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Spacetimes embedded into E_6

J. Yaljá Montiel-Pérez[1], J. H. Caltenco[2], R. Linares y M.[2], J. López-Bonilla[2]

[1]Centro de Investigación en Computación, Instituto Politécnico Nacional, CDMX, México

[2]ESIME-Zacatenco, Instituto Politécnico Nacional, Col. Lindavista 07738, CDMX, México

*E-mail address: jlopezb@ipn.mx

ABSTRACT

We study the local and isometric embedding of Riemannian spacetimes into the pseudo-Euclidean flat E_6. Our main purpose is to pinpoint the known results with the corresponding references, and to indicate the main routes and connections from here to the still open problems in the field. Some new results are also included in order to round off our discussion.

Keywords**:** Embedding of Riemannian 4-spaces, Local and isometric embedding, E_6

1. INTRODUCTION

Here we study spacetimes admitting local and isometric embedding into E_6, i.e. 4-spaces of class two [1, 2]. When we conceive a certain V_4 as a subspace of a flat N-dimensional space $(N \leq 10)$, new geometric objects arise (like various second fundamental forms and Ricci vectors) which enrich the Riemannian structure and offer the possibility [3-6] of reinterpreting physical fields using them. Unfortunately to date such hope has not been realized since it has been extremely difficult to establish a natural correspondence between the quantities governing the extrinsic geometry of V_4 and physical fields. In spite of this, one cannot but accept the great value of the embedding process, for it combines harmoniously such themes as the Petrov [1, 7-12] and the Churchill-Plebanski [13-18] classifications, exact

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solutions and their symmetries [1, 2, 19-34], the Newman-Penrose formalism [1, 35-38], and the kinematics of time-like and null congruences [1, 2, 23, 24, 36, 39, 40]. On the other hand, it offers the possibility of obtaining exact solutions that cannot be obtained by any other means [1].

In this work we study spacetimes (here denoted as V_4) admitting a local and isometric embedding into the pseudo-Euclidian flat space E_6, that is, spacetimes of class two [1, 2]. The article is organized as follows. In Sec. 2 we expound the Gauss, Codazzi and Ricci equations for an spacetime V_4 embedded into E_6. In Sec. 3 we analyse those equations to state the necessary algebraic conditions for a spacetime to be of class two. Some of these stated conditions are already known but some others are new. In Sec. 4, we again employ the results of the prevoius sections to investigate a vacuum V_4 which leads us to the Collinson [36] and the Yakupov [41] theorems; our results are then used in some metrics obtaining known results and a few new ones.

2. GAUSS-CODAZZI-RICCI EQUATIONS

In this Section we expound the governing equations of the embedding of a V_4 into a flat 6-dimensional space. These are the Gauss-Codazzi-Ricci equations (EGCR), which constitute an algebraic and differential system which usually is not very easy to solve to obtain the three important quantities for the embedding process, namely, the two second fundamental form tensors and the Ricci vector. Let us mention that in all of our discussion we use the tensorial form of the EGCR.

In the embedding problem, let us recall, the intrinsic geometry of the spacetime (mainly determined through the metric tensor g_{ab}) is assumed given, what is required is the extrinsic geometry of V_4 respect to E_6. In the stated case, we have two additional dimensions which means that V_4 now posses two normals (in E_6) with indicators $\varepsilon_1 = \pm 1$ and $\varepsilon_2 = \pm 1$; this requires two second fundamental form tensors ${}^a b_{ij} = {}^a b_{ji}$, $a = 1,2$ and a Ricci vector A_r, these quantities cannot be prescribed arbitrarily since they determine the extrinsic geometry of the spacetime embedded into E_6. For a local and isometric embedding to be realizable, it is necessary and sufficient that the EGCR equations hold [2, 25, 36, 42]. These algebraic and differential relationships between the new geometric degrees of freedom ${}^c b_{ji}$ and A_r, are the conditions for their existence; the set of equations are:

$$R_{acpq} = \sum_{r=1}^{2} \varepsilon_r ({}^r b_{ap}\, {}^r b_{cq} - {}^r b_{aq}\, {}^r b_{cp}), \qquad \text{Gauss} \qquad (1a)$$

$${}^1 b_{ac;r} - {}^1 b_{ar;c} = \varepsilon_2 (A_c\, {}^2 b_{ar} - A_r\, {}^2 b_{ac}), \qquad \text{Codazzi} \qquad (1b)$$

$${}^2 b_{ac;r} - {}^2 b_{ar;c} = -\varepsilon_1 (A_c\, {}^1 b_{ar} - A_r\, {}^1 b_{ac}), \qquad \text{Codazzi} \qquad (1c)$$

$$F_{jr} \equiv A_{r,j} - A_{j,r} = {}^1b_j{}^{c2}b_{cr} - {}^1b_r{}^{c2}b_{cj}, \quad \text{Ricci} \tag{1d}$$

where a semicolon indicates a covariant (with respect to the coordinates of V_4) derivative, a comma an ordinary derivative and R_{abcd} stands for the V_4 Riemann curvature tensor [1].

Notice the similitude of (1*b*) and (1*c*): if in the former we replace, respectively, ${}^1b_{cr}$ and ε_2 by ${}^2b_{cr}$ and $-\varepsilon_1$, we get the latter. The set of equations (1*a*) to (1*d*) is, are written entirely in terms of the V_4 we try to embed and, in general, are rather difficult to solve; it is natural thus to look for a simplification. This can be achieved reasoning by analogy with the Thomas theorem, valid for any spacetime V_4 embedded into E_5 [38, 43, 44], namely, if a class one spacetime has $\det(b_{ij}) \neq 0$ then Gauss equation (1*a*) implies Codazzi's (1*b*)-(1*c*), therefore, in the case of class one spacetimes, the search of the second fundamental form b_{ij} is mainly algebraic simplifying the embedding process. For class two spacetimes the process is more difficult, for, unfortunately, the consideration of at least one differential equation becomes necessary.

Gupta-Goel [30] showed that, when $\det({}^2b_{cr}) \neq 0$, the second of Codazzi equations (1*c*) and the Ricci equation (1*d*) follow from the equations of Gauss (1*a*) and the first of Codazzi (1*b*). They used this result to embed every static spherically symmetric spacetime into E_6 (the embedding into E_6 of such spacetimes had been carried out explicitly before by Plebañski [21, 33]). This shows that the metrics of Schwarzchild and of Reissner-Nördstrom can be embedded into E_6 in spite of the fact that it is not possible to embed them into E_5 [38, 39, 44, 45].

Goenner [46] has performed a complete study of the interdependence of the ECGR; for example, he proved two theorems that are more general than the Gupta-Goel [30] result, namely:

If the rank of ${}^2b_{at} \geq 3$, *then (1a), (1b) and (1c) imply (1d)*; (2*a*)

the second result says that:

if the rank of ${}^2b_{at} \geq 4$, *then (1a) and (1b) imply (1c) and (1d).* (2*b*)

These two theorems are valid for any R_n embedded into E_{n+2} and are a consequence of the Bianchi identities [1] for the curvature tensor.

For class-2 spacetimes the intrinsic geometry of V_4 is given, we are left thus with the problem of constructing ${}^rb_{ac}$, ε_r, $r = 1,2$ and A_p as solutions to equations (1*a*)-(1*d*). Before we address the problem of finding solutions to those equations it can be convenient to verify certain other conditions that every class 2 spacetime should satisfy.

If such conditions are found not to hold, it is pointless to try to find the second fundamental forms or the Ricci vector.

3. NECESSARY CONDITIONS FOR THE EMBEDDING OF V_4 INTO E_6

To show that a certain spacetime cannot be embedded into E_6 needs showing that the EGCR admit no solution; but proving this directly is usually a very complex task, therefore, it is better to search for indirect evidence in the form of necessary conditions for the embedding. If these conditions are not satisfied we can be sure that no solution of the EGCR exist. In this section we address some necessary conditions for a V_4 to be a class two spacetime.

First a result that Matsumoto proved [47] for a Riemannian R_4 (that is, a space with a positive definite metric) embedded into E_6, whose general validity for spacetimes (that is pseudo-Riemannian V_4 spaces with a non-positive definite metric) was first noticed by Goenner [42]. For every class-2 V_4, we must have:

$$F^{ij}F^{kr}+F^{ik}F^{rj}+F^{ir}F^{jk}=-\frac{\varepsilon_1\varepsilon_2}{2}(R^{acij}R_{ac}{}^{kr}+R^{acik}R_{ac}{}^{rj}+R^{acir}R_{ac}{}^{jk}), \qquad (3a)$$

with F^{ab} defined as in (1*d*). If we multiply Matsumoto's expression times the Levi-Civita tensor [1, 37] η_{pjkr} we get:

$$^*C_2\equiv{}^*C_{abcd}C^{abcd}=-2\varepsilon_1\varepsilon_2 F_2\equiv-2\varepsilon_1\varepsilon_2 F_{ab}{}^*F^{ab}, \qquad (3b)$$

where C_{abcd} is the Weyl conformal tensor and $^*C_{abcd}$ is its dual [1, 11, 48]. F_{ab} is an extrinsic quantity but equation (3*b*) indicates that the invariant $F_{ab}{}^*F^{ab}$ has to be intrinsic because it is simply proportional to an invariant of the Weyl tensor.

Not mattering what the class of a spacetime, it is very easy to show the interesting identity (that we believe has not been previously noticed):

$$^*R^{*tjkc*}R_{arkc}R_{pj}{}^{ar}=\frac{Y}{4}\delta^t_p, \qquad (4a)$$

where we have defined:

$$Y\equiv{}^*R^{*tjkc*}R_{arkc}R_{tj}{}^{ar}=-{}^*C_3+\frac{R}{2}{}^*C_2+6{}^*R_3,$$
$$^*C_3\equiv{}^*C_{abcd}C^{cdpq}C_{pq}{}^{ab} \quad , \quad {}^*R_3\equiv{}^*R_{ijab}R^{ia}R^{jb}, \qquad (4b)$$

where $R_{ab}\equiv R^i_{abi}$ is the Ricci tensor, $R\equiv R^b_b$ is the scalar curvature, $^*R_{abcd}$ is the simple dual and $^*R^*_{abcd}$ the double dual of the Riemann curvature tensor R_{abcd} [1, 49, 50].

Employing only the Gauss equation (1*a*), Yakupov [51] has been able to obtain a very general necessary condition for a spacetime to be of class two, see also Goenner [2]. Yakupov result asserts that:

$$\text{Every } V_4 \text{ embedded into } E_6 \text{ should have } Y = 0; \tag{4c}$$

this is a restriction upon the intrinsic geometry of a class two spacetime. Using it, we can ascertain that if a V_4 is such that $Y \neq 0$, then its embedment into E_6 is not possible. We can take as an example the Kerr metric [1, 52-56] which has $R = 0$, ${}^*R_3 = 0$ and ${}^*C_3 \neq 0$; with such values, it is clear that $Y \neq 0$ and then it is not possible to embed a spinning black hole into E_6. Kerr's metric do accept [57] embedding into E_9 also but it is not yet known whether it can be embedded into E_7 or into E_8.

One has to keep in mind that Yakupov's result is only a necessary condition since the mere fact that $Y = 0$ cannot guarantee the embedding since only Gauss equation is needed to prove it [51]. Gödel's metric [1, 58] is an example in which ${}^*R_3 = {}^*C_2 = {}^*C_3 = 0$ and $Y = 0$, but even so it is not known yet whether it is possible to embed it into E_6 or not [2, 22, 25, 32, 38, 39, 59-64].

Let us consider now equation (1*b*), first take its covariant derivative with respect to x^p, then rotate cyclically the indices *c*, *r* and *p* obtaining in this way three equations, finally sum these equations to get:

$$R^q_{arp}\,{}^1b_{qc} + R^q_{apc}\,{}^1b_{qr} + R^q_{acr}\,{}^1b_{qp} = \varepsilon_2 (F_{rp}\,{}^2b_{ac} + F_{pc}\,{}^2b_{ar} + F_{cr}\,{}^2b_{ap}), \tag{5a}$$

and, doing similarly with (1*c*), we also get:

$$R^q_{arp}\,{}^2b_{qc} + R^q_{apc}\,{}^2b_{qr} + R^q_{acr}\,{}^2b_{qp} = -\varepsilon_1 (F_{rp}\,{}^1b_{ac} + F_{pc}\,{}^1b_{ar} + F_{cr}\,{}^1b_{ap)}. \tag{5b}$$

These equations are equivalent to equation (8) in [51] and to equations (A2.5), (A2.6) in Hodgkinson [65] although you must note that he only considers the case $R_{ab} = 0$.

This equations offer a way for evaluating the second second-fundamental form ${}^2b_{ab}$ as a function of F^{ab}, ${}^*R_{abcd}$ and ${}^1b_{ab}$, which can be seen as follows, multiply (5*a*) and (5*b*) times η^{trpc}, to obtain:

$${}^*R^{ijkr\,1}b_{jk} = \varepsilon_2\, {}^*F^{ij\,2}b^r{}_j, \qquad {}^*R^{ijkr\,2}b_{jk} = -\varepsilon_1 {}^*F^{ij\,1}b^r{}_j, \tag{5c}$$

take the product of equations (5*c*) times F_{ic}, and keeping in mind the identity [66-69] ${}^*F_{ac}F^{ic} = F_2 \delta^i_a / 4$, we get:

$$F_2\,{}^2b_{ij} = 4\varepsilon_2\,{}^*R^{acr}{}_i\,{}^1b_{cr}F_{aj}\,, \qquad F_2\,{}^1b_{ij} = -4\varepsilon_1\,{}^*R^{acr}{}_i\,{}^2b_{cr}F_{aj}\,, \tag{5d}$$

from which, if $F_2 = 0$, we can evaluate ${}^1b_{ac}$ in terms of the quantities mentioned above. Notice that (5*d*) reduces to a trivial identity for vacuum class-2 spacetimes as follows from Yakupov result that, if we are in a vacuum, $F_{ij} = 0$ always [41]. Other point worth mentioning is that, on multiplying (5*a*) times $\varepsilon_1\,{}^1b^a{}_t$ and (5*b*) times $\varepsilon_2\,{}^2b^a{}_t$ and summing to each other the resulting equations, we obtain equation (3*a*) again.

Furthermore, if in equations (5*a*) and (5*b*) we contract *a* with *r*, we get $({}^s b = {}^s b^c{}_c, s = 1, 2)$:

$$R^q{}_p\,{}^1b_{qc} - R^q{}_c\,{}^1b_{qp} = \varepsilon_2(F_{qp}\,{}^2b^q{}_c - F_{qc}\,{}^2b^q{}_p + {}^2bF_{pc}), \tag{5e}$$

$$R^q{}_p\,{}^2b_{qc} - R^q{}_c\,{}^2b_{qp} = -\varepsilon_1(F_{qp}\,{}^1b^q{}_c - F_{qc}\,{}^1b^q{}_p + {}^1bF_{pc}); \tag{5f}$$

these equations reproduce equation (9) in Yakupov [51] - though this author employs $R_{ab} \equiv Rg_{ab}/4$ - and equation (A2.7) in [65] for the case $R_{ab} = 0$. Using (5*e*) and (5*f*) it is elementary to obtain:

$${}^*F^{pc}R^q{}_p\,{}^1b_{qc} = \frac{\varepsilon_2}{4}F_2\,{}^2b, \qquad \text{and} \qquad {}^*F^{pc}R^q{}_p\,{}^2b_{qc} = \frac{-\varepsilon_1}{4}F_2\,{}^1b, \tag{5g}$$

equations (5*e*), (5*f*) and (5*g*) are interesting for General Relativity because they involve the Ricci tensor which is directly related with the sources of the gravitational field.

On multiplying (5*e*), times ${}^2b^p{}_t$ and antisymmetrizing indices *c* and *t*, and times ${}^1b^p{}_t$ and antisymmetrizing the same indices than before, and finally substracting the former equation form the latter, we get:

$$R^{ap}{}_{ct}F_{ap} = 2[R^q{}_tF_{qc} - R^q{}_cF_{qt} + R_{pq}({}^1b^q{}_c\,{}^2b^p{}_t - {}^1b^q{}_t\,{}^2b^p{}_c)] \tag{6a}$$

or, using the expression for the Weyl tensor [1],

$$C_{apct}F^{ap} = 2R_{pq}({}^1b^q{}_c\,{}^2b^p{}_t - {}^1b^q{}_t\,{}^2b^q{}_c) + R_{qt}F^q{}_c - R_{qc}F^q{}_t - \frac{R}{3}F_{ct}\,. \tag{6b}$$

For Einstein spaces, in which $R_{ab} \equiv Rg_{ab}/4$, (6*b*) implies that F^{ap} is eigentensor of the conformal tensor [8, 70]:

$$C_{ijap}F^{ap} = -\frac{R}{3}F_{ij}\,; \tag{6c}$$

the three relations (6*a*), (6*b*) and (6*c*) are contributions of this work though have been anticipated by Hodgkinson [65] for the case in which $R_{ab} = 0$.

In terms of C_{ijkr}, equations (5*c*) become:

$$\left({}^{*}C^{ijkr} + \frac{1}{2}\eta^{ijra}R^{k}_{a}\right){}^{1}b_{jk} = +\varepsilon_2 {}^{*}F^{ij}\ {}^{2}b^{r}{}_{j}\,,$$
$$\left({}^{*}C^{ijkr} + \frac{1}{2}\eta^{ijra}R^{k}_{a}\right){}^{2}b_{jk} = -\varepsilon_1 {}^{*}F^{ij}\ {}^{1}b^{r}{}_{j}\,, \qquad (7a)$$

from here, is straightforward to obtain:

$${}^{*}C^{ijkcr}b_{jk}{}^{r}b_{ic} = \frac{\varepsilon_r}{4}{}^{*}C_2, \quad r = 1,2, \qquad {}^{*}C^{ijkc2}b_{ic}{}^{1}b_{jk} = 0 \qquad (7b)$$

$${}^{*}C^{ijkc}\ {}^{1}b_{jk} = \frac{\varepsilon_2}{2}({}^{*}F^{ij2}b^{c}{}_{j} + {}^{*}F^{cj2}b^{i}{}_{j}), \qquad (7c)$$

to finalize, from (4b), (4c) and (6b) we can obtain the starting identity:

$${}^{*}C_{apct}F^{ap}F^{ct} = \frac{1}{3}\varepsilon_1\varepsilon_2({}^{*}C_3 - 6{}^{*}R_3) = \frac{\varepsilon_1\varepsilon_2}{6}R\,{}^{*}C_2\,. \qquad (7d)$$

Given equations (3)-(7), the main algebraic relations which must hold in any class-2 4-spacetimes have been established; to this date nobody has been capable of establishing any (perhaps because they do not exist!) necessary differential conditions for the embedding of V_4 into E_6. For spacetimes embedded into E_5 the only known necessary differential condition was advanced by our group [62]. Nor have been found any necessary algebraic and/or differential conditions for 4-spacetimes embedded into E_7, if other were the case, these would permit the study of the embedding problem for the Kerr metric mentioned before.

4. NECESSARY CONDITIONS FOR THE EMBEDDING OF VACUUM CLASS-2 SPACETIMES

In this section we study the embedding properties of vacuum ($R_{ab} = 0$) spacetimes. The conditions for being class-2 can be best described in terms of properties of the null geodesic congruences spanned by Debever-Penrose vectors (DP) [1, 8, 11, 35, 55, 71].

Collinson has studied the problem of embedding of a vacuum 4-spacetime in which a doubly degenerate DP vector n^r spanning a null geodesic congruence exist, and has obtained the following two necessary conditions [36]:

In every vacuum class-2 V_4 with Petrov type-II a null geodesic congruence should exist with the three optical scalars equal to zero (8*a*)

that is, if κ, σ and ρ are such NP spin coefficients [1, 12, 35, 37, 55, 72], we must have for the congruence: $\kappa = 0$ (geodesic), $\sigma = 0$ (shearfree), $\rho - \bar{\rho} = 0$ (no rotation), and $\rho + \bar{\rho} = 0$ (no expansion); also

In Petrov-type D or N vacuum 4-spacetimes embedded into E_6,
a null geodesic congruence with no shear and no rotation exist; (8*b*)

that is, spacetimes with $\kappa = \sigma = \rho - \bar{\rho} = 0$.

Studying the embedding problem Yakupov came across [41, 65] the following two results published without proof in a Russian journal ([1], p, 369):

No Petrov type III vacuum spacetime V_4 *may be embedded into* E_6, (8*c*)

together with (8*c*), Yakupov got [1] that:

in all class-2 vacuum spacetimes $F_{ac} = 0$, (8*d*)

then from equation (1*d*), it follows that ${}^1b^r{}_c$ and ${}^2b^r{}_c$ commute with each other or, in other words, that the Ricci vector A_r is a gradient. From section 3 and Yakupov's result (4*c*), we get:

$$ {}^*C_2 = 0 \tag{9a}$$

and:

$$ {}^*C_3 = F_2 = 0, \quad {}^*C^{ijkr}\ {}^p b_{jk} = 0, \quad p = 1,2. \tag{9b}$$

Let us pinpoint that (9*a*) is valid for every vacuum spacetime (including the Petrov type-I), Conditions (9*a*) and (9*b*) may be extended to all Einstein spaces, the proof is not given here, as:

any algebraically special V_4 *embedded into* E_6 *must have* $F_{ab} = 0$, (10)

and, from (3*b*), this can be seen to imply that ${}^*C_2 = 0$. On the other hand, Goenner [2, 42, 73] claimed without proof that:

${}^*C_2 = 0$ *in every spacetime of class two and Petrov type different from I*, (11)

however, as he recognize later [74] this is only true for Einstein and vacuum spacetimes.

5. APPLICATIONS

The results obtained in the previous sections have applications to known metrics in GR as we exhibit in the following cases.

a) The type D Schwarzschild metric [1] ($R_{ab}=0$).

We know [21, 30, 33, 38, 39, 44, 75-77] that this metric can be embedded into E_6.

b) The type D Kerr metric ($R_{ab}=0$).

This metric generalizes [78] the previous one and corresponds to a black hole with no electric charge. This cannot be embedded into E_6 because, as $Y\neq 0$ and ${}^*C_2\neq 0$, it violates Yakupov condition.

c) Petrov type III vacuum metric [7].

The metric:

$$ds^2=\exp(x^2)[\exp(-2x^4)(dx^1)^2+(dx^2)^2]+2dx^3dx^4-x^2(x^3+\exp(x^2))(dx^4)^2\,, \qquad (12)$$

has been embedded into E_7 by Collinson [25, p. 410] therefore, by condition (8*c*) we must conclude that (12) is of class three

d) The vacuum C metric [79, 80].

The vacuum type D metric:

$$ds^2=(x+y)^{-2}(f^{-1}dx^2+h^{-1}dy^2+fd\phi^2-hdt^2)\,, \qquad (13)$$

$f\equiv x^3+ax+b$, $h\equiv y^3+ay-b$, where *a*, *b* are constants.

satisfies the necessary conditions (4*c*) and (9*a*). In [31, 34] it was shown that we cannot embed the metric into E_6; however, Rosen [22] has explicitly embedded (13) into E_8. We still do not known whether the C-metric can be embedded into E_7 or not.

e) The type D Taub metric ($R_{ab}=0$).

The line element is [1, 37, 80, 81]:

$$ds^2=f^{-1}(dx^2-dt^2)+f^2(dy^2+dz^2)\,, \qquad f\equiv(1+kx)^{1/2}\,, \qquad (14a)$$

where the constant $k \neq 0$. According to Goenner [2, p. 455] this metric is of class two, we next exhibit explicitly the embedding of (14*a*) into E_6 since apparently it has not been previously published:

$$z^1 = A + \frac{f}{2}(y^2 + z^2 - 1), \qquad z^2 = A + \frac{f}{2}(y^2 + z^2 + 1),$$
$$z^3 = yf, \quad z^4 = zf \quad , \quad z^5 = f^{-1/2}\cosh t \quad , \quad z^6 = f^{-1/2}\sinh t, \tag{14b}$$

where $A = x/k - f^{-2}/16$. Therefore (14*a*) reduces to:

$$ds^2 = (dz^1)^2 - (dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2 - (dz^6)^2. \tag{14c}$$

and the embedding into E_6 is explicit.

f) Type D Kasner metric ($R_{ab} = 0$) [82].

The type D Kasner metric [37] is of class two (see [1, p. 370]). The embedding class of the Kasner type I (2 $\leq$class $\leq$ 3, according to Goenner [2, p. 455]) is not known.

g) The vacuum type III Siklos metric [83].

The explicit form of the solution is (see [1, p. 378]):

$$ds^2 = r^2 x^{-3}(dx^2 + dy^2) - 2dudr + \frac{3}{2}xdu^2; \tag{15}$$

given condition (8*c*) this metric cannot be embedded into E_6, but it can be embedded into E_8 because it is of the type of Robinson-Trautman (see J9 of Collinson [25]); as it is not known whether it is a subspace of E_7 or not, we do not know either its embedding class.

h) Held [84]-Robinson [85].

Held and Robinson have derived type III metrics with $R_{ab} = 0$ whose degenerate null congruence has rotation, thus by (8*c*) these spacetimes do not admit embedding into E_6.

i) Petrov type N [7, p. 384], $R_{ab} = 0$.

The spacetimes with line element:

$$ds^2 = -2dx^1dx^4 + \sin^2 x^4 (dx^2)^2 + \sinh x^4 (dx^3)^2, \tag{16}$$

is of class two as proved in J11 of Collinson [25].

j) Gravitational waves [1,37] ($R_{ab} = 0$).

The metric for gravitational waves along the axis x^3 is [86]:

$$ds^2 = (dx^1)^2 + (dx^2)^2 - 2dx^3dx^4 + 2H(x^1, x^2, x^4)(dx^4)^2, \tag{17}$$

where $H_{,11} + H_{,22} = 0$, has the Petrov type *N* and it can be embedded into E_6 (see J8 of [25].

k) Hauser [87].

Hauser has derived a type *N* metric with $R_{ab} = 0$ but its principal degenerate congruence posses rotation, $(\rho - \overline{\rho}) \neq 0$, thus (8*b*) does not hold and thence it is not of class two. This is a biparametric metric, therefore it could be embedded, depending on the values taken by its parameters, into some E_r with $r = 7, \ldots, 10$.

6. CONCLUSIONS

Let us end this article pinpointing some open problems in the field which have been mentioned in the text:

1) There have not been found any-with reasonable physical sources-type I or II metrics embedded into E_6.
2) The affirmation (11) has not been proved but, otherwise, not a single counterexample is known [88].
3) Condition (8*c*) lacks an explicit proof.
4) It is not known whether Gödel's metric [89] can be embedded into E_6 or not.
5) No a single differential necessary condition for the embedding of V_4 into E_6 is known.
6) It is necessary to analyse if for class two it is possible to obtain analogous identities to those valid for class one 4-spaces (obtained in [44, 61, 90-92]) expressing ${}^1b_{ac}$, ${}^2b_{ac}$, and A_j in terms of the intrinsic geometry of the spacetime.
7) To determine if Kerr, Siklos and *C* metrics can be embedded into E_7.
8) To make a complete study of Petrov type *D* Einstein spacetimes of class two [93].

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