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Riesz triple probabilisitic of almost lacunary Cesáro C_{111} statistical convergence of Γ^3 defined by a Musielak Orlicz function

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ABSTRACT

In this paper we study the concept of almost lacunary statistical Cesáro of Γ^3 over probabilistic $p-$ metric spaces defined by Musielak Orlicz function. Since the study of convergence in PP-spaces is fundamental to probabilistic functional analysis, we feel that the concept of almost lacunary statistical Cesáro of Γ^3 over probabilistic $p-$ metric spaces defined by Musielak in a PP-space would provide a more general framework for the subject.

Keywords: analytic sequence, Orlicz function, triple sequences, entire sequence, Riesz space, statistical convergence, Cesáro C_{111} - statistical convergence

1. INTRODUCTION

Throughout w, Γ and Λ denote the classes of all, entire and analytic scalar valued single sequences, respectively. We write w^3 for the set of all complex triple sequences (x_{mnk}), where $m, n, k \in \mathbb{N}$, the set of positive integers. Then, w^3 is a linear space under the coordinate wise addition and scalar multiplication.

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Some initial work on double series is found in Apostol [1] and double sequence spaces is found in Hardy [7], Deepmala et al. [6, 5] and many others. The initial work on triple sequence spaces is found in Sahiner et al. [13], Esi [2] and Esi et al. [3, 4,17,18], Shri Prakash et al. [14], Subramanian et al. [15,16] and many others.

Let (x_{mnk}) be a triple sequence of real or complex numbers. Then the series $\sum_{m,n,k=1}^{\infty} x_{mnk}$ is called a triple series. Then the triple series is said to be convergent if and only if the triple sequence (S_{mnk}) is convergent, where

$$S_{mnk} = \sum_{i,j,q=1}^{m,n,k} x_{ijq} (m,n,k = 1,2,3,\ldots).$$

A sequence $x = (x_{mnk})$ is said to be triple analytic if

$$sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty.$$

The vector space of all triple analytic sequence is usually denoted by Λ^3. A sequence $x = (x_{mnk})$ is called triple entire sequence if

$$|x_{mnk}|^{\frac{1}{m+n+k}} \to 0 \text{ as } m,n,k \to \infty.$$

A sequence $x = (x_{mnk})$ is called triple entire sequence if $(|x_{mnk}|)^{\frac{1}{m+n+k}} \to 0$ as $m,n,k \to \infty$. The triple entire sequences will be denoted by Γ^3.

Consider a triple sequence $x = (x_{mnk})$. The $(m,n,k)^{th}$ section $x^{[m,n,k]}$ of the sequence is defined by $x^{[m,n,k]} = \sum_{i,j,q=0}^{m,n,k} x_{ijq} \mathfrak{I}_{ijq}$ for all $m,n,k \in \mathbb{N}$,

$$\mathfrak{I}_{ijq} = \begin{bmatrix} 0 & 0 & \ldots 0 & 0 & \ldots \\ 0 & 0 & \ldots 0 & 0 & \ldots \\ . & & & & \\ . & & & & \\ . & & & & \\ 0 & 0 & \ldots 1 & 0 & \ldots \\ 0 & 0 & \ldots 0 & 0 & \ldots \end{bmatrix}$$

with 1 in the $(i,j,q)^{th}$ position and zero otherwise.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [10] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\}$$

where: $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$.

Consider the triple difference sequence space is defined as

$$\Delta_{mnk} = x_{mnk} - x_{m,n+1,k} - x_{m,n,k+1} + x_{m,n+1,k+1} - x_{m+1,n,k} + x_{m+1,n+1,k} + x_{m+1,n,k+1} - x_{m+1,n+1,k+1}$$

and

$$\Delta^0 x_{mnk} = \langle x_{mnk} \rangle.$$

2. DEFINITIONS AND PRELIMINARIES

Definition 2.1: *An Orlicz function ([see [9]) is a function $M: [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called modulus function.*

Lindenstrauss and Tzafriri ([11]) used the idea of Orlicz function to construct Orlicz sequence space.

A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = sup\{|v|u - (f_{mnk})(u) : u \geq 0\}, m, n, k = 1,2, \ldots$$

is called the complementary function of a Musielak-Orlicz function f. For a given Musielak-Orlicz function f, (see [12]) the Musielak-Orlicz sequence space t_f is defined as follows

$$t_f = \left\{x \in w^3 : I_f(|x_{mnk}|)^{1/m+n+k} \to 0 \quad as \quad m, n, k \to \infty\right\},$$

where: I_f is a convex modular defined by

$$I_f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mnk}(|x_{mnk}|)^{1/m+n+k}, x = (x_{mnk}) \in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mnk}\left(\frac{|x_{mnk}|^{1/m+n+k}}{mnk}\right)$$

is an exteneded real number.

Definition 2.2: *A triple sequence $x = (x_{mnk})$ of real numbers is called almost $P -$ convergent to a limit 0 if*

$$P - lim_{p,q,u\to\infty} sup_{r,s,t\geq 0} \frac{1}{pqu} \sum_{m=r}^{r+p-1} \sum_{n=s}^{s+q-1} \sum_{k=t}^{t+u-1} |x_{mnk}|^{1/m+n+k} \to 0.$$

That is, the average value of (x_{mnk}) taken over any rectangle

$$\{(m,n,k): r \leq m \leq r+p-1, s \leq n \leq s+q-1, t \leq k \leq t+u-1\}$$

tends to 0 as both p, q and u to ∞, and this $P -$ convergence is uniform in i, ℓ and j. Let denote the set of sequences with this property as $\left[\widehat{\chi^3}\right]$.

Definition 2.3: *Let $(Q_r), (\overline{Q_s}), (\overline{\overline{Q_t}})$ be sequences of positive numbers and*

$$Q_r = \begin{bmatrix} q_{11} & q_{12} & \dots & q_{1s} & 0\dots \\ q_{21} & q_{22} & \dots & q_{2s} & 0\dots \\ . & & & & \\ . & & & & \\ . & & & & \\ q_{r1} & q_{r2} & \dots & q_{rs} & 0\dots \\ 0 & 0 & \dots 0 & 0 & 0\dots \end{bmatrix} = q_{11} + q_{12} + \cdots + q_{rs} \neq 0,$$

$$\overline{Q}_s = \begin{bmatrix} \overline{q}_{11} & \overline{q}_{12} & \dots & \overline{q}_{1s} & 0\dots \\ \overline{q}_{21} & \overline{q}_{22} & \dots & \overline{q}_{2s} & 0\dots \\ . & & & & \\ . & & & & \\ . & & & & \\ \overline{q}_{r1} & \overline{q}_{r2} & \dots & \overline{q}_{rs} & 0\dots \\ 0 & 0 & \dots 0 & 0 & 0\dots \end{bmatrix} = \overline{q}_{11} + \overline{q}_{12} + \cdots + \overline{q}_{rs} \neq 0,$$

$$\overline{\overline{Q}}_t = \begin{bmatrix} \overline{\overline{q}}_{11} & \overline{\overline{q}}_{12} & \dots & \overline{\overline{q}}_{1s} & 0\dots \\ \overline{\overline{q}}_{21} & \overline{\overline{q}}_{22} & \dots & \overline{\overline{q}}_{2s} & 0\dots \\ . & & & & \\ . & & & & \\ . & & & & \\ \overline{\overline{q}}_{r1} & \overline{\overline{q}}_{r2} & \dots & \overline{\overline{q}}_{rs} & 0\dots \\ 0 & 0 & \dots 0 & 0 & 0\dots \end{bmatrix} = \overline{\overline{q}}_{11} + \overline{\overline{q}}_{12} + \cdots + \overline{\overline{q}}_{rs} \neq 0$$

and is given by

$$T_{rst} = \frac{1}{Q_r \overline{Q}_s \overline{\overline{Q}}_t} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} q_m \overline{q}_n \overline{\overline{q}}_k |x_{mnk}|^{1/m+n+k}$$

is called the Riesz mean of triple sequence $x = (x_{mnk})$. If $P - lim_{rst} T_{rst}(x) = 0, 0 \in \mathbb{R}$, then the sequence $x = (x_{mnk})$ is said to be Riesz convergent to 0. If $x = (x_{mnk})$ is Riesz convergent to 0, then we write $P_R - lim x = 0$.

Definition 2.4: *The four dimensional matrix A is said to be RH-regular if it maps every bounded $P -$ convergent sequence into a $P -$ convergent sequence with the same $P -$ limit.*

Definition 2.5: *The triple sequence $\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ is called triple lacunary if there exist three increasing sequences of integers such that*

$$m_0 = 0, h_i = m_i - m_{r-1} \to \infty \text{ as } i \to \infty \text{ and}$$

$$n_0 = 0, \overline{h_\ell} = n_\ell - n_{\ell-1} \to \infty \text{ as } \ell \to \infty.$$

$$k_0 = 0, \overline{h_j} = k_j - k_{j-1} \to \infty \text{ as } j \to \infty.$$

Let $m_{i,\ell,j} = m_i n_\ell k_j, h_{i,\ell,j} = h_i \overline{h_\ell h_j}$, and $\theta_{i,\ell,j}$ is determine by

$$I_{i,\ell,j} = \{(m,n,k): m_{i-1} < m < m_i \quad \text{and} \quad n_{\ell-1} < n \le n_\ell \quad \text{and} \quad k_{j-1} < k \le k_j\},$$

$$q_k = \frac{m_k}{m_{k-1}}, \overline{q_\ell} = \frac{n_\ell}{n_{\ell-1}}, \overline{q_j} = \frac{k_j}{k_{j-1}}.$$

Using the notations of lacunary Fuzzy sequence and Riesz mean for triple sequences. $\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ be a triple lacunary sequence and $q_m \overline{q}_n \overline{\overline{q}}_k$ be sequences of positive real numbers such that

$$Q_{m_i} = \sum_{m \in (0, m_i]} p_{m_i}, Q_{n_\ell} = \sum_{n \in (0, n_\ell]} p_{n_\ell}, Q_{n_j} = \sum_{k \in (0, k_j]} p_{k_j}$$

and

$$H_i = \sum_{m \in (0, m_i]} p_{m_i}, \overline{H} = \sum_{n \in (0, n_\ell]} p_{n_\ell}, \overline{\overline{H}} = \sum_{k \in (0, k_j]} p_{k_j}.$$

Clearly, $H_i = Q_{m_i} - Q_{m_{i-1}}$, $\overline{H}_\ell = Q_{n_\ell} - Q_{n_{\ell-1}}$, $\overline{\overline{H}}_j = Q_{k_j} - Q_{k_{j-1}}$. If the Riesz transformation of triple sequences is RH-regular, and

$$H_i = Q_{m_i} - Q_{m_{i-1}} \to \infty \text{as} i \to \infty,$$

$$\overline{H} = \sum_{n \in (0, n_\ell]} p_{n_\ell} \to \infty \text{as} \ell \to \infty,$$

$$\overline{\overline{H}} = \sum_{k \in (0, k_j]} p_{k_j} \to \infty \text{as} j \to \infty,$$

then $\theta'_{i,\ell,j} = \{(m_i, n_\ell, k_j)\} = \left\{\left(Q_{m_i} Q_{n_j} Q_{k_k}\right)\right\}$ is a triple lacunary sequence. If the assumptions $Q_r \to \infty$ as $r \to \infty$, $\overline{Q}_s \to \infty$ as $s \to \infty$ and $\overline{\overline{Q}}_t \to \infty$ as $t \to \infty$ may be not enough to obtain the conditions $H_i \to \infty$ as $i \to \infty$, $\overline{H}_\ell \to \infty$ as $\ell \to \infty$ and $\overline{\overline{H}}_j \to \infty$ as $j \to \infty$ respectively. For any lacunary sequences (m_i), (n_ℓ) and (k_j) are integers.

Throughout the paper, we assume that

$$Q_r = q_{11} + q_{12} + \cdots + q_{rs} \to \infty (r \to \infty),$$

$$\overline{Q}_s = \overline{q}_{11} + \overline{q}_{12} + \cdots + \overline{q}_{rs} \to \infty (s \to \infty),$$

$$\overline{\overline{Q}}_t = \overline{\overline{q}}_{11} + \overline{\overline{q}}_{12} + \cdots + \overline{\overline{q}}_{rs} \to \infty (t \to \infty),$$

such that

$$H_i = Q_{m_i} - Q_{m_{i-1}} \to \infty \text{as} i \to \infty,$$

$$\overline{H}_\ell = Q_{n_\ell} - Q_{n_{\ell-1}} \to \infty \text{as} \ell \to \infty$$

and

$$\overline{\overline{H}}_j = Q_{k_j} - Q_{k_{j-1}} \to \infty \text{as} j \to \infty.$$

Let

$$Q_{m_i, n_\ell, k_j} = Q_{m_i} \overline{Q}_{n_\ell} \overline{\overline{Q}}_{k_j}, H_{i\ell j} = H_i \overline{H}_\ell \overline{\overline{H}}_j,$$

$$I'_{i\ell j} = \left\{(m, n, k) \colon Q_{m_{i-1}} < m < Q_{m_i}, \overline{Q}_{n_{\ell-1}} < n < Q \text{ and } \overline{Q}_{k_{j-1}} < k < \overline{Q}_{k_j}\right\},$$

$$V_i = \frac{Q_{m_i}}{Q_{m_{i-1}}}, \overline{V}_\ell = \frac{Q_{n_\ell}}{Q_{n_{\ell-1}}} \text{and} \overline{\overline{V}}_j = \frac{Q_{k_j}}{Q_{k_{j-1}}} \text{and} V_{i\ell j} = V_i \overline{V}_\ell \overline{\overline{V}}_j.$$

If we take $q_m = 1, \overline{q}_n = 1$ and $\overline{\overline{q}}_k = 1$ for all m, n and k then $H_{i\ell j}, Q_{i\ell j}, V_{i\ell j}$ and $I'_{i\ell j}$ reduce to $h_{i\ell j}, q_{i\ell j}, v_{i\ell j}$ and $I_{i\ell j}$.

Let $n \in \mathbb{N}$ and X be a real vector space of dimension m, where $n \leq m$. A real valued function $d_p(x_1, \dots, x_n) = \| (d_1(x_1), \dots, d_n(x_n)) \|_p$ on X satisfying the following four conditions:

(i) $\| (d_1(x_1), \dots, d_n(x_n)) \|_p = 0$ if and and only if $d_1(x_1), \dots, d_n(x_n)$ are linearly dependent,

(ii) $\| (d_1(x_1), \dots, d_n(x_n)) \|_p$ is invariant under permutation,

(iii) $\| (\alpha d_1(x_1), \dots, \alpha d_n(x_n)) \|_p = |\alpha| \, \| (d_1(x_1), \dots, d_n(x_n)) \|_p, \alpha \in \mathbb{R}$

(iv) $d_p((x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)) = (d_X(x_1, x_2, \dots, x_n)^p + d_Y(y_1, y_2, \dots, y_n)^p)^{1/p}$ for $1 \leq p < \infty$; is called the p product metric.

3. ALMOST LACUNARY CESÁRO C_{111} − STATISTICAL CONVERGENCE OF PP-TRIPLE SEQUENCE SPACES

Let $A = \left[a_{mnk}^{pqr}\right]_{m,n,k=0}^{\infty}$ be a triple infinite matrix of real number for $p, q, r = 1,2, \dots$ forming the sum

$$\mu_{pqr}(X) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{mnk}^{pqr} \left(\left(\frac{X_{mnk}}{Y_{mnk}} \right)^{1/m+n+k}, \overline{0} \right) \quad (3.1)$$

is called a triple sequence space of summable to the limit 0, i.e.,

$$lim_{uvw \to \infty} \sum_{m}^{u} \sum_{n}^{v} \sum_{k}^{w} a_{mnk}^{pqr} \left(\left(\frac{X_{mnk}}{Y_{mnk}} \right) \right)^{1/m+n+k} = \mu_{pqr}$$

and

$$\lim_{pqr \to \infty} \mu_{pqr} = 0$$

Define the means

$$\sigma_{pqr}^{X} = \frac{1}{pqr} \sum_{m=0}^{p} \sum_{n=0}^{q} \sum_{k=0}^{r} \left(\left(\frac{X_{mnk}}{Y_{mnk}} \right) \right)^{1/m+n+k}$$

and

$$A\sigma_{pqr}^{X} = \frac{1}{pqr}\sum_{m=0}^{p}\sum_{n=0}^{q}\sum_{k=0}^{r} a_{mnk}^{pqr}\left(\left(\left(\frac{X_{mnk}}{Y_{mnk}}\right)\right)^{1/m+n+k}, \bar{0}\right).$$

We say that $\left(\frac{X_{mnk}}{Y_{mnk}}\right)$ is statistically lacunary equivalent summable $(C,1,1,1)$ to 0, if the sequence $\sigma = (\sigma_{mnk}^{X})$ is statistically convergent to $\bar{0}$, that is, $st_3 - lim_{pqr}\sigma_{pqr}^{X} = 0$. It is denoted by $C_{111}(st_3)$.

Let $q_m, \overline{q}_n$ and $\overline{\overline{q}}_k$ be sequences of positive numbers and $Q_r = q_{11} + \cdots + q_{rs}$, $\overline{Q}_s = \overline{q}_{11} + \cdots + \overline{q}_{rs}$ and $\overline{\overline{Q}}_t = \overline{\overline{q}}_{11} + \cdots + \overline{\overline{q}}_{rs}$.

Definition 3.1: *A triple $(X, P, *)$ be a $PP -$ space. Then a triple sequence $X = (X_{mnk})$ is said to statistically convergent to $\bar{0}$ with respect to the probabilistic $p -$ metric $P -$ provided that for every $\varepsilon > 0$ and $\gamma \in (0,1)$*

$$\delta\left(\left\{m, n, k \in \mathbb{N}: P - lim_{r,s,t\to\infty}\frac{1}{Q_r\overline{Q}_s\overline{\overline{Q}}_t}\sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{k=1}^{t} q_m\overline{q}_n\overline{\overline{q}}_k\left[f\left(A\sigma_{pqr}^{X}\right)(\varepsilon)\right] \le 1 - \gamma\right\}\right) = 0$$

or equivalently

$$lim_{k\ell v}\frac{1}{k\ell v} m \le k, n \le \ell, k \le v:$$

$$P - lim_{r,s,t\to\infty}\frac{1}{Q_r\overline{Q}_s\overline{\overline{Q}}_t}\sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{k=1}^{t} q_m\overline{q}_n\overline{\overline{q}}_k\left[f\left(A\sigma_{pqr}^{X}\right)(\varepsilon)\right] \le 1 - \gamma = 0$$

In this case we write $St_{PP} - lim_X = \bar{0}$.

Definition 3.2: *A triple $(X, P, *)$ be a $PP -$ space. The two non-negative sequences $X = (X_{mnk})$ and $Y = (Y_{mnk})$ are said to be almost asymptotically statistical equivalent of multiple $\bar{0}$ in $PP -$ space X if for every $\varepsilon > 0$ and $\gamma \in (0,1)$.*

$$\delta\left(\left\{m, n, k \in \mathbb{N}: P - lim_{r,s,t\to\infty}\frac{1}{Q_r\overline{Q}_s\overline{\overline{Q}}_t}\sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{k=1}^{t} q_m\overline{q}_n\overline{\overline{q}}_k\left[f\left(A\sigma_{pqr}^{X}\right)(\varepsilon), \bar{0}\right] \le 1 - \gamma\right\}\right) = 0$$

or equivalently

$$lim_{k\ell v}\frac{1}{k\ell}\left|\left\{m\le k, n\le \ell, k\le v: P_{\left(\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k}-\bar{0}}(\varepsilon)\le 1-\gamma\right\}\right| = 0.$$

In this case we write $X\overset{\hat{S}(PP)}{\equiv}Y$.

Definition 3.3: *A triple $(X, P, *)$ be a $PP-$ space and $\theta=(m_r n_s k_t)$ be a lacunary sequence. The two non-negative sequences $X=(X_{mnk})$ and $Y=(Y_{mnk})$ are said to be a almost asymptotically lacunary statistical equivalent of multiple $\bar{0}$ in $PP-$ space X if for every $\varepsilon>0$ and $\gamma\in(0,1)$*

$$\delta_\theta\left(\left\{m,n,k\in I_{r,s,t}: P_{\left(\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k}-\bar{0}}(\varepsilon)\le 1-\gamma\right\}\right) = 0 \quad (3.2)$$

or equivalently

$$lim_{rst}\frac{1}{h_{rst}}\left|\left\{m,n\in I_{rst}: P_{\left(\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k}-\bar{0}}(\varepsilon)\le 1-\gamma\right\}\right| = 0.$$

In this case we write $X\overset{\widehat{S_\theta}(PP)}{\equiv}Y$.

Lemma 3.1: *A triple $(X, P, *)$ be a $PP-$ space. Then for every $\varepsilon>0$ and $\gamma\in(0,1)$, the following statements are equivalent:*

$$(1)\ lim_{rst}\frac{1}{h_{rst}}\left|\left\{m,n,k\in I_{rst}: P_{\left(\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k}-\bar{0}}(\varepsilon)\le 1-\gamma\right\}\right| = 0,$$

$$(2)\ \delta_\theta\left(\left\{m,n,k\in I_{r,s,t}: P_{\left(\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k}-\bar{0}}(\varepsilon)\le 1-\gamma\right\}\right) = 0,$$

$$(3)\ \delta_\theta\left(\left\{m,n,k\in I_{r,s,t}: P_{\left(\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k}-\bar{0}}(\varepsilon)\le 1-\gamma\right\}\right) = 1,$$

$$(4)\ lim_{rst}\frac{1}{h_{rst}}\left|\left\{m,n,k\in I_{rst}: P_{\left(\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k}-\bar{0}}(\varepsilon)\le 1-\gamma\right\}\right| = 1.$$

4. MAIN RESULTS

Theorem 4.1: *Let f be a Musielak Orlicz function and a triple $(X, P, *)$ be a $PP-$ space. If two triple sequences $X = (X_{mnk})$ and $Y = (Y_{mnk})$ are almost asympototically lacunary statistical equivalent of multiple $\bar{0}$ with respect to the probabilistic $p-$ metric P, then $\bar{0}$ is unique sequence.*

Proof. Assume that $X \overset{\widehat{S_{\theta}^{\bar{0}}}(PP)}{\equiv} Y$. For a given $\lambda > 0$ choose $\gamma \in (0,1)$ such that $(1-\gamma) > 1-\lambda$. Then, for any $\varepsilon > 0$, define the following set:

$$K = \left\{ m, n, k \in I_{r,s,t} : P_{\left(\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k} - \bar{0}}(\varepsilon) \leq 1-\gamma \right\}$$

Then, clearly

$$lim_{rst} \frac{K \cap \bar{0}}{h_{rst}} = 1,$$

so K is non-empty set,since $x \overset{\widehat{S_{\theta}^{\bar{0}}}(PP)}{\equiv} y, \delta_{\theta}(K) = 0$ for all $\varepsilon > 0$, which implies $\delta_{\theta}(\mathbb{N} - K) = 1$. If $m, n, k \in \mathbb{N} - K$, then we have

$$P_{\bar{0}}(\varepsilon) = P_{\left(\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k} - \bar{0}}(\varepsilon) > (1-\gamma) \geq 1-\lambda$$

since λ is arbitrary, we get $P_{\bar{0}}(\varepsilon) = 1$.
This completes the proof.

Theorem 4.2: *Let f be a Musielak Orlicz function and a triple $(X, P, *)$ be a $PP-$ space. For any lacunary sequence $\theta = (m_r n_s k_t), \widehat{S_{\theta}}(PP) \subset \hat{S}(PP)$ if $limsup_{rst} q_{rst} < \infty$.*

Proof. If $limsup_{rst} q_{rst} < \infty$. then there exists a $B > 0$ such that $q_{rst} < B$ for all $r, s, t \geq 1$.

Let $X \overset{\widehat{S_{\theta}}(PP)}{\equiv} Y$ and $\varepsilon > 0$. Now we have to prove $\hat{S}(PP)$. Set

$$K_{rst} = \left| \left\{ m, n, k \in I_{r,s,t} : P_{\left(\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k} - \bar{0}}(\varepsilon) > 1-\gamma \right\} \right|.$$

Then by definition, for given $\varepsilon > 0$, there exists $r_0 s_0 t_0 \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that

$$\frac{K_{rst}}{h_{rst}} < \frac{\varepsilon}{2B} \text{forall} r > r_0, s > s_0 \text{and} t > t_0.$$

Let $M = max\{K_{rst}: 1 \leq r \leq r_0, 1 \leq s \leq s_0, 1 \leq t \leq t_0\}$ and let uvw be any positive integer with $m_{r-1} < u \leq m_r, n_{s-1} < v \leq n_s$ and $k_{t-1} < w \leq k_t$.

Then

$$\frac{1}{uvw}\left|\left\{m \leq u, n \leq v, k \leq w: P_{\left(\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k}-\bar{0}}(\varepsilon) > 1-\gamma\right\}\right| \leq$$

$$\frac{1}{m_{r-1}n_{s-1}k_{t-1}}\left|\left\{m \leq m_r, n \leq n_s, k \leq k_t: P_{\left(\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k}-\bar{0}}(\varepsilon) > 1-\gamma\right\}\right| =$$

$$\frac{1}{m_{r-1}n_{s-1}k_{t-1}}\{K_{111} + \cdots + K_{rst}\}$$

$$\leq \frac{M}{m_{r-1}n_{s-1}k_{t-1}} r_0 s_0 t_0 + \frac{\varepsilon}{2B} q_{rst} \leq \frac{M}{m_{r-1}n_{s-1}k_{t-1}} r_0 s_0 t_0 + \frac{\varepsilon}{2}.$$

This completes the proof.

Theorem 4.3: *Let f be a Musielak Orlicz function and a triple $(X, P, *)$ be a $PP-$ space. For any lacunary sequence $\theta = (m_r n_s k_t), \hat{S}(PP) \subset \widehat{S_\theta}(PP)$ if $liminf_{rst} q_{rst} > 1$.*

Proof. If $liminf_{rst} q_{rst} > 1$, then there exists a $\beta > 0$ such that $q_{rst} > 1 + \beta$ for sufficiently large rst, which implies

$$\frac{h_{rst}}{K_{rst}} \geq \frac{\beta}{1+\beta}.$$

Let $X \overset{\widehat{S^{\bar{0}}}(pp)}{\equiv} Y$, then for every $\varepsilon > 0$ and for sufficiently large r, s, t we have

$$\frac{1}{m_r n_s k_t}\left|\left\{m \leq m_r, n \leq n_s, k \leq k_t: P_{\left(\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k}-\bar{0}}(\varepsilon) > 1-\gamma\right\}\right| \geq$$

$$\frac{1}{m_r n_s k_t}\left|\left\{m, n, k \in I_{rst}: P_{\left(\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k}-\bar{0}}(\varepsilon) > 1-\gamma\right\}\right| \geq$$

$$\frac{\beta}{1+\beta}\frac{1}{h_{rst}}\left|\left\{m, n, k \in I_{rst}: P_{\left(\left|\frac{X_{mnk}}{Y_{mnk}}\right|\right)^{1/m+n+k}-\bar{0}}(\varepsilon) > 1-\gamma\right\}\right|.$$

Therefore $X \overset{\widehat{S^{\bar{0}}}\theta(pp)}{\equiv} Y$.

This completes the proof.

Corollary 4.1: *Let f be a Musielak Orlicz function and a triple $(X, P, *)$ be a $PP-$ space. For any lacunary sequence $\theta = (m_r n_s)$, with $1 < liminf_{rs} q_{rs} \leq limsup_{rst} q_{rst} < \infty$, then $\hat{S}(PP) = \widehat{S_\theta}(PP)$.*

Proof. The result clearly follows from Theorem 4.2 and Theorem 4.3.

5. CONCLUSION

In this paper we have studied the concept of almost lacunary statistical Cesáro of Γ^3 over probabilistic $p-$ metric spaces defined by Musielak Orlicz function. The results of this study in PP-spaces are more general than earlier results.

Competing Interests

The authors declare that there is not any conflict of interests regarding the publication of this manuscript.

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