Riesz triple probabilisitic of almost lacunary Cesáro $C_{111}$ statistical convergence of $\Gamma^3$ defined by a Musielak Orlicz function

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ABSTRACT

In this paper we study the concept of almost lacunary statistical Cesáro of $\Gamma^3$ over probabilistic $p$-metric spaces defined by Musielak Orlicz function. Since the study of convergence in PP-spaces is fundamental to probabilistic functional analysis, we feel that the concept of almost lacunary statistical Cesáro of $\Gamma^3$ over probabilistic $p$-metric spaces defined by Musielak in a PP-space would provide a more general framework for the subject.

Keywords: analytic sequence, Orlicz function, triple sequences, entire sequence, Riesz space, statistical convergence, Cesáro $C_{111}$ - statistical convergence

1. INTRODUCTION

Throughout $w$, $\Gamma$ and $\Lambda$ denote the classes of all, entire and analytic scalar valued single sequences, respectively. We write $w^3$ for the set of all complex triple sequences $(x_{mnk})$, where $m,n,k \in \mathbb{N}$, the set of positive integers. Then, $w^3$ is a linear space under the coordinate wise addition and scalar multiplication.
Some initial work on double series is found in Apostol [1] and double sequence spaces is found in Hardy [7], Deepmala et al. [6, 5] and many others. The initial work on triple sequence spaces is found in Sahiner et al. [13], Esi [2] and Esi et al. [3, 4,17,18], Shri Prakash et al. [14], Subramanian et al. [15,16] and many others.

Let \((x_{m,n,k})\) be a triple sequence of real or complex numbers. Then the series \(\sum_{m,n,k=1}^{\infty} x_{ijq}(m,n,k = 1,2,3,\ldots)\) is called a triple series. Then the triple series is said to be convergent if and only if the triple sequence \((S_{m,n,k})\) is convergent, where

\[
S_{m,n,k} = \sum_{i,j,q=1}^{m,n,k} x_{ijq}(m,n,k = 1,2,3,\ldots).
\]

A sequence \(x = (x_{m,n,k})\) is said to be triple analytic if

\[
\sup_{m,n,k}|x_{m,n,k}|^{\frac{1}{m+n+k}} < \infty.
\]

The vector space of all triple analytic sequence is usually denoted by \(\Lambda^3\). A sequence \(x = (x_{m,n,k})\) is called triple entire sequence if

\[
|x_{m,n,k}|^{\frac{1}{m+n+k}} \to 0 \text{ as } m,n,k \to \infty.
\]

A sequence \(x = (x_{m,n,k})\) is called triple entire sequence if \((|x_{m,n,k}|^{\frac{1}{m+n+k}} \to 0 \text{ as } m,n,k \to \infty\). The triple entire sequences will be denoted by \(\Gamma^3\).

Consider a triple sequence \(x = (x_{m,n,k})\). The \((m,n,k)\)th section \(x^{m,n,k}\) of the sequence is defined by \(x^{m,n,k} = \sum_{i,j,q=0}^{m,n,k} x_{ijq} S_{ijq}\) for all \(m,n,k \in \mathbb{N}\),

\[
S_{ijq} = \begin{bmatrix}
0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0 \\
0 & 0 & \ldots & 0 & 0 \\
\end{bmatrix}
\]

with 1 in the \((i,j,q)^{th}\) position and zero otherwise.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [10] as follows

\[Z(\Delta) = \{x = (x_k) \in w: (\Delta x_k) \in Z\}\]

for \(Z = c, c_0\) and \(\ell_\infty\), where \(\Delta x_k = x_k - x_{k+1}\) for all \(k \in \mathbb{N}\).

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by
\[ Z(\Delta) = \{ x = (x_{mn}) \in w^2: (\Delta x_{mn}) \in Z \} \]

where: \( Z = \Lambda^2, \chi^2 \) and \( \Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1} \) for all \( m, n \in \mathbb{N} \).

Consider the triple difference sequence space is defined as
\[
\Delta_{mnk} = x_{mnk} - x_{m,n+1,k} - x_{m,n,k+1} + x_{m,n+1,k+1} - x_{m+1,n,k} + x_{m+1,n+1,k} + x_{m+1,n+1,k+1} - x_{m+1,k+1} - x_{m+1,n+1,k+1}
\]

and
\[
\Delta^0 x_{mnk} = (x_{mnk}).
\]

2. DEFINITIONS AND PRELIMINARIES

**Definition 2.1:** An Orlicz function ([see [9]]) is a function \( M: [0, \infty) \to [0, \infty) \) which is continuous, non-decreasing and convex with \( M(0) = 0, \ M(x) > 0, \) for \( x > 0 \) and \( M(x) \to \infty \) as \( x \to \infty \). If convexity of Orlicz function \( M \) is replaced by \( M(x + y) \leq M(x) + M(y) \), then this function is called modulus function.

Lindenstrauss and Tzafriri ([11]) used the idea of Orlicz function to construct Orlicz sequence space.

A sequence \( g = (g_{mn}) \) defined by
\[
g_{mn}(v) = \sup \{|v|u - (f_{mnk})(u): u \geq 0\}, \ m, n, k = 1, 2, ...
\]
is called the complementary function of a Musielak-Orlicz function \( f \). For a given Musielak-Orlicz function \( f \), (see [12]) the Musielak-Orlicz sequence space \( t_f \) is defined as follows
\[
t_f = \{ x \in w^3: I_f(|x_{mnk}|^{1/m+n+k} \to 0 \ \text{as} \ \ m, n, k \to \infty \},
\]

where: \( I_f \) is a convex modular defined by
\[
I_f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mnk}(|x_{mnk}|^{1/m+n+k}, x = (x_{mnk}) \in t_f.
\]

We consider \( t_f \) equipped with the Luxemburg metric
\[
d(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mnk} \left( \frac{|x_{mnk}|^{1/m+n+k}}{mnk} \right)
\]
is an extended real number.
Definition 2.2: A triple sequence $x = (x_{mnk})$ of real numbers is called almost $P -$ convergent to a limit 0 if

$$P - \lim_{p,q,u \to \infty} \sup_{s,t \ge 0} \frac{1}{pqu} \sum_{m=r}^{r+p-1} \sum_{n=s}^{s+q-1} \sum_{k=t}^{t+u-1} |x_{mnk}|^{1/m+n+k} \to 0.$$ 

That is, the average value of $(x_{mnk})$ taken over any rectangle 

$$\{(m,n,k): r \le m \le r + p - 1, s \le n \le s + q - 1, t \le k \le t + u - 1\}$$

tends to 0 as both $p, q$ and $u$ to $\infty$, and this $P -$ convergence is uniform in $i, \ell$ and $j$. Let denote the set of sequences with this property as $[\chi^3]$.

Definition 2.3: Let $(Q_r), (\overline{Q}_s), (\overline{Q}_t)$ be sequences of positive numbers and

$$Q_r = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1s} & 0 \cdots \\ q_{21} & q_{22} & \cdots & q_{2s} & 0 \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ q_{r1} & q_{r2} & \cdots & q_{rs} & 0 \cdots \\ 0 & 0 & \cdots & 0 & 0 \cdots \end{bmatrix} = q_{11} + q_{12} + \cdots + q_{rs} \neq 0,$$

$$\overline{Q}_s = \begin{bmatrix} \overline{q}_{11} & \overline{q}_{12} & \cdots & \overline{q}_{1s} & 0 \cdots \\ \overline{q}_{21} & \overline{q}_{22} & \cdots & \overline{q}_{2s} & 0 \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \overline{q}_{r1} & \overline{q}_{r2} & \cdots & \overline{q}_{rs} & 0 \cdots \\ 0 & 0 & \cdots & 0 & 0 \cdots \end{bmatrix} = \overline{q}_{11} + \overline{q}_{12} + \cdots + \overline{q}_{rs} \neq 0,$$

$$\overline{Q}_t = \begin{bmatrix} \overline{q}_{11} & \overline{q}_{12} & \cdots & \overline{q}_{1s} & 0 \cdots \\ \overline{q}_{21} & \overline{q}_{22} & \cdots & \overline{q}_{2s} & 0 \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \overline{q}_{r1} & \overline{q}_{r2} & \cdots & \overline{q}_{rs} & 0 \cdots \\ 0 & 0 & \cdots & 0 & 0 \cdots \end{bmatrix} = \overline{q}_{11} + \overline{q}_{12} + \cdots + \overline{q}_{rs} \neq 0.$$
and is given by

\[
T_{rst} = \frac{1}{Q_r Q_s Q_t} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} q_m-q_n \overline{q_k} |x_{mnk}|^{1/m+n+k}
\]

is called the Riesz mean of triple sequence \( x = (x_{mnk}) \). If \( P - \lim_{rst} T_{rst}(x) = 0, 0 \in \mathbb{R} \), then the sequence \( x = (x_{mnk}) \) is said to be Riesz convergent to 0. If \( x = (x_{mnk}) \) is Riesz convergent to 0, then we write \( P_R - \lim x = 0 \).

**Definition 2.4:** The four dimensional matrix \( A \) is said to be RH-regular if it maps every bounded \( P - \) convergent sequence into a \( P - \) convergent sequence with the same \( P - \) limit.

**Definition 2.5:** The triple sequence \( \theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\} \) is called triple lacunary if there exist three increasing sequences of integers such that

\[
m_0 = 0, \ h_i = m_i - m_{r-1} \to \infty \text{ as } i \to \infty \text{ and } \n_0 = 0, \ n_\ell = n_\ell - n_{\ell-1} \to \infty \text{ as } \ell \to \infty. \]

\[
k_0 = 0, \ h_j = k_j - k_{j-1} \to \infty \text{ as } j \to \infty. \]

Let \( m_{i,\ell,j} = m_i n_\ell k_j, h_{i,\ell,j} = h_i h_\ell h_j, \) and \( \theta_{i,\ell,j} \) is determine by

\[
l_{i,\ell,j} = \{(m,n,k) : m_{i-1} < m < m_i \text{ and } n_{\ell-1} < n \leq n_\ell \text{ and } k_{j-1} < k \leq k_j\},
\]

\[
q_k = \frac{m_k}{m_{k-1}}, \overline{q_\ell} = \frac{n_\ell}{n_{\ell-1}}, \overline{q_j} = \frac{k_j}{k_{j-1}}.
\]

Using the notations of lacunary Fuzzy sequence and Riesz mean for triple sequences. \( \theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\} \) be a triple lacunary sequence and \( q_m \overline{q_n} \overline{q_k} \) be sequences of positive real numbers such that

\[
Q_{m_i} = \sum_{m \in (0, m_i]} p_{m_i}, Q_{n_\ell} = \sum_{n \in (0, n_\ell]} p_{n_\ell}, Q_{n_j} = \sum_{k \in (0, k_j]} p_{k_j}
\]

and

\[
H_i = \sum_{m \in (0, m_i]} p_{m_i} \overline{H} = \sum_{n \in (0, n_\ell]} p_{n_\ell} \overline{H} = \sum_{k \in (0, k_j]} p_{k_j}.
\]
Clearly, \( H_i = Q_{m_i} - Q_{m_{i-1}}, \) \( \overline{H}_\ell = Q_{n_\ell} - Q_{n_{\ell-1}}, \) \( \overline{H}_j = Q_{k_j} - Q_{k_{j-1}}. \) If the Riesz transformation of triple sequences is RH-regular, and
\[
H_i = Q_{m_i} - Q_{m_{i-1}} \to \infty \text{ as } i \to \infty,
\]
\[
\overline{H} = \sum_{n \in (0,n_\ell]} p_{n_\ell} \to \infty \text{ as } \ell \to \infty,
\]
\[
\overline{H} = \sum_{k \in (0,k_j]} p_{k_j} \to \infty \text{ as } j \to \infty,
\]
then \( \theta_{i,\ell,j}' = \{(m_i,n_\ell,k_j)\} = \{(Q_{m_i}Q_{n_\ell}Q_{k_j})\} \) is a triple lacunary sequence. If the assumptions \( Q_r \to \infty \) as \( r \to \infty, \) \( \overline{Q}_s \to \infty \) as \( s \to \infty \) and \( \overline{Q}_t \to \infty \) as \( t \to \infty \) may be not enough to obtain the conditions \( H_i \to \infty \) as \( i \to \infty, \) \( \overline{H}_\ell \to \infty \) as \( \ell \to \infty \) and \( \overline{H}_j \to \infty \) as \( j \to \infty \) respectively. For any lacunary sequences \( (m_i), (n_\ell) \) and \( (k_j) \) are integers.

Throughout the paper, we assume that
\[
Q_r = q_{11} + q_{12} + \cdots + q_{rs} \to \infty (r \to \infty),
\]
\[
\overline{Q}_s = \overline{q}_{11} + \overline{q}_{12} + \cdots + \overline{q}_{rs} \to \infty (s \to \infty),
\]
\[
\overline{Q}_t = \overline{q}_{11} + \overline{q}_{12} + \cdots + \overline{q}_{rs} \to \infty (t \to \infty),
\]
such that
\[
H_i = Q_{m_i} - Q_{m_{i-1}} \to \infty \text{ as } i \to \infty,
\]
\[
\overline{H}_\ell = Q_{n_\ell} - Q_{n_{\ell-1}} \to \infty \text{ as } \ell \to \infty
\]
and
\[
\overline{H}_j = Q_{k_j} - Q_{k_{j-1}} \to \infty \text{ as } j \to \infty.
\]

Let
\[
Q_{m_i,n_\ell,k_j} = Q_{m_i} \overline{Q}_{n_\ell} \overline{Q}_{k_j}, H_{i\ell j} = H_i \overline{H}_\ell \overline{H}_j,
\]
\[
i'_{i\ell j} = \{(m,n,k): Q_{m_{i-1}} < m < Q_{m_i}, \overline{Q}_{n_{\ell-1}} < n < Q, \overline{Q}_{k_{j-1}} < k < \overline{Q}_{k_j}\},
\]

\( -120 - \)
If we take \( q_m = 1, q_n = 1 \) and \( q_k = 1 \) for all \( m, n \) and \( k \) then \( h_{i \ell j}, q_{i \ell j}, v_{i \ell j} \) and \( l_{i \ell j} \) reduce to \( h_{i \ell j}, q_{i \ell j}, v_{i \ell j} \) and \( l_{i \ell j} \).

Let \( n \in \mathbb{N} \) and \( X \) be a real vector space of dimension \( m \), where \( n \leq m \). A real valued function \( d_p(x_1, \ldots, x_n) = \| (d_1(x_1), \ldots, d_n(x_n)) \|_p \) on \( X \) satisfying the following four conditions:

(i) \( \| (d_1(x_1), \ldots, d_n(x_n)) \|_p = 0 \) if and only if \( d_1(x_1), \ldots, d_n(x_n) \) are linearly dependent,

(ii) \( \| (d_1(x_1), \ldots, d_n(x_n)) \|_p \) is invariant under permutation,

(iii) \( \| (\alpha d_1(x_1), \ldots, \alpha d_n(x_n)) \|_p = |\alpha| \| (d_1(x_1), \ldots, d_n(x_n)) \|_p, \alpha \in \mathbb{R} \)

(iv) \( d_p((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)) = (d_X(x_1, x_2, \ldots, x_n)^p + d_Y(y_1, y_2, \ldots, y_n)^p)^{1/p} \) for \( 1 \leq p < \infty \); is called the \( p \) product metric.

3. ALMOST LACUNARY CESÁRO \( C_{II1} \) – STATISTICAL CONVERGENCE OF PP-TRIPLE SEQUENCE SPACES

Let \( A = [a_{mnp}]_{m,n,k=0}^{\infty} \) be a triple infinite matrix of real number for \( p, q, r = 1, 2, \ldots \) forming the sum

\[
\mu_{pqr}(X) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{mnp}^q \left( \frac{x_{mnp}}{y_{mnp}} \right)^{1/m+n+k}, 0 \right) \quad (3.1)
\]

is called a triple sequence space of summable to the limit 0, i.e.,

\[
\lim_{uvw \to \infty} \sum_{m=0}^{u} \sum_{n=0}^{v} \sum_{k=0}^{w} a_{mnp}^q \left( \frac{x_{mnp}}{y_{mnp}} \right)^{1/m+n+k} = \mu_{pqr}
\]

and

\[
\lim_{pqr \to \infty} \mu_{pqr} = 0
\]

Define the means

\[
sigma_{pqr}^X = \frac{1}{pqr} \sum_{m=0}^{p} \sum_{n=0}^{q} \sum_{k=0}^{r} \left( \frac{x_{mnp}}{y_{mnp}} \right)^{1/m+n+k}
\]
and
\[ A\sigma_{pqr}^X = \frac{1}{pqr} \sum_{m=0}^{p} \sum_{n=0}^{q} \sum_{k=0}^{r} q_{mnk}^{pqr} \left( \left( \frac{X_{mnk}}{Y_{mnk}} \right)^{1/m+n+k}, 0 \right). \]

We say that \( \frac{X_{mnk}}{Y_{mnk}} \) is statistically lacunary equivalent sumable \( (C, 1,1,1) \) to 0, if the sequence \( \sigma = (\sigma_{mnk}^X) \) is statistically convergent to \( \overline{0} \), that is, \( st_3 - \lim_{pqr} \sigma_{pqr}^X = 0 \). It is denoted by \( C_{111}(st_3) \).

Let \( q_m, q_n, \) and \( q_k \) be sequences of positive numbers and \( Q_r = q_{11} + \cdots + q_{rs}, \) \( Q_s = \overline{q}_{11} + \cdots + \overline{q}_{rs}, \) and \( Q_t = \overline{q}_{11} + \cdots + \overline{q}_{rs}. \)

**Definition 3.1:** A triple \((X, P,*)\) be a PP − space. Then a triple sequence \( X = (X_{mnk}) \) is said to statistically convergent to \( \overline{0} \) with respect to the probabilistic \( p \) − metric \( P \) − provided that for every \( \varepsilon > 0 \) and \( \gamma \in (0,1) \)

\[
\delta \left( \left\{ m, n, k \in \mathbb{N}: \frac{1}{Q_r Q_s Q_t} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} q_m q_n q_k \left[ f \left( A\sigma_{pqr}^X \right)(\varepsilon) \right] \leq 1 - \gamma \right\} \right) = 0
\]

or equivalently

\[
\lim_{k \ell v} \frac{1}{k \ell v} m \leq k, n \leq \ell, k \leq v:
\]

\[
P - \lim_{r,s,t \to \infty} \frac{1}{Q_r Q_s Q_t} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} q_m q_n q_k \left[ f \left( A\sigma_{pqr}^X \right)(\varepsilon) \right] \leq 1 - \gamma = 0
\]

In this case we write \( St_{PP} - \lim_{X} = \overline{0} \).

**Definition 3.2:** A triple \((X, P,*)\) be a PP − space. The two non-negative sequences \( X = (X_{mnk}) \) and \( Y = (Y_{mnk}) \) are said to be almost asymptotically statistical equivalent of multiple \( 0 \) in PP − space \( X \) if for every \( \varepsilon > 0 \) and \( \gamma \in (0,1) \)

\[
\delta \left( \left\{ m, n, k \in \mathbb{N}: \frac{1}{Q_r Q_s Q_t} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} q_m q_n q_k \left[ f \left( A\sigma_{pqr}^X \right)(\varepsilon), 0 \right] \leq 1 - \gamma \right\} \right) = 0
\]
or equivalently
\[
\lim_{k \to \ell} \frac{1}{k \ell} \left\{ m \leq k, n \leq \ell, k \leq v : P \left( \frac{|X_{mnk}|^{1/m + n + k}}{|Y_{mnk}|} - 0 \right) (\varepsilon) \leq 1 - \gamma \right\} = 0.
\]

In this case we write \( \overset{\delta}{X} \equiv Y \).

**Definition 3.3**: A triple \((X, P, *)\) be a \( PP \) − space and \( \theta = (m, n, k) \) be a lacunary sequence. The two non-negative sequences \( X = (X_{mnk}) \) and \( Y = (Y_{mnk}) \) are said to be a almost asymptotically lacunary statistical equivalent of multiple \( \tilde{0} \) in \( PP \) − space \( X \) if for every \( \varepsilon > 0 \) and \( \gamma \in (0,1) \)
\[
\delta_{\theta} \left( \left\{ m, n, k \in I_{r,s,t} : P \left( \frac{|X_{mnk}|^{1/m + n + k}}{|Y_{mnk}|} - 0 \right) (\varepsilon) \leq 1 - \gamma \right\} = 0 \right) (3.2)
\]
or equivalently
\[
\lim_{r \to s} \frac{1}{h_{rst}} \left\{ m, n \in I_{r,s,t} : P \left( \frac{|X_{mnk}|^{1/m + n + k}}{|Y_{mnk}|} - 0 \right) (\varepsilon) \leq 1 - \gamma \right\} = 0.
\]

In this case we write \( \overset{\tilde{s}_{\theta}(PP)}{X} \equiv Y \).

**Lemma 3.1**: A triple \((X, P, *)\) be a \( PP \) − space. Then for every \( \varepsilon > 0 \) and \( \gamma \in (0,1) \), the following statements are equivalent:

1. \( \lim_{r \to s} \frac{1}{h_{rst}} \left\{ m, n, k \in I_{r,s,t} : P \left( \frac{|X_{mnk}|^{1/m + n + k}}{|Y_{mnk}|} - 0 \right) (\varepsilon) \leq 1 - \gamma \right\} = 0, \)

2. \( \delta_{\theta} \left( \left\{ m, n, k \in I_{r,s,t} : P \left( \frac{|X_{mnk}|^{1/m + n + k}}{|Y_{mnk}|} - 0 \right) (\varepsilon) \leq 1 - \gamma \right\} = 0, \)

3. \( \delta_{\theta} \left( \left\{ m, n, k \in I_{r,s,t} : P \left( \frac{|X_{mnk}|^{1/m + n + k}}{|Y_{mnk}|} - 0 \right) (\varepsilon) \leq 1 - \gamma \right\} = 1, \)

4. \( \lim_{r \to s} \frac{1}{h_{rst}} \left\{ m, n, k \in I_{r,s,t} : P \left( \frac{|X_{mnk}|^{1/m + n + k}}{|Y_{mnk}|} - 0 \right) (\varepsilon) \leq 1 - \gamma \right\} = 1. \)
4. MAIN RESULTS

**Theorem 4.1:** Let $f$ be a Musielak Orlicz function and a triple $(X, P, \ast)$ be a $PP$-space. If two triple sequences $X = (X_{mnk})$ and $Y = (Y_{mnk})$ are almost asymptotically lacunary statistical equivalent of multiple $\tilde{0}$ with respect to the probabilistic $p$-metric $P$, then $\tilde{0}$ is unique sequence.

**Proof.** Assume that $X \equiv Y$. For a given $\lambda > 0$ choose $\gamma \in (0,1)$ such that $(1 - \gamma) > 1 - \lambda$. Then, for any $\varepsilon > 0$, define the following set:

$$K = \left\{ m, n, k \in I_{r,s,t} : P \left( \frac{|X_{mnk}|}{|Y_{mnk}|} \right)^{1/m+n+k} (\varepsilon) \leq 1 - \gamma \right\}$$

Then, clearly

$$\lim_{rst} \frac{K \cap \tilde{0}}{h_{rst}} = 1,$$

so $K$ is non-empty set, since $X \equiv Y$, $\delta_\theta(K) = 0$ for all $\varepsilon > 0$, which implies $\delta_\theta(\mathbb{N} - K) = 1$. If $m, n, k \in \mathbb{N} - K$, then we have

$$P_\tilde{0}(\varepsilon) = P \left( \frac{|X_{mnk}|}{|Y_{mnk}|} \right)^{1/m+n+k} (\varepsilon) > (1 - \gamma) \geq 1 - \lambda$$

since $\lambda$ is arbitrary, we get $P_0(\varepsilon) = 1$.

This completes the proof.

**Theorem 4.2:** Let $f$ be a Musielak Orlicz function and a triple $(X, P, \ast)$ be a $PP$-space. For any lacunary sequence $\theta = (m, n, k), \tilde{S}_\theta(PP) \subset \hat{S}(PP)$ if $\limsup_{rst} q_{rst} < \infty$.

**Proof.** If $\limsup_{rst} q_{rst} < \infty$, then there exists a $B > 0$ such that $q_{rst} < B$ for all $r, s, t \geq 1$.

Let $X \equiv Y$ and $\varepsilon > 0$. Now we have to prove $\hat{S}(PP)$. Set

$$K_{rst} = \left\{ m, n, k \in I_{r,s,t} : P \left( \frac{|X_{mnk}|}{|Y_{mnk}|} \right)^{1/m+n+k} (\varepsilon) > 1 - \gamma \right\}.$$  

Then by definition, for given $\varepsilon > 0$, there exists $r_0 s_0 t_0 \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that

$$\frac{K_{rst}}{h_{rst}} < \frac{\varepsilon}{2B} \text{ for all } r > r_0, s > s_0 \text{ and } t > t_0.$$
Let \( M = \max\{K_{rst} : 1 \leq r \leq r_0, 1 \leq s \leq s_0, 1 \leq t \leq t_0\} \) and let \( uvw \) be any positive integer with \( m_{r-1} < u \leq m_r, n_{s-1} < v \leq n_s \) and \( k_{t-1} < w \leq k_t \).

Then

\[
\begin{align*}
\frac{1}{uvw} \left\{ m \leq u, n \leq v, k \leq w : P\left( \frac{|X_{mnk}|}{|Y_{mnk}|}^{1/m+n+k-\bar{0}} (\varepsilon) > 1 - \gamma \right) \right\} \leq \\
\frac{1}{m_r-1 n_{s-1} k_{t-1}} \left\{ m \leq m_r, n \leq n_s, k \leq k_t : P\left( \frac{|X_{mnk}|}{|Y_{mnk}|}^{1/m+n+k-\bar{0}} (\varepsilon) > 1 - \gamma \right) \right\} = \\
\frac{1}{m_r-1 n_{s-1} k_{t-1}} \{K_{111} + \cdots + K_{rst}\} \\
\leq \frac{M}{m_r-1 n_{s-1} k_{t-1}} r_0 s_0 t_0 + \frac{\varepsilon}{2B q_{rst}} \leq \frac{M}{m_r-1 n_{s-1} k_{t-1}} r_0 s_0 t_0 + \frac{\varepsilon}{2}.
\end{align*}
\]

This completes the proof.

**Theorem 4.3:** Let \( f \) be a Musielak Orlicz function and a triple \((X, P, \cdot)\) be a \( PP - \) space. For any lacunary sequence \( \theta = (m_r n_s k_t) \), \( \tilde{S}(PP) \subset \tilde{S}_\theta(PP) \) if \( \liminf_{rst} q_{rst} > 1 \).

**Proof.** If \( \liminf_{rst} q_{rst} > 1 \), then there exists a \( \beta > 0 \) such that \( q_{rst} > 1 + \beta \) for sufficiently large \( rst \), which implies

\[
\frac{h_{rst}}{K_{rst}} \geq \frac{\beta}{1 + \beta}.
\]

Let \( X \equiv Y \), then for every \( \varepsilon > 0 \) and for sufficiently large \( r, s, t \) we have

\[
\begin{align*}
\frac{1}{m_r-1 n_{s-1} k_{t-1}} \left\{ m \leq m_r, n \leq n_s, k \leq k_t : P\left( \frac{|X_{mnk}|}{|Y_{mnk}|}^{1/m+n+k-\bar{0}} (\varepsilon) > 1 - \gamma \right) \right\} \geq \\
\frac{1}{m_r-1 n_{s-1} k_{t-1}} \left\{ m, n, k \in I_{rst} : P\left( \frac{|X_{mnk}|}{|Y_{mnk}|}^{1/m+n+k-\bar{0}} (\varepsilon) > 1 - \gamma \right) \right\} \geq \\
\frac{\beta}{1 + \beta h_{rst}} \frac{1}{m_r-1 n_{s-1} k_{t-1}} \left\{ m, n, k \in I_{rst} : P\left( \frac{|X_{mnk}|}{|Y_{mnk}|}^{1/m+n+k-\bar{0}} (\varepsilon) > 1 - \gamma \right) \right\}.
\end{align*}
\]

Therefore \( X \equiv Y \).

This completes the proof.
Corollary 4.1: Let $f$ be a Musielak Orlicz function and a triple $(X, P, \ast)$ be a $PP$ space. For any lacunary sequence $\theta = (m_r n_s)$, with $1 < \liminf_{r,s} q_{rs} \leq \limsup_{r,s,t} q_{rst} < \infty$, then $\hat{S}(PP) = \hat{S}_\theta(PP)$.

Proof. The result clearly follows from Theorem 4.2 and Theorem 4.3.

5. CONCLUSION

In this paper we have studied the concept of almost lacunary statistical Cesáro of $\Gamma^3$ over probabilistic $p$-metric spaces defined by Musielak Orlicz function. The results of this study in PP-spaces are more general than earlier results.

Competing Interests

The authors declare that there is not any conflict of interests regarding the publication of this manuscript.

References


