1)^2(2n+3)}-\frac{(5n+8)t^{2n+4}}{(n+1)^2(2n+3)(n+2)(2n+4)}$$

$$+\frac{2t^{3n+5}}{(n+1)^3(2n+3)(3n+5)}-\frac{2\Gamma(n+2)t^{n-\alpha+2}}{\Gamma(n-\alpha+3)(n+1)}-\frac{2\Gamma(n+3)t^{n-\alpha+3}}{\Gamma(n-\alpha+4)(n+1)(n+2)}$$

$$+\frac{2\Gamma(n+2)t^{2n-\alpha+3}}{\Gamma(n-\alpha+2)(n+1)^2(2n-\alpha+3)}+\frac{\Gamma(2n+4)t^{2n-\alpha+4}}{\Gamma(2n-\alpha+5)(n+1)^2(2n+3)}+\frac{\Gamma(n+2)t^{n-2\alpha+3}}{\Gamma(n-2\alpha+4)(n+1)}$$

$$\vdots$$

Therefore, the solution for (9) is

$$u(t)=u_0(t)+u_1(t)+u_2(t)+\dots$$

$$=\frac{3t^{n+1}}{n+1}+\frac{3t^{n+2}}{(n+1)(n+2)}+\frac{t^{n+3}}{(n+1)(n+2)(n+3)}-\frac{4t^{2n+3}}{(n+1)^2(2n+3)}-\frac{(5n+8)t^{2n+4}}{(n+1)^2(2n+3)(n+2)(2n+4)}$$

$$+\frac{2t^{3n+5}}{(n+1)^3(2n+3)(3n+5)}-\frac{3\Gamma(n+2)t^{n-\alpha+2}}{\Gamma(n-\alpha+3)(n+1)}-\frac{2\Gamma(n+3)t^{n-\alpha+3}}{\Gamma(n-\alpha+4)(n+1)(n+2)}$$

$$+\frac{2\Gamma(n+2)t^{2n-\alpha+3}}{\Gamma(n-\alpha+2)(n+1)^2(2n-\alpha+3)}+\frac{\Gamma(2n+4)t^{2n-\alpha+4}}{\Gamma(2n-\alpha+5)(n+1)^2(2n+3)}+\frac{\Gamma(n+2)t^{n-2\alpha+3}}{\Gamma(n-2\alpha+4)(n+1)}+\dots \quad (13)$$

where $u(t)$ is the general form of the RFDE (9) solution.

3. 2. Convergences Analysis of the solution function sequences of RFDE α_i order with *A(t) = t*

Next, we will analyze the convergences of solution function sequences with $\alpha_i=\frac{i}{i+1}$ and $A(t)=t$. By substituting $\alpha=\alpha_i$ and $n=1$ to (9) then obtained

$$\frac{d^{\left(\frac{i}{i+1}\right)}u}{dt^{\left(\frac{i}{i+1}\right)}}=t^n+u(t)-u^2(t),\qquad t>0,\quad i=1,2,3,\dots,\quad 0<\alpha\le 1 \quad (14)$$

with initial condition $u(0)=0$.

Based on (13), the solution function sequences of RFDE (14) denoted by $g_i(t)$ is

$$g_i(t)=\frac{3}{2}t^2+\frac{1}{2}t^3+\frac{1}{24}t^4-\frac{17}{10}t^5-\frac{13}{360}t^6+\frac{1}{160}t^8-\frac{3t^{3-\frac{i}{i+1}}}{\Gamma\left(4-\frac{i}{i+1}\right)}-\frac{2t^{4-\frac{i}{i+1}}}{\Gamma\left(5-\frac{i}{i+1}\right)}$$

$$+\frac{t^{5-\frac{i}{i+1}}}{\Gamma\left(3-\frac{i}{i+1}\right)\left(5-\frac{i}{i+1}\right)}+\frac{6t^{6-\frac{i}{i+1}}}{\Gamma\left(7-\frac{i}{i+1}\right)}+\frac{t^{4-\frac{2i}{i+1}}}{\Gamma\left(5-\frac{2i}{i+1}\right)}+\dots .$$

If written in detail, obtained

$$g_1(t)=\frac{3}{2}t^2+\frac{1}{2}t^3+\frac{1}{24}t^4-\frac{17}{10}t^5-\frac{13}{360}t^6+\frac{1}{160}t^8-\frac{8t^{\frac{5}{2}}}{5\sqrt{\pi}}-\frac{32t^{\frac{7}{2}}}{105\sqrt{\pi}}+\frac{8t^{\frac{9}{2}}}{27\sqrt{\pi}}+\frac{128t^{\frac{11}{2}}}{3465\sqrt{\pi}}+\dots .$$

$$g_2(t)=\frac{3}{2}t^2+\frac{1}{2}t^3+\frac{1}{24}t^4-\frac{17}{10}t^5-\frac{13}{360}t^6+\frac{1}{160}t^8-\frac{81\sqrt{3}t^{\frac{7}{3}}\Gamma\left(\frac{2}{3}\right)}{56\pi}-\frac{81\sqrt{3}t^{\frac{10}{3}}\Gamma\left(\frac{2}{3}\right)}{280\pi}+\frac{27\sqrt{3}t^{\frac{13}{3}}\Gamma\left(\frac{2}{3}\right)}{104\pi}$$

$$+\frac{2187\sqrt{3}t^{\frac{16}{3}}\Gamma\left(\frac{2}{3}\right)}{58240\pi}+\frac{27t^{\frac{8}{3}}}{80\Gamma\left(\frac{2}{3}\right)}+\dots\ .$$

$$g_3(t)=\frac{3}{2}t^2+\frac{1}{2}t^3+\frac{1}{24}t^4-\frac{17}{10}t^5-\frac{13}{360}t^6+\frac{1}{160}t^8-\frac{32\sqrt{2}t^{\frac{9}{4}}\Gamma\left(\frac{3}{4}\right)}{15\pi}-\frac{256\sqrt{2}t^{\frac{13}{4}}\Gamma\left(\frac{3}{4}\right)}{585\pi}+\frac{32\sqrt{2}t^{\frac{17}{4}}\Gamma\left(\frac{3}{4}\right)}{85\pi}$$

$$.\ +\frac{4096\sqrt{2}t^{\frac{21}{4}}\Gamma\left(\frac{3}{4}\right)}{69615\pi}+\frac{8t^{\frac{5}{2}}}{15\sqrt{\pi}}+\dots\ .$$

and so on for $g_4(t), g_5(t), g_6(t),\dots$.

Because $(\alpha_i)=\left(\frac{i}{i+1}\right)$ converges to $\alpha=1$, then we have to find the solution of the RFDE with $\alpha=1$ and $A(t)=t$. By substituting $\alpha=1$ and $n=1$ to (13), obtained the solution $u(t)$ then in this case it is called $g(t)$,

$$g(t)=\frac{t^2}{2}+\frac{t^3}{6}+\frac{7t^4}{24}-\frac{33t^5}{20}-\frac{13t^6}{360}+\frac{t^8}{160}+\dots\ .$$

This part will be shown the sequences of solution function $g_i(t)$ converges to solution function $g(t)$.

$$\lim_{i\to\infty} g_i(t)=\lim_{i\to\infty}\left\{\frac{3}{2}t^2+\frac{1}{2}t^3+\frac{1}{24}t^4-\frac{17}{10}t^5-\frac{13}{360}t^6+\frac{1}{160}t^8-\frac{3t^{3-\frac{i}{i+1}}}{\Gamma\left(4-\frac{i}{i+1}\right)}-\frac{2t^{4-\frac{i}{i+1}}}{\Gamma\left(5-\frac{i}{i+1}\right)}\right.$$

$$\left.+\frac{t^{5-\frac{i}{i+1}}}{\Gamma\left(3-\frac{i}{i+1}\right)\left(5-\frac{i}{i+1}\right)}+\frac{6t^{6-\frac{i}{i+1}}}{\Gamma\left(7-\frac{i}{i+1}\right)}+\frac{t^{4-\frac{2i}{i+1}}}{\Gamma\left(5-\frac{2i}{i+1}\right)}+\dots\right\}$$

$$= \frac{3}{2}t^2 + \frac{1}{2}t^3 + \frac{1}{24}t^4 - \frac{17}{10}t^5 - \frac{13}{360}t^6 + \frac{1}{160}t^8 + \lim_{i\to\infty}\left\{ -\frac{3t^{3-\frac{i}{i+1}}}{\Gamma\left(4-\frac{i}{i+1}\right)} - \frac{2t^{4-\frac{i}{i+1}}}{\Gamma\left(5-\frac{i}{i+1}\right)} \right.$$

$$\left. + \frac{t^{5-\frac{i}{i+1}}}{\Gamma\left(3-\frac{i}{i+1}\right)\left(5-\frac{i}{i+1}\right)} + \frac{6t^{6-\frac{i}{i+1}}}{\Gamma\left(7-\frac{i}{i+1}\right)} + \frac{t^{4-\frac{2i}{i+1}}}{\Gamma\left(5-\frac{2i}{i+1}\right)} + \dots \right\}$$

$$= \frac{3}{2}t^2 + \frac{1}{2}t^3 + \frac{1}{24}t^4 - \frac{17}{10}t^5 - \frac{13}{360}t^6 + \frac{1}{160}t^8 - \frac{3}{\Gamma(4-1)}t^{3-1} - \frac{2}{\Gamma(5-1)}t^{4-1}$$

$$+ \frac{1}{\Gamma(3-1)(5-1)}t^{5-1} + \frac{6}{\Gamma(7-1)}t^{6-1} + \frac{1}{\Gamma(5-2)}t^{4-2} + \dots$$

$$= \frac{3}{2}t^2 + \frac{1}{2}t^3 + \frac{1}{24}t^4 - \frac{17}{10}t^5 - \frac{13}{360}t^6 + \frac{1}{160}t^8 - \frac{3}{\Gamma(3)}t^2 - \frac{2}{\Gamma(4)}t^3 + \frac{1}{4\Gamma(2)}t^4 + \frac{6}{\Gamma(6)}t^5$$

$$+ \frac{1}{\Gamma(3)}t^2 + \dots$$

$$= \frac{3}{2}t^2 + \frac{1}{2}t^3 + \frac{1}{24}t^4 - \frac{17}{10}t^5 - \frac{13}{360}t^6 + \frac{1}{160}t^8 - \frac{3}{2}t^2 - \frac{2}{3}t^3 + \frac{1}{4}t^4 + \frac{6}{120}t^5 + \frac{1}{2}t^2 + \dots$$

$$+ \frac{1}{\Gamma(3)}t^2 + \dots$$

$$= \frac{t^2}{2} + \frac{t^3}{6} + \frac{7t^4}{24} - \frac{33t^5}{20} - \frac{13t^6}{360} + \frac{t^8}{160} + \dots = g(t).$$

It is obtained that $\lim_{i\to\infty} g_i(t) = g(t)$, so the sequences of solution function RFDE (14) converges to the solution function of RFDE with $\alpha = 1$ and $A(t) = t$.

The sequences of solution function RFDE (14) can be described in graphical form which is shown in Figure 1.a and Figure 1.b as follows:

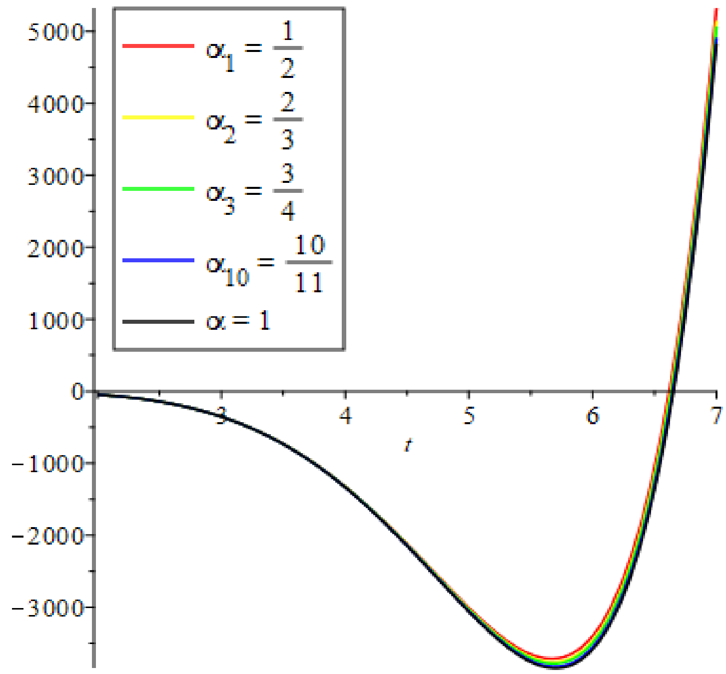

Figure 1.a. Graphic of RFDE solution with $(\alpha_i) = \left(\frac{i}{i+1}\right)$ and $A(t) = t \quad (0 \le t \le 7)$

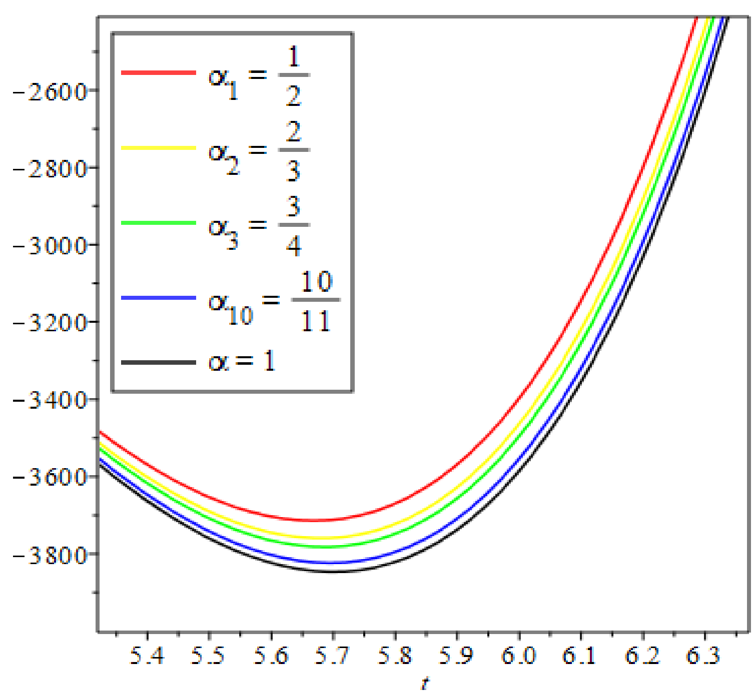

Figure 1.b. Graphic of RFDE solution with $(\alpha_i) = \left(\frac{i}{i+1}\right)$ and $A(t) = t \quad (5.3 \le t \le 6.4)$

Based on Figure 1.a and 1.b it can be seen that graphic of the sequences of solution function RFDE $(\alpha_i)=\left(\frac{i}{i+1}\right)$ order heading to the graph function solution RFDE $\alpha=1$ order which is shown in black. In other words, the sequences of solution function RFDE (14) converges to the solution function RFDE with $\alpha = \lim_{i\to\infty}\frac{i}{i+1}=1$ and $n=1$.

3. 3. Convergences Analysis of the solution function sequences of RFDE α_i order with *A(t)* = t^2

Next, we will analyze the convergences of solution function sequences with $\alpha_i=\frac{i}{i+1}$ and $A(t)=t^2$. By substituting $\alpha=\alpha_i$ and $n=2$ to (9) then obtained

$$\frac{d^{\left(\frac{i}{i+1}\right)}u}{dt^{\left(\frac{i}{i+1}\right)}}=t^n+u(t)-u^2(t), \qquad t>0, \quad i=1,2,3,\ldots, \quad 0<\alpha\le 1 \tag{15}$$

with initial condition $u(0)=0$.

Based on (13), the solution function sequences of RFDE (15) denoted by $h_i(t)$ is

$$h_i(t)=t^3+\frac{1}{4}t^4+\frac{1}{60}t^5-\frac{4}{63}t^7-\frac{1}{112}t^8+\frac{2}{2079}t^{11}-\frac{6t^{4-\frac{i}{i+1}}}{\Gamma\left(5-\frac{i}{i+1}\right)}-\frac{4t^{5-\frac{i}{i+1}}}{\Gamma\left(6-\frac{i}{i+1}\right)}$$

$$+\frac{4t^{7-\frac{i}{i+1}}}{3\Gamma\left(4-\frac{i}{i+1}\right)\left(7-\frac{i}{i+1}\right)}+\frac{80t^{8-\frac{i}{i+1}}}{\Gamma\left(9-\frac{i}{i+1}\right)}+\frac{2t^{5-\frac{2i}{i+1}}}{\Gamma\left(6-\frac{2i}{i+1}\right)}+\ldots\ .$$

If written in detail, obtained

$$h_1(t)=t^3+\frac{1}{4}t^4+\frac{1}{60}t^5-\frac{4}{63}t^7-\frac{1}{112}t^8+\frac{2}{2079}t^{11}-\frac{32t^{\frac{7}{2}}}{35\sqrt{\pi}}-\frac{128t^{\frac{9}{2}}}{945\sqrt{\pi}}+\frac{64t^{\frac{13}{2}}}{585\sqrt{\pi}}+\frac{4096t^{\frac{15}{2}}}{405405\sqrt{\pi}}+\ldots$$

$$h_2(t)=t^3+\frac{1}{4}t^4+\frac{1}{60}t^5-\frac{4}{63}t^7-\frac{1}{112}t^8+\frac{2}{2079}t^{11}-\frac{243\sqrt{3}t^{\frac{10}{3}}\Gamma\left(\frac{2}{3}\right)}{280\pi}-\frac{243\sqrt{3}t^{\frac{13}{3}}\Gamma\left(\frac{2}{3}\right)}{1820\pi}+\frac{27\sqrt{3}t^{\frac{19}{3}}\Gamma\left(\frac{2}{3}\right)}{266\pi}$$

$$+\frac{6561\sqrt{3}t^{\frac{22}{3}}\Gamma\left(\frac{2}{3}\right)}{608608\pi}+\frac{81t^{\frac{11}{3}}}{440\Gamma\left(\frac{2}{3}\right)}+\dots.$$

$$h_3(t)=t^3+\frac{1}{4}t^4+\frac{1}{60}t^5-\frac{4}{63}t^7-\frac{1}{112}t^8+\frac{2}{2079}t^{11}-\frac{256\sqrt{2}t^{\frac{13}{4}}\Gamma\left(\frac{3}{4}\right)}{195\pi}-\frac{2048\sqrt{2}t^{\frac{17}{4}}\Gamma\left(\frac{3}{4}\right)}{9945\pi}+\frac{512\sqrt{2}t^{\frac{25}{4}}\Gamma\left(\frac{3}{4}\right)}{3375\pi}$$

$$.+\frac{524288\sqrt{2}t^{\frac{29}{4}}\Gamma\left(\frac{3}{4}\right)}{30282525\pi}+\frac{32t^{\frac{7}{2}}}{105\sqrt{\pi}}+\dots .$$

and so on for $h_4(t), h_5(t), h_6(t), \dots$.

Because $(\alpha_i)=\left(\frac{i}{i+1}\right)$ converges to $\alpha=1$, then we have to find the solution of the RFDE with $\alpha=1$ and $A(t)=t^2$. By substituting $\alpha=1$ and $n=2$ to (13), obtained the solution $u(t)$ then in this case it is called $h(t)$,

$$h(t)=t^3+\frac{1}{12}t^4+\frac{1}{60}t^5-\frac{1}{9}t^6-\frac{1}{21}t^7-\frac{1}{112}t^8+\frac{2}{2079}t^{11}+\dots .$$

This part will be shown the sequences of solution function $h_i(t)$ converges to solution function $h(t)$.

$$\lim_{i\to\infty}h_i(t)=\lim_{i\to\infty}\left\{h_i(t)=t^3+\frac{1}{4}t^4+\frac{1}{60}t^5-\frac{4}{63}t^7-\frac{1}{112}t^8+\frac{2}{2079}t^{11}-\frac{6t^{4-\frac{i}{i+1}}}{\Gamma\left(5-\frac{i}{i+1}\right)}-\frac{4t^{5-\frac{i}{i+1}}}{\Gamma\left(6-\frac{i}{i+1}\right)}\right.$$

$$\left.+\frac{4t^{7-\frac{i}{i+1}}}{3\Gamma\left(4-\frac{i}{i+1}\right)\left(7-\frac{i}{i+1}\right)}+\frac{80t^{8-\frac{i}{i+1}}}{\Gamma\left(9-\frac{i}{i+1}\right)}+\frac{2t^{5-\frac{2i}{i+1}}}{\Gamma\left(6-\frac{2i}{i+1}\right)}+\dots\right\}$$

$$=t^3+\frac{1}{4}t^4+\frac{1}{60}t^5-\frac{4}{63}t^7-\frac{1}{112}t^8+\frac{2}{2079}t^{11}+\lim_{i\to\infty}\left\{-\frac{6t^{4-\frac{i}{i+1}}}{\Gamma\left(5-\frac{i}{i+1}\right)}-\frac{4t^{5-\frac{i}{i+1}}}{\Gamma\left(6-\frac{i}{i+1}\right)}\right.$$

$$\left. + \frac{4t^{7-\frac{i}{i+1}}}{3\Gamma\left(4-\frac{i}{i+1}\right)\left(7-\frac{i}{i+1}\right)} + \frac{80t^{8-\frac{i}{i+1}}}{\Gamma\left(9-\frac{i}{i+1}\right)} + \frac{2t^{5-\frac{2i}{i+1}}}{\Gamma\left(6-\frac{2i}{i+1}\right)} + \dots \right\}$$

$$= t^3 + \frac{1}{4}t^4 + \frac{1}{60}t^5 - \frac{4}{63}t^7 - \frac{1}{112}t^8 + \frac{2}{2079}t^{11} - \frac{6}{\Gamma(5-1)}t^{4-1} - \frac{4}{\Gamma(6-1)}t^{5-1}$$

$$+ \frac{4}{3\Gamma(4-1)(7-1)}t^{7-1} + \frac{80}{\Gamma(9-1)}t^{8-1} + \frac{2}{\Gamma(6-2)}t^{5-2} + \dots$$

$$= t^3 + \frac{1}{4}t^4 + \frac{1}{60}t^5 - \frac{4}{63}t^7 - \frac{1}{112}t^8 + \frac{2}{2079}t^{11} - \frac{6}{\Gamma(4)}t^3 - \frac{4}{\Gamma(5)}t^4 + \frac{4}{18\Gamma(3)}t^6 + \frac{80}{\Gamma(8)}t^7$$

$$+ \frac{2}{\Gamma(4)}t^3 + \dots$$

$$= t^3 + \frac{1}{4}t^4 + \frac{1}{60}t^5 - \frac{4}{63}t^7 - \frac{1}{112}t^8 + \frac{2}{2079}t^{11} - t^3 - \frac{1}{6}t^4 + \frac{1}{9}t^6 + \frac{1}{63}t^7 + \frac{1}{3}t^3 + \dots$$

$$= t^3 + \frac{1}{12}t^4 + \frac{1}{60}t^5 - \frac{1}{9}t^6 - \frac{1}{21}t^7 - \frac{1}{112}t^8 + \frac{2}{2079}t^{11} + \dots = h(t).$$

It is obtained that $\lim_{i\to\infty} h_i(t) = h(t)$, so the sequences of solution function RFDE (15) converges to the solution function of RFDE with $\alpha = 1$ and $A(t) = t^2$.

The sequences of solution function RFDE (15) can be described in graphical form which is shown in Figure 2.

Based on Figure 2 it can be seen that graphic of the sequences of solution function RFDE $(\alpha_i) = \left(\frac{i}{i+1}\right)$ order heading to the graph function solution RFDE $\alpha = 1$ order which is shown in black. In other words, the sequences of solution function RFDE (15) converges to the solution function RFDE with $\alpha = \lim_{i\to\infty} \frac{i}{i+1} = 1$ and $n = 2$.

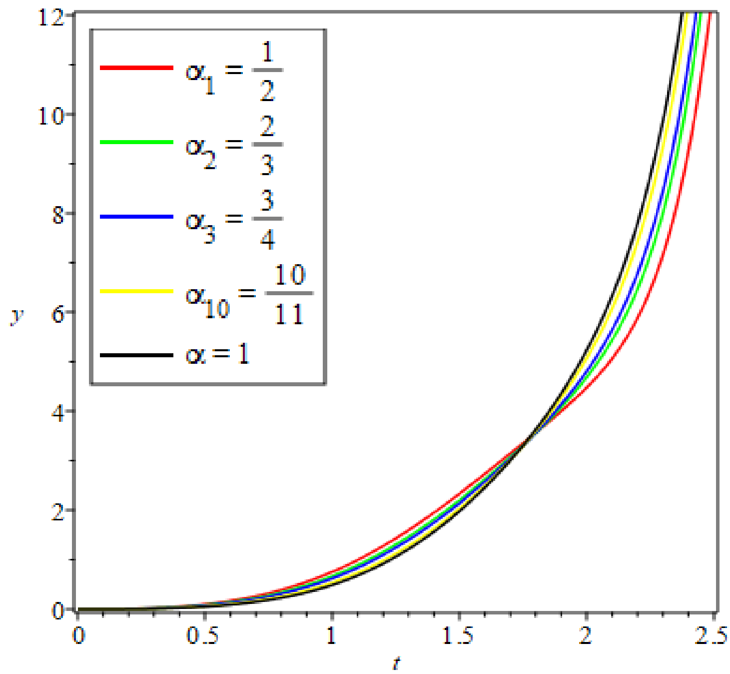

Figure 2. Graphic of RFDE solution with $(\alpha_i) = \left(\dfrac{i}{i+1}\right)$ and $A(t) = t^2$

4. CONCLUSION

The Modified Homotopy Perturbation Method can be used to find a solution of RFDE and if the sequence of order α_i converges to a number called α then the sequence of solution function RFDE α_i order will converge to the RFDE's solution function α order.

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