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SHORT COMMUNICATION

Linearization of (second order operator)$^{1/2}$

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ABSTRACT

We illustrate by means of the Pauli matrices how to linearize (second order operator)$^{1/2}$ associated to the Weyl and Dirac equations. Besides, we exhibit the square root of the two dimensional Laplacian operator.

Keywords**:** Pauli matrices, Weyl and Dirac equations, Laplacian operator, Cauchy-Riemann conditions

1. INTRODUCTION

In complex variable the holomorphic character of $f(z) = u + i\,v$ implies the Cauchy [1]-Riemann [2] conditions:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad , \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\,, \tag{1}$$

as obtained by D'Alembert [3], Euler [4], Lagrange [5], and Hamilton [6]; such conditions are equivalent to [7, 8]:

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$$\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right)(u + i\,v) = 0, \tag{2}$$

which is important to generalize (1) to the quaternionic case [9].

Moreover, we can construct the matrix version of (1) [10]:

$$\hat{O}\binom{u}{v} = \vec{0} \qquad , \qquad \hat{O} = \begin{pmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & -\frac{\partial}{\partial y} \end{pmatrix}, \tag{3}$$

thus is immediate to see that for an arbitrary vector $\vec{\omega}$:

$$\hat{O}^2\,\vec{\omega} = \nabla^2\,\vec{\omega} \qquad \therefore \qquad \sqrt{\nabla^2}\,I_{2x2} = \hat{O}\,, \tag{4}$$

then $\hat{O}$ is the square root of the Laplacian operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, that is, it was possible to linearize the operator $(\nabla^2)^{1/2}$.

The Pauli matrices [11, 12]:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5}$$

first obtained by Cayley [13] and Sylvester [14], were used [11, 15] in the Schrödinger equation to incorporate the electron spin. So the operator (3) adopts the form $\hat{O} = \sigma_1\,\frac{\partial}{\partial x} + \sigma_3\,\frac{\partial}{\partial y}$, and in quantum mechanics the Laplacian operator is related to the square of the linear momentum operator, then from (4) is natural to think that operators of the type $(quadratic\ in\ \hat{p})^{1/2}$ can be linearized in quantum physics. In Sec. 2 and 3 we apply this idea to motivate the Weyl [16] and Dirac [17] equations, respectively.

2. THE WEYL EQUATION

With the matrices (5) is easy to see that:

$$\sigma_1\,p_1 + \sigma_2\,p_2 + \sigma_3\,p_3 = \begin{pmatrix} p_3 & p_1 - i\,p_2 \\ p_1 + i\,p_2 & -p_3 \end{pmatrix}, \tag{6}$$

or after squaring, $(\vec{\sigma}\cdot\vec{p})^2 = ({p_1}^2 + {p_2}^2 + {p_3}^2)\,I_{2x2}$, then is natural to propose the following linearization [18]:

$$\vec{\sigma}\cdot\vec{p} = \pm\sqrt{{p_1}^2 + {p_2}^2 + {p_3}^2}\,I\,, \tag{7}$$

with p_r as quantum mechanical operators. A relativistic free particle with zero rest mass has the energy $E = c({p_1}^2 + {p_2}^2 + {p_3}^2)^{1/2}$ and due to (7):

$$c\,(\vec{\sigma}\cdot\vec{p}) = \pm i\,\hbar\,I\,\frac{\partial}{\partial t}\,, \tag{8}$$

that for each sign operates on a two-component vector as:

$$c\,(\vec{\sigma}\cdot\vec{p})\,\phi_R = i\,\hbar\,\frac{\partial}{\partial t}\,\phi_R \qquad , \qquad c\,(\vec{\sigma}\cdot\vec{p})\,\phi_L = -i\,\hbar\,\frac{\partial}{\partial t}\,\phi_L\,, \tag{9}$$

which are the equations proposed by Weyl [16, 19] for the neutrino, spin $1/2$; the next Section shows how the Dirac equation for a free particle leads to (9).

3. THE DIRAC EQUATION

With the 4×4 matrices [20-22]:

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \qquad , \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{10}$$

we have that:

$$m_0\,c\,\beta + \vec{\alpha}\cdot\vec{p} = \begin{pmatrix} m_0 c\,I & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m_0 c\,I \end{pmatrix},$$

that is, $(m_0 c\,\beta + \vec{\alpha}\cdot\vec{p})^2 = ({m_0}^2 c^2 + {p_1}^2 + {p_2}^2 + {p_3}^2)\,I_{4x4}$ which generalizes the relation (7):

$$m_0 c\,\beta + \vec{\alpha}\cdot\vec{p} = \sqrt{{m_0}^2 c^2 + {p_1}^2 + {p_2}^2 + {p_3}^2}\;\, I\,, \tag{11}$$

this in turn, together with $E = c\sqrt{{m_0}^2 c^2 + p^2} = i\,\hbar\,\frac{\partial}{\partial t}$, generates the Dirac equation [17, 20, 23-25] for a free electron, spin $1/2$ [26]:

$$(m_0 c^2\,\beta + c\,\vec{\alpha}\cdot\vec{p})\,\psi_{4x1} = i\,\hbar\,\frac{\partial}{\partial t}\,\psi_{4x1}\,. \tag{12}$$

when $m_0 = 0$, the equation (12) acquires the form:

$$\begin{pmatrix} 0 & c\,\vec{\sigma}\cdot\vec{p} \\ c\,\vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix}\psi = i\,\hbar\,\frac{\partial}{\partial t}\,\psi\,, \qquad \psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{13}$$

that is:

$$c\,(\vec{\sigma}\cdot\vec{p})\,\xi = i\,\hbar\,\frac{\partial\eta}{\partial t} \qquad , \qquad c\,(\vec{\sigma}\cdot\vec{p})\,\eta = i\,\hbar\,\frac{\partial\xi}{\partial t}\,,$$

and by adding and subtracting these equations we get the Weyl equations (9) [22], with $\phi_R = (\xi + \eta)/\sqrt{2}$ and $\phi_L = (\xi - \eta)/\sqrt{2}$.

4. CONCLUSIONS

The expressions (4), (7) and (11) exhibit linear forms of the square root of important operators in Mathematical Physics, and from here it follows the relevant role of the Pauli matrices. Eberlein [10] comments that in the first volume of his treatise on spinors, Cartan remarks that the equations describing the so-called spin spherical harmonics form the simplest example of the Dirac square-root process.

References

[1] A. Cauchy, Mémoire sur les intégrales définies, Paris (1825).

[2] B. Riemann, Grundlagen für eine allgemeine theorie der functionen einer veränderlichen complexen grösse, Göttingen (1851).

[3] J. D'Alembert, Essai d'une nouvelle théorie de la résistance del fluides, Paris (1752) 316-361.

[4] L. Euler, Continuation des recherches sur la théorie du movement des fluides*, Mém. Acad. Berlin* 11 (1757).

[5] J. Lagrange, Solutions de différents problems du calcul integral, *Miscellanea Taurinensia* 3 (1766) 179-380.

[6] W. R. Hamilton, Theory of conjugate functions or algebraic couples, *Trans. Roy. Irish Acad.* 48 (1837) 293-422.

[7] A. Cauchy, Résumé des lecons données á l'Ecole Polytechnique sur les applications du calcul infinitésimal, Paris (1823).

[8] C. Lanczos, The tensor analytical relationships of Dirac's equation, *Zeits. für Physik* 57 (1929) 447-473.

[9] B. E. Carvajal-Gámez, F. Gallegos-Funes, J. López-Bonilla, On the Fueter-Lanczos conditions, *J. Vect. Rel.* 4, No. 3 (2009) 137-141.

[10] W. F. Eberlein, Cauchy-Riemann operator, *Am. J. Phys.* 35, No. 1 (1967) 53.

[11] W. Pauli, Zur quantenmechanik des magnetischen electrons, *Zeits. für Physik* 43 (1927) 601-623.

[12] P. Jordan, E. Wigner, Über das paulische aquivalenzverbot, *Zeits. für Physik* 47 (1927) 631-651.

[13] A. Cayley, A memoir on the theory of matrices, *London Phil. Trans.* 148 (1858) 17-37.

[14] J. Sylvester, On quaternions, nonions and sedenions, *Johns Hopkins Circ.* 3 (1884) 7-9.

[15] C. Darwin, The electron as a vector wave, *Proc. Roy. Soc. London* A116 (1927) 227-253.

[16] H. Weyl, Electron and gravitation, *Zeits. für Physik* 56 (1929) 330-352.

[17] P. A. M. Dirac, The quantum theory of the electron, *Proc. Roy. Soc. London* A117 (1928) 610-624 and A118 (1928) 351-361.

P. A. M. Dirac, The quantum theory of the electron, *Proc. Roy. Soc. London* A118 (1928) 351-361.

[18] P. A. M. Dirac, Pretty Mathematics, *Int. J. Theor. Phys.* 21, No. 8-9 (1982) 603-605.

[19] B. L. van der Waerden, Group theory and quantum mechanics, Springer-Verlag, Berlin (1974).

[20] W. Greiner, Relativistic quantum mechanics, Springer-Verlag, Berlin (1990).

[21] J. López-Bonilla, L. Rosales-Roldán, A. Zúñiga-Segundo, Dirac matrices via quaternions, *J. Sci. Res.* 53 (2009) 253-255.

[22] P. Lam-Estrada, M. Maldonado, J. López-Bonilla, R. López-Vázquez, Transformations of Dirac spinor under boosts and 3-rotations, *Prespacetime Journal* 7, No. 1 (2016) 148-153.

[23] H. S. Kragh, The genesis of Dirac's relativistic theory of electrons, *Archive for History of Exact Sciences* 24, No. 1 (1981) 31-67.

[24] H. S. Kragh, Relativity and quantum theory from Sommerfeld to Dirac, Ann. der Phys. 9, No. 11-12 (2000) 961-974.

[25] G. Farmelo (Editor), It must be beautiful: Great equations of modern science, Granta Books, London (2002).

[26] N. Hamdan, A. Chamaa, J. López-Bonilla, On the relativistic concept of the Dirac's electron spin, *Lat. Am. J. Phys. Educ.* 2, No. 1 (2008) 65-70.