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SHORT COMMUNICATION

Todorov's formula involving Stirling numbers of the second kind and Nörlund polynomials

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ABSTRACT

We use the Todorov's formula for the generalized Bernoulli numbers, in terms of Stirling numbers of the second kind, to deduce the Guo-Qi and Schläfli identities. Our approach is based in the duality property between the Stirling numbers, and in the Nörlund polynomials. We also consider the Janjic's definitions for the Stirling numbers.

***Keywords*:** Bernoulli numbers, Nörlund polynomials, Stirling numbers, Duality property, Hoppe's formula, Generalized chain rule of differentiation

1. INTRODUCTION

Todorov [1, 2] deduced the relation:

$$B_n^{(z)} = \sum_{k=0}^{n} (-1)^k \, \frac{\binom{n+z}{n-k}\binom{z+k-1}{k}}{\binom{n+k}{k}} \, S_{n+k}^{[k]}\,, \tag{1}$$

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where: $S_r^{[k]}$ are the Stirling numbers of the second kind [2-6], $B_n^{(z)}$ are the generalized Bernoulli numbers or Nörlund polynomials defined via the following generating function [7, 8]:

$$\sum_{n=0}^{\infty} B_n^{(z)} \frac{x^n}{n!} = \left(\frac{x}{e^x-1}\right)^z , \tag{2}$$

and $B_n = B_n^{(1)}$ are the Faulhaber [9]-Bernoulli [10] numbers [2-4, 11, 12].

In Sec. 2 we show that (1) and (2) allow obtain the Guo-Qi and Schläfli identities, and also the generating function for $S_r^{[k]}$. In Sec. 3 we exhibit that the definitions of Janjic for the Stirling numbers are consequences of the formulas of Hoppe and Quaintance-Gould [4].

2. NÖRLUND POLYNOMIALS

The expression (1) with $z = 1$ implies the following relation of Guo-Qi [13, 14] involving the Bernoulli and Stirling numbers of the second kind:

$$B_n = \sum_{k=0}^{n} (-1)^k \frac{\binom{n+1}{k+1}}{\binom{n+k}{k}} S_{n+k}^{[k]} , \tag{3}$$

this formula (3) also was proved in [4, 15-17].

If in (2) we employ $z = -k$ and apply the relation of Carlitz [8, 18]:

$$B_n^{(-k)} = \frac{1}{\binom{n+k}{k}} S_{n+k}^{[k]} , \tag{4}$$

we obtain the generating function for Stirling numbers [4]:

$$\sum_{m=k}^{\infty} S_m^{[k]} \frac{x^m}{m!} = \frac{1}{k!} (e^x - 1)^k . \tag{5}$$

We know the duality property [4, 19-23]:

$$S_{-n}^{(-N)} = (-1)^{N+n} S_N^{[n]} , \qquad S_{-n}^{[-N]} = (-1)^{N+n} S_N^{(n)} , \tag{6}$$

where $S_r^{(k)}$ are the Stirling numbers of the first kind [4, 11, 24]. Then, from (4) and (6):

$$B_k^{(n)} = \frac{1}{\binom{k-n}{-n}} S_{-(n-k)}^{[-n]} = \frac{(-1)^k}{\binom{k-n}{-n}} S_n^{(n-k)} , \tag{7}$$

but $\binom{k-n}{-n} = (-1)^k \binom{n-1}{k}$, thus (7) implies the following identity [4, 8]:

$$S_n^{(n-k)} = \binom{n-1}{k} B_k^{(n)}. \tag{8}$$

Now, we apply (1) with $z = -k$, and (4) to deduce:

$$S_{k+m}^{[m]} = (-1)^k \sum_{j=0}^{k} (-1)^j \binom{m+k}{m-j}\binom{m-j-1}{k-j} S_{k+j}^{[j]} = (-1)^k\, S_{-m}^{(-(k+m))},$$

where we make the change $m \rightarrow -n$ to obtain:

$$S_n^{(n-k)} = \sum_{j=0}^{k} (-1)^j \binom{-n+k}{-n-j}\binom{-n-j-1}{k-j} S_{j+k}^{[j]}, \tag{9}$$

but:

$$\binom{-n+k}{-n-j} = \binom{k-n}{k+j}, \qquad \binom{-n-j-1}{k-j} = (-1)^{k+j}\binom{k+n}{k-j},$$

then (9) implies the Schläfli's identity [4, 25]:

$$S_n^{(n-k)} = (-1)^k \sum_{j=0}^{k} \binom{k-n}{k+j}\binom{k+n}{k-j} S_{j+k}^{[j]}; \tag{10}$$

Gould [4, 26, 27] deduced the inverse of (10):

$$S_n^{[n-k]} = (-1)^k \sum_{j=0}^{k} \binom{k-n}{k+j}\binom{k+n}{k-j} S_{j+k}^{(j)}. \tag{11}$$

On the other hand, Choi [28] used the multiple Hurwitz zeta function to obtain the following explicit expression for the generalized Bernoulli polynomials [2]:

$$B_{n+k}^{(n)}(x) = n\binom{n+k}{n} \sum_{j=0}^{n-1} (-1)^j \frac{B_{j+k+1}(x)}{j+k+1} \sum_{l=j}^{n-1} \binom{l}{j} S_n^{(l+1)}\, x^{l-j}, \qquad n \geq 1, \tag{12}$$

such that $B_r^{(m)}(0) = B_r^{(m)}$, $B_m(x) = B_m^{(1)}(x)$, and the Bernoulli numbers are given by $B_m = B_m(0)$. Therefore, (12) with $x = 0$ implies the relation:

$$B_{n+k}^{(n)} = n\binom{n+k}{n} \sum_{r=1}^{n} \frac{(-1)^{r+1}}{k+r} B_{k+r}\, S_n^{(r)}, \tag{13}$$

but from (3):

$$B_{n+k}^{(n)} = \sum_{j=1}^{n+k} (-1)^j \frac{\binom{2n+k}{n+k-j}\binom{n+j-1}{j}}{\binom{n+k+j}{j}} S_{n+k+j}^{[j]}, \tag{14}$$

then (13) and (14) allow to deduce the interesting identity:

$$\sum_{r=1}^{n} \frac{(-1)^{r+1}}{k+r} B_{k+r}\, S_n^{(r)} = \sum_{j=1}^{n+k} \frac{(-1)^j}{n+j} \frac{\binom{2n+k}{n+j}}{\binom{n+k+j}{k}} S_{n+k+j}^{[j]}, \qquad n \geq 1. \tag{15}$$

For example, if $n = 1$ the expression (15) reproduces (3); if $k = 0$ then (15) gives the property:

$$\sum_{r=1}^{n} \frac{(-1)^{r+1}}{r} B_r\, S_n^{(r)} = \sum_{r=1}^{n} \frac{(-1)^r}{n+r} \binom{2n}{n+r} S_{n+r}^{[r]}, \qquad n \geq 1. \tag{16}$$

3. JANJIC'S DEFINITIONS FOR STIRLING NUMBERS

Janjic [29] employs the relations:

$$\frac{d^n}{dx^n} f(u) = \sum_{k=0}^{n} S_n^{[k]}\, u^k \frac{d^k}{du^k} f(u), \qquad u = e^x, \tag{17}$$

$$\frac{d^n}{dx^n} f(v) = \frac{1}{x^n} \sum_{k=0}^{n} S_n^{(k)} \frac{d^k}{dv^k} f(v), \qquad v = \ln x, \tag{18}$$

where $f(t)$ is an arbitrary function, to define the Stirling numbers $S_n^{[k]}$ and $S_n^{(k)}$ [4]. Here we show that (17) and (18) are consequences of the formulas of Hoppe [30, 31] and Quaintance-Gould [4], respectively. In fact, into the Hoppe's expression:

$$\frac{d^n}{dx^n} f(u) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \frac{d^k}{du^k} f(u) \sum_{j=0}^{k} (-1)^j \binom{k}{j} u^{k-j} \frac{d^n}{dx^n} u^j, \tag{19}$$

we apply $u = e^x$, thus $\frac{d^n}{dx^n} u^j = j^n u^j$, then (17) is immediate by the Euler's result [4]:

$$S_n^{[k]} = \frac{(-1)^k}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^n. \tag{20}$$

We know the property [4]:

$$n! \binom{y}{n} = \sum_{k=0}^{n} S_n^{(k)} y^k, \tag{21}$$

where we can use $y = \frac{d}{dv}$ to deduce:

$$n! \binom{\frac{d}{dv}}{n} = \sum_{k=0}^{n} S_n^{(k)} \frac{d^k}{dv^k}, \tag{22}$$

besides, we have the relation [4]:

$$\frac{d^n}{dx^n} f(v) = \frac{n!}{x^n} \binom{\frac{d}{dv}}{n} f(v), \qquad v = \ln x, \tag{23}$$

therefore (22) and (23) imply (18). The expressions (17) and (18) are the formulas (8.13) and (12.34) in [4], respectively.

4. CONCLUSIONS

Our procedure shows the usefulness of (1), (2), (4) and (6), that is, of the Todorov's formula, the Nörlund polynomials [generalized Bernoulli numbers], and the Stirling numbers and binomial coefficients with negative indices [duality property]. The application of (6) in (10) and (11) gives the Sun's identities [23, 32] for Stirling numbers of the first and second kind.

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