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Analytical Solution to Linear Conformable Fractional Partial Differential Equations

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ABSTRACT

An analytical solution is better than an approximate or series solution of a problem. Here we develop an analytical formulation to solve linear fractional order partial differential equations with given boundary conditions. We discuss the method for the simultaneous fractional derivative, in space as well as time and up to order two. Examples reflect the effectiveness and simplicity of the method. First we convert the fractional derivative into integer order derivative and then use method of separation of variables in usual sense to get the complete solution. The fractional derivative has been taken in the sense of Katugampola's derivative.

Keywords: Fractional partial differential equation, Conformable fractional derivative, Katugampola's derivative, Method of separation of variables

1. INTRODUCTION

Fractional derivative, which is the generalization of ordinary derivative to an arbitrary order, are used in modelling in various field in science and engineering. It has given an idea to the researchers to develop new models in the field of mathematics and its applications which are realistic and successful in our real life. It covers Able's integral equation, capacitance

theory, electro-electrolyte interface models, visco-elasticity analysis of feedback amplifiers, fractional multipoles electric conductance of biological systems, fractional-order models of neurons, etc. In general, fractional order differential equations do not have any known method to get exact solution but there are approximate and numerical solutions like Variation iteration method, Adomian's decomposition method, Taylor's series etc.

Fractional derivative [1-4, 15] was discovered in a discussion between L. Hospital and Leibnitz through a letter. Many mathematicians like Hadmard, Erdelyi-Kobe, Fourier, Euler, Mittag- Leffler, Laplace, Riemann, Grunewald etc. tried to develop the definition of fractional derivative in discrete form/continuous form using backward difference operator, integral, exponential function, Gamma function etc. There are about 25 definitions of fractional derivative. But none of them satisfies fundamental properties of derivative and also, to obtain the solution of fractional differential equations using these definitions is a difficult task. Recently concept of conformable derivative [5] has taken place and researchers are taking keen interest to develop the theory of fractional derivative as they feel to be comfortable with this definition the reason is, the definition posses some properties [10, 12] of a derivative. A number of applications of conformable derivative have been presented in [6-8] and properties has analysed. Further the generalised form of Conformable fractional derivative has presented by U. N. Katugampola [9] and corresponding properties [9, 14] have discussed.

The work on fractional order partial differential equation has been already done as series solution or Mittag leffler function but not analytically. Here we present the solution of fractional order partial differential equation in the form of boundary value problem in which fractional derivative is involved in both time as well as space and we use Katugampola's derivative to form a usual boundary value problem which is solved by method of separation of variables.

2. PRELIMINARIES

R. Khalil and colleagues introduce a definition of fractional derivative so called "Conformable fractional derivative" by doing some appropriate modification in the classical definition of an ordinary derivative. This definition satisfies some fundamental properties of an ordinary derivative.

Definition 2.1. Let $f : [0, \infty) \rightarrow R$, and $x > 0$ then conformable fractional derivative of order α is given by

$$D^\beta f(x) = \lim_{\epsilon \rightarrow 0} \frac{f(x + \epsilon x^{1-\beta}) - f(x)}{\epsilon} \quad \forall x > 0, \beta \in (0,1]$$

If f is β differential in some $(0, a)$, and $\lim_{t \rightarrow 0^+} f^\beta(x)$ exists, then $\lim_{t \rightarrow 0^+} f^\beta(x) = f^\beta(0)$

The definition of Conformable derivative is further improved by U. N Katugampola. He used exponential function in the definition of R. Khalil which generate the same results and properties as that of Conformable derivative which generate the same results and properties as that of Conformable derivative

Definition 2.2. Let $f : [0, \infty) \rightarrow R$, and $x > 0$ then conformable fractional derivative of order α is given by

$$D^\beta f(x) = \lim_{\epsilon \rightarrow 0} \frac{f(xe^{\epsilon x^{-\beta}}) - f(x)}{\epsilon} \quad \forall x > 0, \beta \in (0,1]$$

If f is β differential in some $(0, a)$, and $\lim_{t \rightarrow 0^+} f^\beta(x)$ exists, then $\lim_{t \rightarrow 0^+} f^\beta(x) = f^\beta(0)$

Definition 2.3. Let $0 \leq a \leq x$, and f be a function defined on $(a, x]$, then β – fractional integral is defined by $I_a^\beta(x) = \int_a^x \frac{f(t)}{t^{1-\beta}} dt$ provided integral exists.

It is interesting to note that for $\beta = 1$ the definition coincides with the classical definition of first order derivative

Theorem 2.1. Let $\beta \in (0,1]$ and f, g be β differentiable at point $x > 0$. The

- (i) $D^\beta(cf + dg) = cD^\beta(f) + dD^\beta(g)$
- (ii) $D^\beta(t^\alpha) = \alpha t^{\alpha-\beta}, \alpha \in R$
- (iii) $D^\beta(\eta) = 0$ For all constant function η
- (iv) $D^\beta(fg) = fD^\beta(g) + gD^\beta(f)$.
- (v) $D^\beta\left(\frac{f}{g}\right) = \frac{gD^\beta(f) - fD^\beta(g)}{g^2}, g(x) \neq 0$.
- (vi) If f is differentiable, then $D^\beta(f)(x) = x^{1-\beta} \frac{df}{dx}$.

Theorem 2.2. Let $\beta \in (0,1]$ and $x > 0$. Then we have following results

- (a) $D^\beta(x^\sigma) = \sigma x^{\sigma-1}, \forall \sigma \in R$.
- (b) $D^\beta(c) = 0$ where c is constant
- (c) $D^\beta(e^{ax}) = ax^{1-\beta} e^{ax}, a \in R$.
- (d) $D^\beta(\sin bx) = bx^{1-\beta} \cos bx, b \in R$.
- (e) $D^\beta(\cos bx) = -bx^{1-\beta} \sin bx, b \in R$.
- (f) $D^\beta(x^\beta) = \beta$

It should be noted that (i) at $\beta = 1$ we get the corresponding ordinary derivative, but β -differentiable function is not necessarily differentiable (ii) Conformable derivative possess

invariant parameter property that is $D^\beta\left(f\left(\frac{x^\beta}{\beta}\right)\right) = D(f(x))_{x=\frac{x^\beta}{\beta}}$

In the next section we solve partial differential equation of fractional order according to definition of conformable fractional derivative as an application.

3. SOLUTION OF LINEAR CONFORMABLE PARTIAL DIFFERENTIAL EQUATIONS

Here we have used method of separation of variables to solve the partial differential equations of fractional order. First we separate the variable and make differential equations choosing some suitable constant and then solve each of them using the definition $\frac{d^\beta}{dx^\beta}(f)(x) = x^{1-\beta} \frac{df}{dx}$, and the respective boundary condition as illustrate from the following examples:

Example 3.1. Solve $\frac{\partial^\alpha u}{\partial x^\alpha} = 3 \frac{\partial^\alpha u}{\partial y^\alpha}$; $u(0, y) = 8e^{-3y}, 0 < \alpha \leq 1$.

Solution: Take $u = XY$ where X, Y are functions of x, y respectively

Then equation becomes $X^\alpha Y = 3XY^\alpha$, $X^\alpha \equiv \frac{d^\alpha}{dx^\alpha}$, $Y^\alpha \equiv \frac{d^\alpha}{dy^\alpha}$

Choosing constant k such that $\frac{X^\alpha}{3X} = \frac{Y^\alpha}{y} = k$

we get differential equations

$$X^\alpha - 3kX = 0 \text{ and } Y^\alpha - kY = 0$$

In first equation $\frac{d^\alpha X}{dx^\alpha} = 3kX$ using the definition of conformable fractional derivative we have

$$x^{1-\alpha} \frac{dX}{dx} = 3kX \text{ which gives } X = Ae^{\frac{3kx^\alpha}{\alpha}} \text{ on solving,}$$

Similarly from 2nd equation we obtain $Y = Be^{\frac{ky^\alpha}{\alpha}}$ and therefore $u = XY = \lambda e^{\frac{k}{\alpha}(3x^\alpha + y^\alpha)}$; $\lambda = AB$

Applying boundary condition $u(0, y) = 8e^{-3y} = \lambda e^{\frac{k}{\alpha}y^\alpha}$, we get $\lambda = 8, k = -3\alpha$

Hence $u = 8e^{-3(3x^\alpha + y^\alpha)}$ is the desired solution.

Example 3.2. Solve the boundary problem $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^{1/2} u}{\partial t^{1/2}}$, given that $z = 0$ when $x = 0$ and $x = \pi$ and $u(x, 0) = \eta \sin 3x$.

Solution: Using method of separation of variables, consider $u = X(x)T(t)$ so that equation is $X''T = XT^{1/2}$ where $X'' \equiv \frac{d^2}{dx^2}, T^{1/2} \equiv \frac{d^{1/2}}{dt^{1/2}}$ choosing constant k such that $\frac{X''}{X} = \frac{T^{1/2}}{T} = k$ we get differential equations $X'' - kT = 0$ and $T^{1/2} - kT = 0$ respectively. One may easily find that $X = Ae^{\sqrt{k}x} + Be^{-\sqrt{k}x}$ and $T = Ce^{2ky^{1/2}}$ and therefore $u(x,t) = (\psi_1 e^{\sqrt{k}x} + \psi_2 e^{-\sqrt{k}x})e^{2ky^{1/2}}$ where $\psi_1 = AC, \psi_2 = BC$

For different values of k we get different solutions as:

- (i) For $k = 0, u = (\psi_1 + \psi_2) = \text{constant}$
- (ii) For $k = \mu^2, u = (\psi_1 e^{\mu x} + \psi_2 e^{-\mu x})e^{2\mu^2 y^{1/2}}$
- (iii) For $k = -\mu^2, u = (\psi_1 \sin \mu x + \psi_2 \cos \mu x)e^{-2\mu^2 y^{1/2}}$

But from the given conditions, (iii) is considered the most consistent solution

And so $u(0,t) = 0 \Rightarrow \psi_2 = 0$ and $u(\pi,t) = 0 \Rightarrow \mu = n$, therefore $u(x,t) = \psi_1 \sin nx e^{-2n^2 y^{1/2}}$

Now $u(x,0) = \eta \sin 3x = \psi_1 \sin nx \Rightarrow \psi_1 = \eta, n = 3$

Hence $u = \eta \sin 3x e^{-18y^{1/2}}$

Example 3.3. Solve $\frac{\partial^\alpha u}{\partial x^\alpha} - 2u = 3 \frac{\partial^\beta u}{\partial y^\beta}, 0 < \alpha, \beta \leq 1 \quad u(x,0) = \theta_0 e^{\frac{4x^\alpha}{\alpha}}$

Solution: Consider $u = X(x)Y(y)$ so that equation becomes $X^\alpha Y - 2XY = 3XY^\beta$ or

$$\frac{X^\alpha}{X} - 2 = 3 \frac{Y^\beta}{Y} = \omega, \text{ where } \omega \text{ is a constant}$$

we get differential equations $X^\alpha = (\omega + 2)X$ and $3Y^\beta = \omega Y$

from first equation $x^{1-\alpha} \frac{dX}{dx} = (\omega + 2)X$, solving we get $X = Ae^{(\omega+2)\frac{x^\alpha}{\alpha}}$ and

from second equation $3y^{1-\beta} \frac{dY}{dy} = \omega Y$, solving

we get $Y = Be^{\frac{\omega y^\beta}{3\beta}}$

Therefore, $u = \xi e^{\left[(\omega+2)\frac{x^\alpha}{\alpha} + \frac{\omega y^\beta}{3\beta} \right]}$ ($\xi = AB$) and $u(x,0) = \theta_0 e^{\frac{4x^\alpha}{\alpha}} \Rightarrow \xi = \theta_0$ and $\omega = 2$

Hence $z = \theta_0 e^{\left[4\frac{x^\alpha}{\alpha} + \frac{2y^\beta}{3\beta} \right]}$

Example 3.4. Solve $\frac{\partial^{1/2} f}{\partial t^{1/2}} = \frac{\partial^{1/3} f}{\partial x^{1/3}}$ ($0 < \alpha, \beta \leq 1$) with the following boundary conditions

$$f(0,t) = f(l,t) = 0 = f(x,0) \text{ and } f(x,1) = g(x).$$

Solution: Consider the separable solution $f = X(x)T(t)$, substituting in the above equation we get

$$XT^{1/2} = X^{1/3}T \text{ and for constant } \kappa \text{ we may write } \frac{T^{1/2}}{T} = \frac{X^{1/3}}{X} = \kappa \text{ to get the different equations}$$

$$T^{1/2} - \kappa T = 0 \text{ and } X^{1/3} - \kappa X = 0$$

Now we consider the following cases:

- (i) If $\kappa = 0$, then $T^{1/2} = X^{1/3} = 0 \Rightarrow X, T$ both must be constant and therefore the same f is.
- (ii) If $\kappa = \lambda^2$, then $X^{1/3} - \lambda^2 X = 0$ gives $X = Ae^{3\lambda x^{1/3}} + Be^{-3\lambda x^{1/3}}$ and using first two boundary conditions we get $X = 0$ which is not admissible.
- (iii) If $\kappa = -\lambda^2$, we get periodic solution
- (iv) $f = XT = (C_1 \sin 3\lambda x^{1/3} + C_2 \cos 3\lambda x^{1/3}) (C_3 \sin 2\lambda t^{1/2} + C_4 \cos 2\lambda t^{1/2})$ applying boundary conditions,

$$f(0,t) = 0 \Rightarrow C_2 = 0$$

$$\text{and } f(l,t) = 0 \Rightarrow \sin 3\lambda l^{1/3} = 0$$

$$\text{that is } \lambda = \frac{n\pi}{3l^{1/3}}.$$

$f(x,0) = 0 \Rightarrow C_4 = 0$ Therefore general solution is

$$f(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x^{1/3}}{l^{1/3}} \sin \frac{2n\pi t^{1/2}}{3l^{1/3}}.$$

$$\text{And } f(x,1) = g(x) \Rightarrow \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x^{1/3}}{l^{1/3}} = g(x),$$

b_n is determined with the help of fractional Fourier series and hence we will get the final solution.

Example 3.5. Solve the time fractional partial differential equation $D_t^\alpha u = x^2 u_{xx} + xu_x$ with the following conditions

$$u(x,t) = x, \quad u(0,t) = 0, \quad u(1,t) = e^t$$

and u is bounded.

Solution: With the help of method of separation of variables we may write above equation as $XT^\alpha = x^2 X''T + xX'T$

$$\text{or } t^{1-\alpha} XT' = x^2 X''T + xX'T$$

for constant k , we get

$$t^{1-\alpha} \frac{T'}{T} = x^2 \frac{X''}{X} + x \frac{X'}{X} = k,$$

on separating the variables

$$\text{Now for } t^{1-\alpha} \frac{T'}{T} = k, \quad T = C_1 e^{k \frac{t^\alpha}{\alpha}}$$
 is the obvious solution

$$\text{And for } x^2 \frac{X''}{X} + x \frac{X'}{X} = k, \text{ we get Cauchy-Euler equation } x^2 X'' + xX' - kX = 0$$

Substituting $x = e^z$ it transformed

$$\text{into } (D_1(D_1 - 1) + D_1 - k) = 0$$

$$\Rightarrow (D_1^2 - k)X = 0 \quad \text{where } D_1 \equiv \frac{d}{dz}$$

Which gives

$$X = (C_2 e^{-\sqrt{k}z} + C_3 e^{\sqrt{k}z}) = (C_2 x^{-\sqrt{k}} + C_3 x^{\sqrt{k}})$$

$$\text{Therefore } u = (Ax^{-\sqrt{k}} + Bx^{\sqrt{k}}) e^{k \frac{t^\alpha}{\alpha}}$$

Using all the conditions we get desired solution $u = xe^t$

4. CONCLUSION

In this paper we have discussed the solution of partial differential equation involving derivatives of fractional order with respect to time or space by converting it into PDE of integer ordered and we see that conformable fractional derivative converts the fractional PDE into of the classical order which can be solved easily by known methods. From the procedure of the solutions it is clear that definition is convenient and there is no complexity in the computation to get exact solution. Here we have considered only up to two dimension problems and within certain range of fractional order but in future it may be extended for higher dimensional problems.

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