C-compactness Via Grills

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ABSTRACT

In the present paper, we study C-compactness with respect to a grill, which simultaneously generalizes C-compactness and G-compactness and term it as C(G)-compact space. Several of its properties are investigated and effects of various kinds of functions on them are studied.

Keywords: Grill, G-compact, C-compact, Quasi-H-closed

1. INTRODUCTION

In the present paper, we consider a topological space equipped with a grill, a brilliant notion that has been initiated by Choquet [1]. A grill \( \mathcal{G} \) on a topological space \( X \) is a collection of subsets of \( X \) satisfying the following conditions: (1) \( \emptyset \notin \mathcal{G} \), (2) \( A \in \mathcal{G} \) and \( A \subseteq B \Rightarrow B \in \mathcal{G} \), and (3) \( A \notin \mathcal{G} \) and \( B \notin \mathcal{G} \Rightarrow A \cup B \notin \mathcal{G} \). 

\( \mathcal{G}(\{\emptyset\}) := \{\emptyset\} \), \( \emptyset \) is a trivial example of grills. Some useful grills are (i) \( \mathcal{G}_\infty \), the grill of all infinite subsets of \( X \), (ii) \( \mathcal{G}_{\co} \), the grill of all uncountable subsets of \( X \), (iii) \( \mathcal{G}_p = \{X : A \subseteq X \in P(X) \} \), (iv) \( \mathcal{G}_\sigma = \{X \in X : \text{int}(\text{cl}(A)) \neq \emptyset \} \). For a grill \( \mathcal{G} \) on \( X \) and \( A \subseteq X \), we denote the grill \( \{G \cap A : G \notin \mathcal{G}\} \) by \( \mathcal{G}_A \).

A topological space \((X, \tau)\) with a grill \( \mathcal{G} \) on \( X \) will be denoted by \((X, \tau, \mathcal{G})\). Roy and Mukherjee [6] defined a topology obtained as an associated structure on a topological space \((X, \tau)\) induced by a grill on \( X \). According to them, for \( A \in P(X) \), \( \Phi_{\mathcal{G}}(A, \tau) \) or \( \Phi_{\mathcal{G}}(A) \) or simply
Φ(A) is the set \( \{ x \in X : A \cap U \in \mathcal{G}, \text{ for every open neighborhood } U \text{ of } x \} \). We can easily check that (i) for the grill \( \phi \), Φ(A) is \( \phi \) (ii) for the grill \( \mathcal{G}(\{ \phi \}) \), Φ(A) is \( \text{cl}(A)\), (iii) for the grill \( \mathcal{G}_c \), \( \Phi(A) \) is the set of all \( \omega \)-accumulation points of \( A \) (iv) for the grill \( \mathcal{G}_{co} \), \( \Phi(A) \) is the set of all condensation points of \( A \). Consider the operator \( \Psi : P(X) \to P(X) \), where \( \Psi(A) = A \cup \Phi(A) \), then \( \Psi \) is a Kuratowski closure operator and hence induces a topology on \( X \), strictly finer than \( \tau \), in general. Also \( \tau_G = \{ U \subseteq X : \Psi(X - U) = X - U \} \). We can easily check, \( \tau_G(\phi) \) = the discrete topology and \( \tau_G(\mathcal{G}(\{ \phi \})) = \tau \). For a grill space \( (X, \tau, \mathcal{G}) \), the \( \mathcal{B} = \{ U - A : U \in \tau \text{ and } A \notin \mathcal{G} \} \) is the base for the topology \( \tau_G \) on \( X \), finer than \( \tau \). Gupta and Noiri [3] defined C-compactness in an ideal topological space. Here we will define and explore C-compactness in a topological space by using the notion of grills. Some interesting illustrations of \( \tau_G \) are as follows:

1. If \( \tau \) is the topology generated by the partition \( \{ \{2n-1, 2n\} : n \in \mathbb{N} \} \) on the set \( \mathbb{N} \) of natural numbers, then \( \tau_G \) for \( \mathcal{G}_c \) is the discrete topology.
2. If \( \tau \) is the indiscrete topology on a set \( X \), then \( \tau_G \) for \( \mathcal{G}_c \) is the cofinite topology on \( X \).
3. For any topological space \( (X, \tau) \), \( \tau_G \) for \( \mathcal{G}_c \) is the \( \tau^c \) topology of Njastad [5].

We recall that a subset \( A \) of a grill space \( (X, \tau, \mathcal{G}) \) is said to be \( \mathcal{G} \)-compact [7] if for every cover \( \mathcal{U} \) of \( A \) by elements of \( \tau \), there exists a finite subfamily \( \{ U_1, U_2, U_3, \ldots, U_n \} \) such that \( A - \bigcup_{i=1}^{n} U_i \notin \mathcal{G} \). The grill space \( (X, \tau, \mathcal{G}) \) is said to be \( \mathcal{G} \)-compact if \( X \) is \( \mathcal{G} \)-compact.

It is clear that \( (X, \tau) \) is compact if and only if \( (X, \tau, \mathcal{G}(\{ \phi \})) \) is \( \mathcal{G}(\{ \phi \}) \)-compact. If \( (X, \tau) \) is compact then \( (X, \tau, \mathcal{G}) \) is \( \mathcal{G} \)-compact for any grill \( \mathcal{G} \).

2. QUASI-H-CLOSED WITH RESPECT TO A GRILL SPACE

A topological space \( (X, \tau) \) is said to be Quasi-H-closed or simply QHC, if for every open cover \( \mathcal{U} \) of \( X \), there exists a finite subfamily \( \{ U_1, U_2, U_3, \ldots, U_n \} \) such that \( X = \bigcup_{i=1}^{n} \text{cl}(U_i) \).

In this section, we define quasi-H-closedness via grills and study some of its properties.

**Definition 2.1.** Let \( (X, \tau) \) be a topological space and \( \mathcal{G} \) be a grill on \( X \). \( X \) is quasi-H-closed with respect to \( \mathcal{G} \) or just \( (\mathcal{G}) \text{QHC} \) if for every open cover \( \mathcal{U} \) of \( X \), there exists a finite subfamily \( \{ U_1, U_2, U_3, \ldots, U_n \} \) of \( \mathcal{U} \) such that \( X - \bigcup_{i=1}^{n} \text{cl}(U_i) \notin \mathcal{G} \). Such a subfamily is said to be proximate subcover modulo \( \mathcal{G} \) or just \( (\mathcal{G}) \text{proximate subcover} \).

**Definition 2.2.** A grill \( \mathcal{G} \) of subsets of a topological space \( (X, \tau) \) is said to be co non-dense if the complement of each of its members is non-dense.

**Theorem 2.3.** For a space \( (X, \tau) \), the following are equivalent:

(a) \( (X, \tau) \) is quasi-H-closed.
(b) \( (X, \tau) \) is \( (\phi) \text{QHC} \).
(c) \( (X, \tau) \) is \( (\mathcal{G}_c) \text{QHC} \).
(d) \((X, \tau)\) is \((G_0)\) QHC.
(e) \((X, \tau)\) is \((G)\) QHC for every co non-dense grill \(G\).

**Proof:** It is easy to check from the above discussion.

A family \(F\) of subsets of \(X\) is said to have the *finite-intersection property with respect to a grill* \(G\) on \(X\) or just \((G)\)FIP if the intersection of finite subfamily of \(F\) is a member of \(G\). Recall that a subset in a space is called regular open if it is the interior of its own closure. The complement of a regular open set is called regular closed.

**Theorem 2.4.** For a space \((X, \tau)\) and a grill \(G\) on \(X\), the following are equivalent:

(a) \((X, \tau)\) is \((G)\) QHC;
(b) For each family \(F\) of closed sets having empty intersection, there is a finite subfamily \(\{F_1, F_2, F_3, \ldots, F_n\}\) such that \(\bigcap_{i=1}^{n} \text{int}(F_i) \notin G\);
(c) For each family \(F\) of closed sets such that \(\{\text{int}(F) : F \in F\}\) has \((G)\)FIP, one has \(\bigcap \{F : F \in F\} \neq \phi\);
(d) Every regular open cover has a \((G)\) proximate cover;
(e) For each family \(F\) of nonempty regular closed sets having empty intersection, there is a finite subfamily \(\{F_1, F_2, F_3, \ldots, F_n\}\) such that \(\bigcap_{i=1}^{n} \text{int}(F_i) \notin G\);
(f) For each collection \(F\) of nonempty regular closed sets such that \(\{\text{int}(F) : F \in F\}\) has \((G)\)FIP, one has \(\bigcap \{F : F \in F\} \neq \phi\);
(g) For each open filter base \(B\) on \(G\), \(\bigcap \{\text{cl}(B) : B \in B\} \neq \phi\);
(h) Every open ultra filter on \(G\) converges.

**3. C-COMPACTNESS WITH RESPECT TO A GRILL**

In this section, we generalize the concept of C-compactness of Viglino [9] and compactness via grills of Roy and Mukherjee [7].

Herrington and Long [4] characterized C-compact spaces. A space \((X, \tau)\) is said to be C-compact if for each closed set \(A\) and each \(\tau\)-open covering \(U\) of \(A\), there exists a finite subfamily \(\{U_1, U_2, U_3, \ldots, U_n\}\) such that \(A \subseteq \bigcup_{i=1}^{n} \text{cl}(U_i)\).

**Definition 3.1.** Let \((X, \tau)\) be a topological space and \(G\) be a grill on \(X\). \((X, \tau)\) is said to be C-compact with respect to grill or just \(C(G)\)-compact if for every \(\tau\)-open covering \(U\) of \(A\), there exists a finite subfamily \(\{U_1, U_2, U_3, \ldots, U_n\}\) such that \(A \subseteq \bigcup_{i=1}^{n} \text{cl}(U_i) \notin G\).

Every C-compact space \((X, \tau)\) is \(C(G)\)-compact for any grill \(G\) on \(X\). It is clear from the following example that the converse of it is not true.

**Example 3.2.** Consider Example 3. of [8]. Let \(G\) be a grill of all supersets of \(X\)–\(A\). Then \((X, \tau, G)\) is \(C(G)\)-compact, but \(X\) is not C-compact.
**Theorem 3.3.** For a space \((X, \tau)\), the following are equivalent:

(a) \((X, \tau)\) is C-compact.

(b) \((X, \tau)\) is C(\(\phi\))-compact.

(c) \((X, \tau)\) is C(\(G_x\))-compact.

**Theorem 3.4.** If a space is \(G\)-compact then it is C(\(G\))-compact.

**Proof:** Let \(X\) be a \(G\)-compact space, \(A\) a closed subset of \(X\) and \(\{V_a\}_{a \in A}\) an open cover of \(A\). Then \((X-A) \cup \bigcup_{a \in A} (V_a)\) is an open cover of \(X\). Since \(X\) is \(G\)-compact, therefore there exists finite \(\Lambda_0 \subseteq \Lambda\) such that \(X - \{X - A\} \cup \bigcup_{a \in \Lambda_0} (V_a) \notin G\). This implies \(A - \bigcup_{a \in \Lambda_0} (V_a) \notin G\).

Since \(V_a \subseteq \text{cl}(V_a)\), therefore \(A - \bigcup_{a \in \Lambda_0} \text{cl}(V_a) \notin G\), implying that \(X\) is C(\(G\))-compact.

In view of following example it is clear that the converse of this theorem, in general, is not true.

**Example 3.5.** Consider Example 3.3 of [3]. By Theorem 3.3, \(X\) is C(\(G_x\))-compact, but not (\(G_x\))-compact.

**Theorem 3.6.** Let \((X, \tau)\) be a space and \(G\) be a grill on \(X\). Then the following are equivalent:

(a) \((X, \tau)\) is C(\(G\))-compact;

(b) For each closed subset \(A\) of \(X\) and each family \(F\) of closed subsets of \(X\) such that \(\bigcap \{F \cap A : F \in F\} = \phi\), there is a finite family \(\{F_1, F_2, F_3, \ldots, F_n\}\) such that \(\bigcap_{i=1}^n (\text{int}(F_i)) \cap A \notin G\);

(c) For each closed set \(A\) and each family \(F\) of closed subsets of \(X\) such that \(\{\text{int}(F) \cap A : F \in F\}\) has (\(G\))FIP, one has \(\bigcap \{F \cap A : F \in F\} \neq \phi\);

(d) For each closed set \(A\) and each regular open cover \(U\) of \(A\), there exists a finite subcollection \(\{U_1, U_2, U_3, \ldots, U_n\}\) such that \(A - \bigcup_{i=1}^n \text{cl}(U_i) \notin G\).

(e) For each closed set \(A\) and each family \(F\) of regular closed sets such that \(\bigcap \{F \cap A : F \in F\} = \phi\), there is a finite family \(\{F_1, F_2, F_3, \ldots, F_n\}\) such that \(\bigcap_{i=1}^n (\text{int}(F_i)) \cap A \notin G\);

(f) For each closed set \(A\) and each family \(F\) of regular closed sets such that \(\{\text{int}(F) \cap A : F \in F\}\) has (\(G\))FIP, one has \(\bigcap \{F \cap A : F \in F\} \neq \phi\);

(g) For each closed set \(A\), each open cover \(U\) of \(X-A\) and each open neighborhood \(V\) of \(A\), there exists a finite subfamily \(\{U_1, U_2, U_3, \ldots, U_n\}\) of \(U\) such that \(X - (V \cup (\bigcup_{i=1}^n \text{cl}(U_i))) \notin G\).

(h) For each closed set \(A\) and each open filter base \(B\) on \(X\) such that \(\{B \cap A : B \in B\} \subseteq G\), one has \(\bigcap \{\text{cl}(B) : B \in B\} \cap A \neq \phi\).

**Proof:** (a) \(\Rightarrow\) (b) Let \((X, \tau)\) be C(\(G\))-compact, \(A\) a closed subset, and \(F\) a family of closed subsets with \(\bigcap \{F \cap A : F \in F\} = \phi\). Then \(\{X - F : F \in F\}\) is an open cover of \(A\) and hence...
admits a finite subfamily \( \{ X - F_i : i = 1,2,...,n \} \) such that \( A - \bigcup_{i=1}^n \text{cl}(X - F_i) \notin \mathcal{G} \). This set not in \( \mathcal{G} \) is easily seen to be \( \{ \bigcap_{i=1}^n (\text{int}(F_i)) \cap A \} \).

(b) \( \Rightarrow \) (c) Easy.

(c) \( \Rightarrow \) (a) Let \( A \) be a closed subset. Let \( \mathcal{U} \) be an open cover of \( A \) with the property that for no finite subfamily \( \{ U_1, U_2, U_3, ..., U_n \} \) of \( \mathcal{U} \), one has \( A - \bigcup_{i=1}^n \text{cl}(U_i) \notin \mathcal{G} \). Then \( \{ X - U : U \in \mathcal{U} \} \) is a family of closed sets. Since \( \bigcap_{i=1}^n \{(X - \text{cl}(U_i)) \cap A = \bigcap_{i=1}^n \{A - \text{cl}(U_i)\} = A - \bigcup_{i=1}^n \text{cl}(U_i)\} \), the family \( \{ \text{int}(X - U) \cap A : U \in \mathcal{U} \} \) has \( (\mathcal{G}) \) FIP. By the hypothesis \( \bigcap \{(X - U) \cap A : U \in \mathcal{U} \} \neq \phi \) \( \Rightarrow \) \( \bigcap \{A - U : U \in \mathcal{U} \} \neq \phi \) \( \Rightarrow \) \( A - \bigcup \{U : U \in \mathcal{U} \} \neq \phi \) \( \Rightarrow \) \( \mathcal{U} \) is not a cover of \( A \), a contradiction.

(d) \( \Rightarrow \) (a) Let \( A \) be a closed subset of \( X \) and \( \mathcal{U} \) be an open cover of \( A \). Then \( \{ \text{int}(\text{cl}(U)) : U \in \mathcal{U} \} \) is a regular open cover of \( A \). Let \( \{\text{int}(\text{cl}(U_i)) : i = 1,2,...,n\} \) be a finite subfamily such that \( A - \bigcup_{i=1}^n \text{cl}((\text{int}(\text{cl}(U_i)))) \notin \mathcal{G} \). Since \( U_i \) is open, and for each open set \( U \) we have \( \text{cl}(\text{int}(\text{cl}(U))) = \text{cl}(U) \). We have \( A - \bigcup_{i=1}^n \text{cl}(U_i) \notin \mathcal{G} \). Hence \( X \) is \( C(\mathcal{G}) \)-compact.

(a) \( \Rightarrow \) (d) This is obvious.

(d) \( \Rightarrow \) (e) \( \Rightarrow \) (f) \( \Rightarrow \) (d) are parallel to (a) \( \Rightarrow \) (b) \( \Rightarrow \) (c) \( \Rightarrow \) (a) respectively.

(a) \( \Rightarrow \) (g) Let \( A \) be a closed set, \( V \) an open neighborhood of \( A \), and \( \mathcal{U} \) an open cover of \( X - A \). Since \( X - V \subseteq X - A \), \( \mathcal{U} \) is an open cover of \( X - V \). Let \( \{ U_1, U_2, U_3, ..., U_n \} \) be a finite collection of \( \mathcal{U} \), such that \( (X - V) - \bigcup_{i=1}^n \text{cl}(U_i) \notin \mathcal{G} \). Since \( (X - V) - \bigcup_{i=1}^n \text{cl}(U_i) = X - (V \cup (\bigcup_{i=1}^n \text{cl}(U_i))) \) This shows \( (X - (V \cup (\bigcup_{i=1}^n \text{cl}(U_i)))) \notin \mathcal{G} \).

(g) \( \Rightarrow \) (a) Let \( A \) be a closed subset of \( X \) and \( \mathcal{U} \) an open covering of \( A \). If \( H \) denotes the union of members of \( \mathcal{U} \), then \( F = X - H \) is a closed set and \( X - A \) is an open neighborhood of \( F \). Also \( \mathcal{U} \) is an open cover of \( X - F \). By hypothesis, there is a finite sub-collection \( \{ U_1, U_2, U_3, ..., U_n \} \) of \( \mathcal{U} \), such that \( X - ((X - A) \cup (\bigcup_{i=1}^n \text{cl}(U_i))) \notin \mathcal{G} \). However, this set not in \( \mathcal{G} \) is nothing but \( A - \bigcup_{i=1}^n \text{cl}(U_i) \).

(a) \( \Rightarrow \) (h) Suppose \( A \) is a closed set and \( \mathcal{B} \) is an open filter base on \( X \) with \( \{ B \cap A : B \in \mathcal{B} \} \subset \mathcal{G} \). Suppose, if possible, \( \{ \text{cl}(B) : B \in \mathcal{B} \} \cap A = \phi \). Then \( \{ X - \text{cl}(B) : B \in \mathcal{B} \} \) is an open cover of \( A \). By the hypothesis, there exists a finite subfamily \( \{ X - \text{cl}(B_i) : i = 1,2,3..., n \} \) such that \( A - \bigcup_{i=1}^n \text{cl}(X - \text{cl}(B_i)) \notin \mathcal{G} \). However, this set is \( A \cap (\bigcap_{i=1}^n \text{int}(\text{cl}(B_i))) \) and \( A \cap (\bigcap_{i=1}^n B_i) \) is a subset of it. Therefore, \( A \cap (\bigcap_{i=1}^n B_i) \notin \mathcal{G} \). Since \( \mathcal{B} \) is a filter base, we have a \( B \in \mathcal{B} \) such that \( B \subseteq \bigcap_{i=1}^n B_i \). But then \( A \cap B \notin \mathcal{G} \) which contradicts the fact that \( \{ B \cap A : B \in \mathcal{B} \} \subset \mathcal{G} \).

(h) \( \Rightarrow \) (a) Suppose that \( (X, \tau) \) is not \( C(\mathcal{G}) \)-compact. Then there exist a closed subset \( A \) of \( X \) and an open cover \( \mathcal{U} \) of \( A \) such that for any finite subfamily \( \{ U_1, U_2, U_3, ..., U_n \} \) of \( \mathcal{U} \), we have \( A - \bigcup_{i=1}^n \text{cl}(U_i) \in \mathcal{G} \). We may assume that \( \mathcal{U} \) is closed under finite unions. Then the family \( \mathcal{B} = \{ X \)
–cl(U) : U ∈ U} is an open filter base on X such that \{B ∩ A : B ∈ B\} ⊆ G. So, by the hypothesis, ∩ \{cl(X − cl(U)) : U ∈ U\} ∩ A ≠ φ. Let x be a point in the intersection. Then x ∈ A and x ∈ cl(X − cl(U)) = X − int(cl(U)) ⊆ X − U for each U ∈ U. But this contradicts the fact that U is a cover of A. Hence, (X, τ) is C(G)-compact.

**Definition 3.7.** A filter base B is said to be (G) adherent convergent if for every neighborhood N of the adherent set of B, there exists an element B ∈ B such that (X − N) ∩ B ⋄ G.

**Theorem 3.8.** A space (X, τ) is C(G)-compact if and only if every open filter base on G is (G) adherent convergent.

**Proof:** Let (X, τ) be C(G)-compact and let B be an open filter base on G with A as its adherent set. Let G be an open neighborhood of A. Then A = ∩ \{cl(B) : B ∈ B\}, A ⊆ G, and X − G is closed. Now \{X − cl(B) : B ∈ B\} is an open cover of X − G and so by the hypothesis, it admits a finite subfamily \{X − cl(B_i) : i = 1, 2, 3, ..., n\} such that (X − G) − \bigcup_{i=1}^{n} cl(X − cl(B_i)) ⋄ G. But this implies (X − G) ∩ (\bigcap_{i=1}^{n} int(cl(B_i))) ⋄ G. However B_i ⊆ int(cl(B_i)) implies (X − G) ∩ (\bigcap_{i=1}^{n} (B_i)) ⋄ G. Since B is a filter base and B_i ∈ B, there is a B ∈ B such that B ⊆ \bigcap_{i=1}^{n} B_i. But then (X − G) ∩ B ⋄ G is required.

Conversely, let (X, τ) be not C(G)-compact, and A be a closed set, and U be an open cover of A such that for no finite subfamily \{U_1, U_2, U_3, ..., U_n\} of U, one has A − \bigcup_{i=1}^{n} cl(U_i) ⋄ G. Without loss of generality, we may assume that U is closed for finite unions. Therefore, \mathcal{B} = \{X − cl(U) : U ∈ U\} becomes an open filter base on G. If x is an adherent point of \mathcal{B}, that is, if x ∈ ∩ \{cl(X − cl(U)) : U ∈ U\} = X − ∩ \{int(cl(U)) : U ∈ U\}, then x ⋄ A, because U is an open cover of A and for U ∈ U, U ⊆ int(cl(U)). Therefore, the adherent set of \mathcal{B} is contained in X − A, which is an open set. By the hypothesis, there exists an element B ∈ \mathcal{B} such that (X − (X − A)) ∩ B ⋄ G, that is, A ∩ B ⋄ G, that is A ∩ (X − cl(U)) ⋄ G some U ∈ U. This however contradicts our assumption. This completes the proof.

4. **C(G)-COMPACT SPACES AND FUNCTIONS**

**Definition 4.1.** A function f : (X, τ) → (Y, ζ) is said to be θ-continuous [2] at a point x ∈ X if for every open set V of Y containing f(x), there exists an open set U of X containing x such that f(cl(U)) ⊆ cl(V).

**Theorem 4.2.** Let f : (X, τ, G) → (Y, ζ, \mathcal{H}) be a continuous surjection, (X, τ, G) C(G)-compact, and f(G) ⊆ \mathcal{H}. Then (Y, ζ, \mathcal{H}) is C(\mathcal{H})-compact.

**Proof:** Let A be a closed subset of (Y, ζ) and \mathcal{V} any open cover of A in Y. By continuity of f, f⁻¹(A) is an closed subset of X and is such that \{f⁻¹(V) : V ∈ \mathcal{V}\} is a cover of f⁻¹(A) by open
sets in $X$. Hence, by the $C(\mathcal{G})$-compactness of $X$, there exists a finite subcollection \{ $f^{-1}(V_i) : i = 1, 2, 3,..., n$\} such that $f^{-1}(A) - \bigcup_{i=1}^{n} \text{cl}(f^{-1}(V_i)) \notin \mathcal{G}$. Since $f$ is continuous, $\text{cl}(f^{-1}(B)) \subseteq f^{-1}(\text{cl}(B))$ for every subset $B$ of $Y$. Hence we have $f^{-1}(A) - \bigcup_{i=1}^{n} (f^{-1}\text{cl}(V_i)) = f^{-1}(A - \bigcup_{i=1}^{n} \text{cl}(V_i)) \notin \mathcal{G}$. Since $f$ is surjective, $A - \bigcup_{i=1}^{n} \text{cl}(V_i) \notin f(\mathcal{G}) \subseteq \mathcal{H}$. Hence, $Y$ is $C(\mathcal{H})$-compact.

**Theorem 4.3.** Let $f: (X, \tau, \mathcal{G}) \rightarrow (Y, \varsigma, \mathcal{H})$ be a $\theta$-continuous function, $(X, \tau, \mathcal{G})$ $C(\mathcal{G})$-compact, $(Y, \varsigma)$ Hausdorff, and $f(\mathcal{G}) \subseteq \mathcal{H}$. Then $f(A)$ is $\varsigma_{\mathcal{H}}$-closed.

**Proof:** Let $A$ be any closed set in $X$ and $a \notin f(A)$. For each $x \in A$, there exists a $\varsigma$-open set $V_y$ containing $y = f(x)$ such that $a \notin \text{cl}(V_y)$. Now because $f$ is $\theta$-continuous, there exists an open set $U_x$ containing $x$ such that $f(\text{cl}(U_x)) \subseteq \text{cl}(V_y)$. Now, the family \{ $U_x : x \in A$\} is an open cover of $A$. Therefore, there exists a finite subfamily \{ $U_{x_i} : i = 1, 2, 3,..., n$\} such that $A - \bigcup_{i=1}^{n} \text{cl}(U_{x_i}) \notin \mathcal{G}$. But then $f(\text{cl}(U_{x_i})) \subseteq \text{cl}(f(\mathcal{G}))$, that is, $f(A) - f(\bigcup_{i=1}^{n} \text{cl}(U_{x_i})) \notin f(\mathcal{G}) \subseteq \mathcal{H}$ because $f(\mathcal{G})$ is also a grill. Hence, $f(A) - \bigcup_{i=1}^{n} \text{cl}(V_{y_i}) \notin f(\mathcal{G}) \subseteq \mathcal{H}$. Now $a \notin \text{cl}(V_{y_i})$ for any $i$ implies that $a \in Y - \bigcup_{i=1}^{n} \text{cl}(V_{y_i})$ which is open in $(Y, \varsigma)$. That is $Y - \bigcup_{i=1}^{n} \text{cl}(V_{y_i})$ is a neighborhood of $a$ in $(Y, \varsigma)$. However, $(Y - \bigcup_{i=1}^{n} \text{cl}(V_{y_i})) \cap f(A) = f(A) - \bigcup_{i=1}^{n} \text{cl}(V_{y_i}) \notin f(\mathcal{G}) \subseteq \mathcal{H}$. Hence, $a \notin \Phi_\mathcal{H}(f(A), \varsigma)$. Thus $\Phi_\mathcal{H}(f(A), \varsigma) \subseteq f(A)$. This implies $f(A)$ is $\varsigma_{\mathcal{H}}$-closed.

**References**


