Available online at www.worldscientificnews.com

World Scientific News

An International Scientific Journal

WSN 113 (2018) 130-137 EISSN 2392-2192

C-compactness Via Grills

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ABSTRACT

In the present paper, we study C-compactness with respect to a grill, which simultaneously generalizes C-compactness and G-compactness and term it as C(G)-compact space. Several of its properties are investigated and effects of various kinds of functions on them are studied.

***Keywords*:** Grill, G-compact, C-compact, Quasi-H-closed

1. INTRODUCTION

In the present paper, we consider a topological space equipped with a grill, a brilliant notion that has been initiated by Choquet [1]. A grill $\mathcal{G}$ on a topological space X is a collection of subsets of X satisfying the following conditions: (1) $\phi \notin \mathcal{G}$, (2) $A \in \mathcal{G}$ and $A \subseteq B \Rightarrow B \in \mathcal{G}$, and (3) $A \notin \mathcal{G}$ and $B \notin \mathcal{G} \Rightarrow A \cup B \notin \mathcal{G}$. $\mathcal{G}(\{\phi\}) := P(X) - \{\phi\}$ and ϕ are trivial examples of grills. Some useful grills are (i) $\mathcal{G}_\infty$, the grill of all infinite subsets of X,(ii) $\mathcal{G}_{co}$,the grill of all uncountable subsets of X,(iii) $\mathcal{G}_p = \{A \subseteq X : A \in P(X); p \in A\}$, (iv) $\mathcal{G}_\sigma = \{A \subseteq X : \text{int}(\text{cl}(A)) \neq \phi\}$. For a grill $\mathcal{G}$ on X and $A \subset X$, we denote the grill $\{G \cap A : G \notin \mathcal{G}\}$by $\mathcal{G}_A$.

A topological space (X, τ) with a grill $\mathcal{G}$ on X will be denoted by $(X, \tau, \mathcal{G})$. Roy and Mukherjee [6] defined a topology obtained as an associated structure on a topological space (X, τ) induced by a grill on X. According to them, for $A \in P(X)$, $\Phi_{\mathcal{G}}(A, \tau)$ or $\Phi_{\mathcal{G}}(A)$ or simply

$\Phi(A)$ is the set $\{x \in X : A \cap U \in \mathcal{G}$, for every open neighborhood U of $x\}$. We can easily check that (i) for the grill ϕ, $\Phi(A)$ is ϕ (ii) for the grill $\mathcal{G}(\{\phi\})$, $\Phi(A)$ is cl(A),(iii) for the grill $\mathcal{G}_\infty$, $\Phi(A)$ is the set of all ω-accumulation points of A (iv) for the grill $\mathcal{G}_{co}$, $\Phi(A)$ is the set of all condensation points of A. Consider the operator $\Psi: P(X) \to P(X)$, where $\Psi(A) = A \cup \Phi(A)$, then Ψ is a Kuratowski closure operator and hence induces a topology on X, strictly finer than τ, in general. Also $\tau_\mathcal{G} = \{U \subseteq X : \Psi(X - U) = X - U\}$. We can easily check, $\tau_\mathcal{G}(\phi)$ = the discrete topology and $\tau_\mathcal{G}(\mathcal{G}(\{\phi\})) = \tau$. For a grill space $(X, \tau, \mathcal{G})$, the $\mathcal{B} = \{U - A : U \in \tau$ and $A \notin \mathcal{G}\}$ is the base for the topology $\tau_\mathcal{G}$ on X, finer than τ. Gupta and Noiri [3] defined C-compactness in an ideal topological space. Here we will define and explore C-compactness in a topological space by using the notion of grills. Some interesting illustrations of $\tau_\mathcal{G}$ are as follows:

(1) If τ is the topology generated by the partition $\{\{2n\text{-}1, 2n\} : n \in N\}$ on the set N of natural numbers, then $\tau_\mathcal{G}$ for $\mathcal{G}_\infty$ is the discrete topology.
(2) If τ is the indiscrete topology on a set X, then $\tau_\mathcal{G}$ for $\mathcal{G}_\infty$ is the cofinite topology on X.
(3) For any topological space (X, τ), $\tau_\mathcal{G}$ for $\mathcal{G}_\sigma$ is the τ^α topology of Njastad [5].

We recall that a subset A of a grill space $(X, \tau, \mathcal{G})$ is said to be $\mathcal{G}$-compact [7] if for every cover $\mathcal{U}$ of A by elements of τ, there exists a finite subfamily $\{U_1, U_2, U_3, \ldots, U_n\}$ such that $A - \bigcup_{i=1}^{n} U_i \notin \mathcal{G}$. The grill space $(X, \tau, \mathcal{G})$ is said to be $\mathcal{G}$-compact if X is $\mathcal{G}$-compact.

It is clear that (X, τ) is compact if and only if $(X, \tau, \mathcal{G}(\{\phi\}))$ is $\mathcal{G}(\{\phi\})$-compact. If (X, τ) is compact then $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-compact for any grill $\mathcal{G}$.

2. QUASI-H-CLOSED WITH RESPECT TO A GRILL SPACE

A topological space (X, τ) is said to be Quasi-H-closed or simply QHC, if for every open cover $\mathcal{U}$ of X, there exists a finite subfamily $\{U_1, U_2, U_3, \ldots, U_n\}$ such that $X = \bigcup_{i=1}^{n} \text{cl}(U_i)$.

In this section, we define quasi-H-closedness via grills and study some of its properties.

Definition 2.1. Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X. X is quasi-H-closed with respect to $\mathcal{G}$ or just $(\mathcal{G})$QHC if for every open cover $\mathcal{U}$ of X, there exists a finite subfamily $\{U_1, U_2, U_3, \ldots, U_n\}$ of $\mathcal{U}$ such that $X - \bigcup_{i=1}^{n} \text{cl}(U_i) \notin \mathcal{G}$. Such a subfamily is said to be proximate subcover modulo $\mathcal{G}$ or just $(\mathcal{G})$proximate subcover.

Definition 2.2. A grill $\mathcal{G}$ of subsets of a topological space (X, τ) is said to be co non-dense if the complement of each of its members is non-dense.

Theorem 2.3. For a space (X, τ), the following are equivalent:

(a) (X, τ) is quasi-H-closed.
(b) (X, τ) is (ϕ) QHC.
(c) (X, τ) is $(\mathcal{G}_\infty)$ QHC.

(d) (X, τ) is $(\mathcal{G}_\sigma)$ QHC.
(e) (X, τ) is $(\mathcal{G})$ QHC for every co non-dense grill $\mathcal{G}$.

Proof: It is easy to check from the above discussion.

A family $\mathcal{F}$ of subsets of X is said to have the *finite – intersection property with respect to a grill* $\mathcal{G}$ on X or just $(\mathcal{G})$FIP if the intersection of finite subfamily of $\mathcal{F}$ is a member of $\mathcal{G}$. Recall that a subset in a space is called regular open if it is the interior of its own closure. The complement of a regular open set is called regular closed.

Theorem 2.4. For a space (X, τ) and a grill $\mathcal{G}$ on X, the following are equivalent:

(a) (X, τ) is $(\mathcal{G})$ QHC;
(b) For each family $\mathcal{F}$ of closed sets having empty intersection , there is a finite subfamily $\{F_1, F_2, F_3, \ldots, F_n\}$ such that $\bigcap_{i=1}^{n} \text{int}(F_i) \notin \mathcal{G}$;
(c) For each family $\mathcal{F}$ of closed sets such that $\{\text{int}(F) : F \in \mathcal{F}\}$ has (G)FIP, one has $\cap \{F : F \in \mathcal{F}\} \neq \phi$;
(d) Every regular open cover has a $(\mathcal{G})$ proximate cover;
(e) For each family $\mathcal{F}$ of non empty regular closed sets having empty intersection, there is a finite subfamily $\{F_1, F_2, F_3, \ldots, F_n\}$ such that $\bigcap_{i=1}^{n} \text{int}(F_i) \notin \mathcal{G}$;
(f) For each collection $\mathcal{F}$ of non empty regular closed sets such that $\{\text{int}(F): F \in \mathcal{F}\}$ has $(\mathcal{G})$FIP, one has $\cap \{F : F \in \mathcal{F}\} \neq \phi$;
(g) For each open filter base $\mathcal{B}$ on $\mathcal{G}$, $\cap \{\text{cl}(B) : B \in \mathcal{B}\} \neq \phi$;
(h) Every open ultra filter on $\mathcal{G}$ converges.

3. C-COMPACTNESS WITH RESPECT TO A GRILL

In this section, we generalize the concept of C-compactness of Viglino [9] and compactness via grills of Roy and Mukherjee [7].

Herrington and Long [4] characterized C-compact spaces. A space (X, τ) is said to be C-compact if for each closed set A and each τ-open covering $\mathcal{U}$ of A, there exists a finite subfamily $\{U_1, U_2, U_3, \ldots, U_n\}$ such that $A \subset \bigcup_{i=1}^{n} \text{cl}(U_i)$.

Definition 3.1. Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X. (X, τ) is said to be C-compact with respect to grill or just C($\mathcal{G}$)-compact if for every τ-open covering $\mathcal{U}$ of A, there exists a finite subfamily $\{U_1, U_2, U_3, \ldots, U_n\}$ such that $A - \bigcup_{i=1}^{n} \text{cl}(U_i) \notin \mathcal{G}$.
Every C-compact space (X, τ) is C($\mathcal{G}$)-compact for any grill $\mathcal{G}$ on X. It is clear from the following example that the converse of it is not true.

Example 3.2. Consider Example 3. of [8]. Let $\mathcal{G}$ be a grill of all supersets of $X–A$. Then $(X, \tau, \mathcal{G})$ is C($\mathcal{G}$)-compact, but X is not C-compact.

Theorem 3.3. For a space (X, τ), the following are equivalent:

(a) (X, τ) is C-compact.
(b) (X, τ) is C(ϕ)-compact.
(c) (X, τ) is C($\mathcal{G}_\infty$)-compact.

Theorem 3.4. If a space is $\mathcal{G}$-compact then it is C($\mathcal{G}$)-compact.

Proof: Let X be a $\mathcal{G}$-compact space, A a closed subset of X and $\{V_\alpha\}_{\alpha\in\Lambda}$ an open cover of A. Then $(X–A)\cup\bigcup_{\alpha\in\Lambda}(V_\alpha)$ is an open cover of X. Since X is $\mathcal{G}$-compact, therefore there exists finite $\Lambda_0 \subseteq \Lambda$ such that $X-\{X-A)\cup\bigcup_{\alpha\in\Lambda_0}(V_\alpha)\}\notin\mathcal{G}$. This implies $A-\bigcup_{\alpha\in\Lambda_0}(V_\alpha)\notin\mathcal{G}$. Since $V_\alpha\subset \mathrm{cl}(V_\alpha)$, therefore $A-\bigcup_{\alpha\in\Lambda_0}\mathrm{cl}(V_\alpha)\notin\mathcal{G}$, implying that X is C($\mathcal{G}$)-compact.

In view of following example it is clear that the converse of this theorem, in general, is not true.

Example 3.5. Consider Example 3.3 of [3]. By Theorem 3.3, X is C($\mathcal{G}_\infty$)-compact, but not ($\mathcal{G}_\infty$)-compact.

Theorem 3.6. Let (X, τ) be a space and $\mathcal{G}$ be a grill on X. Then the following are equivalent:

(a) (X, τ) is C($\mathcal{G}$)-compact ;
(b) For each closed subset A of X and each family $\mathcal{F}$ of closed subsets of X such that $\cap\{F\cap A{:}F\in\mathcal{F}\}=\phi$, there is a finite subfamily $\{F_1,F_2,F_3,\ldots,F_n\}$ such that $\bigcap_{i=1}^n(\mathrm{int}(F_i))\cap A\notin\mathcal{G}$;
(c) For each closed set A and each family $\mathcal{F}$ of closed subsets of X such that $\{\mathrm{int}(F)\cap A : F\in\mathcal{F}\}$ has ($\mathcal{G}$)FIP, one has $\cap\{F\cap A : F\in\mathcal{F}\}\neq\phi$;
(d) For each closed set A and each regular open cover $\mathcal{U}$ of A, there exists a finite subcollection $\{U_1,U_2,U_3,\ldots,U_n\}$ such that $A-\bigcup_{i=1}^n\mathrm{cl}(U_i)\notin\mathcal{G}$.
(e) For each closed set A and each family $\mathcal{F}$ of regular closed sets such that $\cap\{F\cap A : F\in\mathcal{F}\}=\phi$, there is a finite subfamily $\{F_1,F_2,F_3,\ldots,F_n\}$ such that $\bigcap_{i=1}^n(\mathrm{int}(F_i))\cap A\notin\mathcal{G}$;
(f) For each closed set A and each family $\mathcal{F}$ of regular closed sets such that $\{\mathrm{int}(F)\cap A : F\in\mathcal{F}\}$ has ($\mathcal{G}$)FIP, one has $\cap\{F\cap A : F\in\mathcal{F}\}\neq\phi$;
(g) For each closed set A, each open cover $\mathcal{U}$ of $X–A$ and each open neighborhood V of A, there exists a finite subfamily $\{U_1,U_2,U_3,\ldots,U_n\}$ of $\mathcal{U}$ such that $X-(V\cup(\bigcup_{i=1}^n\mathrm{cl}(U_i)))\notin\mathcal{G}$.
(h) For each closed set A and each open filter base $\mathcal{B}$ on X such that $\{B\cap A : B\in\mathcal{B}\}\subset\mathcal{G}$, one has $\cap\{\mathrm{cl}(B) : B\in\mathcal{B}\}\cap A\neq\phi$.

Proof: (a)$\Rightarrow$(b) Let (X, τ) be C($\mathcal{G}$)-compact, A a closed subset, and $\mathcal{F}$ a family of closed subsets with $\cap\{F\cap A{:}F\in\mathcal{F}\}=\phi$. Then $\{X-F : F\in\mathcal{F}\}$ is an open cover of A and hence admits a finite subfamily $\{X - F_i : i = 1,2,...n\}$ such that $A - \bigcup_{i=1}^{n} \mathrm{cl}(X - F_i) \notin \mathcal{G}$. This set not in $\mathcal{G}$ is easily seen to be $\{\bigcap_{i=1}^{n}(\mathrm{int}(F_i)) \cap A\}$.

(b) $\Rightarrow$ **(c)** Easy.

(c) $\Rightarrow$ **(a)** Let A be a closed subset. Let $\mathcal{U}$ be an open cover of A with the property that for no finite subfamily $\{U_1, U_2, U_3, \dots, U_n\}$ of $\mathcal{U}$, one has $A - \bigcup_{i=1}^{n} \mathrm{cl}(U_i) \notin \mathcal{G}$. Then $\{X - U : U \in \mathcal{U}\}$ is a family of closed sets .Since $\bigcap_{i=1}^{n}\{(X - \mathrm{cl}(U_i)\} \cap A = \bigcap_{i=1}^{n}\{A - \mathrm{cl}(U_i)\} = A - \bigcup_{i=1}^{n} \mathrm{cl}(U_i)$, the family $\{\mathrm{int}(X - U) \cap A : U \in \mathcal{U}\}$ has $(\mathcal{G})$FIP. By the hypothesis $\cap\{(X - U) \cap A : U \in \mathrm{U}\} \neq \phi$ $\Rightarrow \cap\{A - U : U \in \mathcal{U}\} \neq \phi \Rightarrow A - \cup\{U : U \in \mathcal{U}\} \neq \phi \Rightarrow \mathcal{U}$ is not a cover of A, a contradiction.

(d) $\Rightarrow$ **(a)** Let A be a closed subset of X and $\mathcal{U}$ be an open cover of A. Then $\{\mathrm{int}(\mathrm{cl}(U)) : U \in \mathcal{U}\}$ is a regular open cover of A. Let $\{\mathrm{int}(\mathrm{cl}(U_i)) : i = 1,2,..., n\}$ be a finite subfamily such that $A - \bigcup_{i=1}^{n} \mathrm{cl}(\mathrm{int}(\mathrm{cl}(U_i))) \notin \mathcal{G}$. Since U_i is open, and for each open set U we have cl(int(cl(U))) = cl(U). We have $A - \bigcup_{i=1}^{n} \mathrm{cl}(U_i) \notin \mathcal{G}$. Hence X is C($\mathcal{G}$)-compact.

(a) $\Rightarrow$ **(d)** This is obvious.

(d) $\Rightarrow$ **(e)** $\Rightarrow$ **(f)** $\Rightarrow$ **(d)** are parallel to (a) $\Rightarrow$(b) $\Rightarrow$(c) $\Rightarrow$(a) respectively.

(a) $\Rightarrow$ **(g)** Let A be a closed set, V an open neighborhood of A, and $\mathcal{U}$ an open cover of X–A. Since $X - V \subset X - A$, $\mathcal{U}$ is an open cover of X–V. Let $\{U_1, U_2, U_3, \dots, U_n\}$ be a finite collection of $\mathcal{U}$, such that $(X - V) - \bigcup_{i=1}^{n} \mathrm{cl}(\mathrm{U_i}) \notin \mathcal{G}$. Since $(X - V) - \bigcup_{i=1}^{n} \mathrm{cl}(U_i) = X - (V \cup (\bigcup_{i=1}^{n} \mathrm{cl}(U\ )))$ This shows $(X - (V \cup (\bigcup_{i=1}^{n} \mathrm{cl}(U_i))) \notin \mathcal{G}$.

(g) $\Rightarrow$ **(a)** Let A be a closed subset of X and $\mathcal{U}$ an open covering of A. If H denotes the union of members of $\mathcal{U}$, then $F = X - H$ is a closed set and X–A is an open neighborhood of F. Also $\mathcal{U}$ is an open cover of X–F. By hypothesis, there is a finite sub-collection $\{U_1, U_2, U_3, \dots, U_n\}$ of $\mathcal{U}$, such that $X - ((X - A) \cup (\bigcup_{i=1}^{n} \mathrm{cl}(U_i))) \notin \mathcal{G}$. However, this set not in $\mathcal{G}$ is nothing but $A - \bigcup_{i=1}^{n} \mathrm{cl}(U_i)$.

(a) $\Rightarrow$ **(h)** Suppose A is a closed set and $\mathcal{B}$ is an open filter base on X with $\{B \cap A : B \in \mathcal{B}\} \subset \mathcal{G}$. Suppose, if possible, $\cap\{cl(B) : B \in \mathcal{B}\} \cap A = \phi$. Then $\{X - cl(B) : B \in \mathcal{B}\}$ is an open cover of A. By the hypothesis, there exists a finite subfamily $\{X - \mathrm{cl}(B_i) : i = 1,2,3,..., n\}$ such that $A - \bigcup_{i=1}^{n} \mathrm{cl}(X - \mathrm{cl}(B_i)) \notin \mathcal{G}$. However, this set is $A \cap (\bigcap_{i=1}^{n} \mathrm{int}(\mathrm{cl}(B_i)))$and $A \cap (\bigcap_{i=1}^{n} B_i)$ is a subset of it. Therefore, $A \cap (\bigcap_{i=1}^{n} B_i) \notin \mathcal{G}$. Since $\mathcal{B}$ is a filter base, we have a $B \in \mathcal{B}$ such that $B \subset \bigcap_{i=1}^{n} B_i$. But then $A \cap B \notin \mathcal{G}$ which contradicts the fact that $\{B \cap A : B \in \mathcal{B}\} \subset \mathcal{G}$.

(h) $\Rightarrow$ **(a)** Suppose that (X, τ) is not C($\mathcal{G}$)-compact. Then there exist a closed subset A of X and an open cover $\mathcal{U}$ of A such that for any finite subfamily $\{U_1, U_2, U_3, \dots, U_n\}$ of $\mathcal{U}$, we have $A - \bigcup_{i=1}^{n} \mathrm{cl}(U_i) \in \mathcal{G}$. We may assume that $\mathcal{U}$ is closed under finite unions. Then the family $\mathcal{B} = \{X$

$-\mathrm{cl}(U) : U \in \mathcal{U}\}$ is an open filter base on X such that $\{B \cap A : B \in \mathcal{B}\} \subset \mathcal{G}$. So, by the hypothesis, $\cap\{\mathrm{cl}(X - \mathrm{cl}(U)) : U \in \mathcal{U}\} \cap A \neq \phi$. Let x be a point in the intersection. Then $x \in A$ and $x \in \mathrm{cl}(X - \mathrm{cl}(U)) = X - \mathrm{int}(\mathrm{cl}(U)) \subset X - U$ for each $U \in \mathcal{U}$. But this contradicts the fact that $\mathcal{U}$ is a cover of A. Hence, (X, τ) is C($\mathcal{G}$)-compact.

Definition 3.7. A filter base $\mathcal{B}$ is said to be ($\mathcal{G}$) adherent convergent if for every neighborhood N of the adherent set of $\mathcal{B}$, there exists an element $B \in \mathcal{B}$ such that $(X - N) \cap B \notin \mathcal{G}$.

Theorem 3.8. A space (X, τ) is C($\mathcal{G}$)-compact if and only if every open filter base on $\mathcal{G}$ is ($\mathcal{G}$) adherent convergent.

Proof: Let (X, τ) be C($\mathcal{G}$)-compact and let $\mathcal{B}$ be an open filter base on $\mathcal{G}$ with A as its adherent set. Let G be an open neighborhood of A. Then $A = \cap\{\mathrm{cl}(B) : B \in \mathcal{B}\}$, $A \subset G$, and X- G is closed. Now $\{X\text{-}\ \mathrm{cl}(B) : B \in \mathcal{B}\}$ is an open cover of $X-G$ and so by the hypothesis, it admits a finite subfamily $\{X-\mathrm{cl}(B_i) : i = 1, 2, 3,..., n\}$ such that $(X-G) - \bigcup_{i=1}^{n} \mathrm{cl}(X - \mathrm{cl}(B_i)) \notin \mathcal{G}$. But this implies $(X-G) \cap (\bigcap_{i=1}^{n} \mathrm{int}(\mathrm{cl}(B_i))) \notin \mathcal{G}$. However $B_i \subset \mathrm{int}(\mathrm{cl}(B_i))$ implies $(X\text{-}\ G) \cap (\bigcap_{i=1}^{n} (B_i)) \notin \mathcal{G}$. Since $\mathcal{B}$ is a filter base and $B_i \in \mathcal{B}$, there is a $B \in \mathcal{B}$ such that $B \subset \bigcap_{i=1}^{n} B_i$. But then $(X-G) \cap B \notin \mathcal{G}$ is required.

Conversely, let (X, τ) be not C($\mathcal{G}$)-compact, and A be a closed set, and $\mathcal{U}$ be an open cover of A such that for no finite subfamily $\{U_1, U_2, U_3,..., U_n\}$ of $\mathcal{U}$, one has $A - \bigcup_{i=1}^{n} \mathrm{cl}(U_i) \notin \mathcal{G}$. Without loss of generality, we may assume that $\mathcal{U}$ is closed for finite unions. Therefore, $\mathcal{B} = \{X\text{-}\ \mathrm{cl}(U) : U \in \mathcal{U}\}$ becomes an open filter base on $\mathcal{G}$. If x is an adherent point of $\mathcal{B}$, that is, if $x \in \cap\{\mathrm{cl}(X - \mathrm{cl}(U)) : U \in \mathcal{U}\} = X - \cup\{\mathrm{int}(\mathrm{cl}(U)) : U \in \mathcal{U}\}$, then $x \notin A$, because $\mathcal{U}$ is an open cover of A and for $U \in \mathcal{U}$, $U \subset \mathrm{int}(\mathrm{cl}(U))$. Therefore, the adherent set of $\mathcal{B}$ is contained in $X-A$, which is an open set. By the hypothesis, there exists an element $B \in \mathcal{B}$ such that $(X - (X - A)) \cap B \notin \mathcal{G}$, that is, $A \cap B \notin \mathcal{G}$, that is $A \cap (X - \mathrm{cl}(U)) \notin \mathcal{G}$ some $U \in \mathcal{U}$. This however contradicts our assumption. This completes the proof.

4. C($\mathcal{G}$)-COMPACT SPACES AND FUNCTIONS

Definition 4.1. A function $f : (X, \tau) \to (Y, \varsigma)$ is said to be θ-continuous [2] at a point $x \in X$ if for every open set V of Y containing $f(x)$, there exists an open set U of X containing x such that $f(\mathrm{cl}(U)) \subseteq \mathrm{cl}(V)$.

Theorem 4.2. Let $f : (X, \tau, \mathcal{G}) \to (Y, \varsigma, \mathcal{H})$ be a continuous surjection, $(X, \tau, \mathcal{G})$ C($\mathcal{G}$)-compact, and $f(\mathcal{G}) \subseteq \mathcal{H}$. Then $(Y, \varsigma, \mathcal{H})$ is C($\mathcal{H}$)-compact.

Proof: Let A be a closed subset of (Y, ς) and $\mathcal{V}$ any open cover of A in Y. By continuity of f, $f^{-1}(A)$ is an closed subset of X and is such that $\{f^{-1}(V) : V \in \mathcal{V}\}$ is a cover of $f^{-1}(A)$ by open sets in X. Hence, by the C($\mathcal{G}$)-compactness of X, there exists a finite subcollection $\{ f^{-1}(V_i) : i = 1, 2, 3,..., n\}$ such that $f^{-1}(A) - \bigcup_{i=1}^{n} \mathrm{cl}(f^{-1}(V_i)) \notin \mathcal{G}$. Since f is continuous, $\mathrm{cl}(f^{-1}(B)) \subset f^{-1}(\mathrm{cl}(B))$ for every subset B of Y. Hence we have $f^{-1}(A) - \bigcup_{i=1}^{n} (f^{-1}\mathrm{cl}(V_i)) = f^{-1}(A - \bigcup_{i=1}^{n} \mathrm{cl}(V_i)) \notin \mathcal{G}$. Since f is surjective, $A - \bigcup_{i=1}^{n} \mathrm{cl}(V_i) \notin f(\mathcal{G}) \subseteq \mathcal{H}$. Hence, Y is C($\mathcal{H}$)-compact.

Theorem 4.3. Let $f : (X, \tau, \mathcal{G}) \to (Y, \varsigma, \mathcal{H})$ be a θ-continuous function, $(X, \tau, \mathcal{G})$ C($\mathcal{G}$)-compact, (Y, ς) Hausdorff , and $f(\mathcal{G}) \subseteq \mathcal{H}$. Then $f(A)$ is $\varsigma_{\mathcal{H}}$-closed.

Proof: Let A be any closed set in X and $a \notin f(A)$. For each $x \in A$, there exists a ς-open set V_y containing $y = f(x)$ such that $a \notin \mathrm{cl}(V_y)$. Now because f is θ -continuous, there exists an open set U_x containing x such that $f(\mathrm{cl}(U_x)) \subseteq \mathrm{cl}(V_y)$. Now, the family $\{U_x : x \in A\}$ is an open cover of A. Therefore, there exists a finite subfamily $\{U_{x_i} : i = 1,2,3,\dots,n\}$ suchthat $A - \bigcup_{i=1}^{n} \mathrm{cl}(U_{x_i}) \notin \mathcal{G}$. But then $f(A - \bigcup_{i=1}^{n} \mathrm{cl}(U_{x_i})) \notin f(\mathcal{G}) \subseteq \mathcal{H}$, that is, $f(A) - f(\bigcup_{i=1}^{n} \mathrm{cl}(U_{x_i})) \notin f(\mathcal{G}) \subseteq \mathcal{H}$ because $f(\mathcal{G})$ is also a grill. Hence, $f(A) - \bigcup_{i=1}^{n} \mathrm{cl}(V_{y_i})) \notin f(\mathcal{G}) \subseteq \mathcal{H}$. Now $a \notin \mathrm{cl}(V_{y_i})$ for any i implies that $a \in Y - \bigcup_{i=1}^{n} \mathrm{cl}(V_{y_i})$ which is open in (Y, ς). That is $Y - \bigcup_{i=1}^{n} \mathrm{cl}(V_{y_i})$ is a neighborhood of a in (Y, ς). However, $(Y - \bigcup_{i=1}^{n} \mathrm{cl}(V_{y_i})) \cap f(A) = f(A) - \bigcup_{i=1}^{n} \mathrm{cl}(V_{y_i}) \notin f(\mathcal{G}) \subseteq \mathcal{H}$. Hence, $a \notin \Phi_{\mathcal{H}}(f(A), \varsigma)$. Thus $\Phi_{\mathcal{H}}(f(A),\varsigma) \subset f(A)$. This implies $f(A)$ is $\varsigma_{\mathcal{H}}$-closed.

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