SHORT COMMUNICATION

Todorov’s formula involving Stirling numbers of the second kind and Nörlund polynomials

A. Zúñiga-Segundo¹, J. López-Bonilla²*, S. Vidal-Beltrán²

¹ Depto. Física, ESFM, Instituto Politécnico Nacional, Edif. 9, Col. Lindavista CP 07738, CDMX, México
² ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 4, Col. Lindavista CP 07738, CDMX, México

*E-mail address: jlopezb@ipn.mx

ABSTRACT

We use the Todorov’s formula for the generalized Bernoulli numbers, in terms of Stirling numbers of the second kind, to deduce the Guo-Qi and Schläfli identities. Our approach is based in the duality property between the Stirling numbers, and in the Nörlund polynomials. We also consider the Janjic’s definitions for the Stirling numbers.

Keywords: Bernoulli numbers, Nörlund polynomials, Stirling numbers, Duality property, Hoppe’s formula, Generalized chain rule of differentiation

1. INTRODUCTION

Todorov [1, 2] deduced the relation:

\[ B_n^{(x)} = \sum_{k=0}^{n} (-1)^k \frac{(n+k)}{n+k} \binom{n}{k} S_{n+k}^k, \]  

(Received 30 July 2018; Accepted 11 August 2018; Date of Publication 14 August 2018)
where: \(S_r^{[k]}\) are the Stirling numbers of the second kind [2-6], \(B_n^{(z)}\) are the generalized Bernoulli numbers or Nörlund polynomials defined via the following generating function [7, 8]:

\[
\sum_{n=0}^{\infty} B_n^{(z)} \frac{x^n}{n!} = \left(\frac{x}{e^x-1}\right)^z,
\]

(2)

and \(B_n = B_n^{(1)}\) are the Faulhaber [9]-Bernoulli [10] numbers [2-4, 11, 12].

In Sec. 2 we show that (1) and (2) allow obtain the Guo-Qi and Schläfli identities, and also the generating function for \(S_r^{[k]}\). In Sec. 3 we exhibit that the definitions of Janjic for the Stirling numbers are consequences of the formulas of Hoppe and Quaintance-Gould [4].

2. NÖRLUND POLYNOMIALS

The expression (1) with \(z = 1\) implies the following relation of Guo-Qi [13, 14] involving the Bernoulli and Stirling numbers of the second kind:

\[
B_n = \sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} \frac{S_r^{[k]}}{n+k},
\]

(3)

this formula (3) also was proved in [4, 15-17].

If in (2) we employ \(z = -k\) and apply the relation of Carlitz [8, 18]:

\[
B_n^{(-k)} = \frac{1}{\binom{n+k}{k}} S_r^{[k]},
\]

(4)

we obtain the generating function for Stirling numbers [4]:

\[
\sum_{m=k}^{\infty} S_m^{[k]} \frac{x^m}{m!} = \frac{1}{k!} (e^x - 1)^k.
\]

(5)

We know the duality property [4, 19-23]:

\[
S_{-n}^{[-N]} = (-1)^{N+n} S_N^{[n]}, \quad S_{-n}^{[-N]} = (-1)^{N+n} S_{-n}^{(n)},
\]

(6)

where \(S_r^{(k)}\) are the Stirling numbers of the first kind [4, 11, 24]. Then, from (4) and (6):

\[
B_k^{(n)} = \frac{1}{\binom{k-n}{-n}} S_{-n}^{[-(n-k)]} = \frac{(-1)^k}{\binom{k-n}{-n}} S_{-n}^{(n-k)},
\]

(7)

but \(\binom{k-n}{-n} = (-1)^k \binom{n-1}{k}\), thus (7) implies the following identity [4, 8]:

-202-
Now, we apply (1) with \( z = -k \), and (4) to deduce:
\[
S_{k+m}^{[m]} = (-1)^k \sum_{j=0}^{k} (-1)^j \binom{m+k}{m-j} \binom{m-j-1}{k-j} S_{k+j}^{[j]} = (-1)^k S_{-m}^{(-k-m)} ,
\]
where we make the change \( m \to -n \) to obtain:
\[
S_{n}^{(n-k)} = \sum_{j=0}^{k} (-1)^j \binom{-n+k}{-n-j} \binom{-n-j-1}{k-j} S_{j+k}^{[j]} , \tag{9}
\]
but:
\[
\binom{-n+k}{-n-j} = \binom{k-n}{k+j} , \quad \binom{-n-j-1}{k-j} = (-1)^{k+j} \binom{k+n}{k-j} ,
\]
then (9) implies the Schl"afli’s identity [4, 25]:
\[
S_{n}^{(n-k)} = (-1)^k \sum_{j=0}^{k} \binom{k-n}{k+j} \binom{k+n}{k-j} S_{j+k}^{[j]} ; \tag{10}
\]
Gould [4, 26, 27] deduced the inverse of (10):
\[
S_{n}^{[n-k]} = (-1)^k \sum_{j=0}^{k} \binom{k-n}{k+j} \binom{k+n}{k-j} S_{j+k}^{(j)} . \tag{11}
\]
On the other hand, Choi [28] used the multiple Hurwitz zeta function to obtain the following explicit expression for the generalized Bernoulli polynomials [2]:
\[
B_{n+k}^{(n)}(x) = n \binom{n+k}{n} \sum_{j=0}^{n-1} (-1)^j \frac{B_{j+k+1}^{(j)}(x)}{j+k+1} \sum_{l=j}^{n-1} \binom{l+1}{j} S_{n}^{(l+1)} x^{l-j} , \quad n \geq 1 , \tag{12}
\]
such that \( B_r^{(m)}(0) = B_r^{(m)} , \quad B_m(x) = B_m^{(1)}(x) \), and the Bernoulli numbers are given by \( B_m = B_m(0) \). Therefore, (12) with \( x = 0 \) implies the relation:
\[
B_{n+k}^{(n)}(0) = n \binom{n+k}{n} \sum_{r=1}^{n} \frac{(-1)^{r+1}}{k+r} B_{k+r}^{(r)} S_{n}^{(r)} , \tag{13}
\]
but from (3):
\[
B_{n+k}^{(n)} = \sum_{j=1}^{n+k} (-1)^j \binom{2n+k}{n+k-j} \binom{n+j-1}{j} \binom{n+k+j}{j} S_{n+k+j}^{[j]} , \tag{14}
\]
then (13) and (14) allow to deduce the interesting identity:
\begin{equation}
\sum_{r=1}^{n} \frac{(-1)^{r+1}}{k+r} B_{k+r} S_{n}^{(r)} = \sum_{j=1}^{n+k} \left( \frac{2n+k}{n+j} \right) \left( \frac{n+j}{n+k+j} \right)^{r} s_{n+k+j}^{[j]}, \quad n \geq 1. \tag{15}
\end{equation}

For example, if \( n = 1 \) the expression (15) reproduces (3); if \( k = 0 \) then (15) gives the property:

\begin{equation}
\sum_{r=1}^{n} \frac{(-1)^{r+1}}{r} B_{r} S_{n}^{(r)} = \sum_{r=1}^{n} \frac{(-1)^{r}}{n+r} \left( \frac{2n}{n+r} \right) s_{n+r}^{[r]}, \quad n \geq 1. \tag{16}
\end{equation}

3. JANJIC’S DEFINITIONS FOR STIRLING NUMBERS

Janjic [29] employs the relations:

\begin{equation}
\frac{d^n}{dx^n} f(u) = \sum_{k=0}^{n} S_n^{[k]} u^k \frac{d^k}{du^k} f(u), \quad u = e^x, \tag{17}
\end{equation}

\begin{equation}
\frac{d^n}{dx^n} f(v) = \frac{1}{x^n} \sum_{k=0}^{n} S_n^{(k)} \frac{d^k}{dv^k} f(v), \quad v = \ln x, \tag{18}
\end{equation}

where \( f(t) \) is an arbitrary function, to define the Stirling numbers \( S_n^{[k]} \) and \( S_n^{(k)} \) [4]. Here we show that (17) and (18) are consequences of the formulas of Hoppe [30, 31] and Quaintance-Gould [4], respectively. In fact, into the Hoppe’s expression:

\begin{equation}
\frac{d^n}{dx^n} f(u) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \frac{d^k}{du^k} f(u) \sum_{j=0}^{k} (-1)^j \left( \begin{array}{c} k \\ j \end{array} \right) u^{k-j} \frac{d^n}{dx^n} u^j, \tag{19}
\end{equation}

we apply \( u = e^x \), thus \( \frac{d^n}{dx^n} u^j = j^n u^j \), then (17) is immediate by the Euler’s result [4]:

\begin{equation}
S_n^{[k]} = \frac{(-1)^k}{k!} \sum_{j=0}^{k} (-1)^j \left( \begin{array}{c} k \\ j \end{array} \right) j^n. \tag{20}
\end{equation}

We know the property [4]:

\begin{equation}
n! \left( \begin{array}{c} y \\ n \end{array} \right) = \sum_{k=0}^{n} S_n^{(k)} y^k, \tag{21}
\end{equation}

where we can use \( y = \frac{d}{dv} \) to deduce:

\begin{equation}
n! \left( \frac{d}{dv} \right) = \sum_{k=0}^{n} S_n^{(k)} \frac{d^k}{dv^k}, \tag{22}
\end{equation}

besides, we have the relation [4]:

-204-
therefore (22) and (23) imply (18). The expressions (17) and (18) are the formulas (8.13) and (12.34) in [4], respectively.

4. CONCLUSIONS

Our procedure shows the usefulness of (1), (2), (4) and (6), that is, of the Todorov’s formula, the Nörlund polynomials [generalized Bernoulli numbers], and the Stirling numbers and binomial coefficients with negative indices [duality property]. The application of (6) in (10) and (11) gives the Sun’s identities [23, 32] for Stirling numbers of the first and second kind.

References


